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Langevin equation for the rotation of a magnetic particle

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We investigate Brownian motion on a curved surface with the help of the covariant Langevin and Fokker–Planck equations. As a special example we address the orientational diffusion of a ferromagnetic particle immersed in a viscous fluid under the influence of a time-dependent magnetic field. This system may model the recently discovered thermal ratchet behaviour in ferrofluids. Numerical simulations of the Langevin equation are shown to be in good agreement with results from the solution of the Fokker–Planck equation reported previously. Copyright © 2004 John Wiley & Sons, Ltd.

KEYWORDS: Brownian motion; ferrofluids; thermal ratchet; Langevin equation

INTRODUCTION

Independent of its macroscopic motion, matter is always in chaotic motion on the microscopic level. To extract macroscopic work from these random fluctuations under thermal equilibrium is excluded by the second law of thermodynamics. However, directed transport *can* be achieved in so-called Brownian motors or ratchets. These are devices that combine dissipative motion in the presence of thermal fluctuations and some time-dependent deterministic force that drives the system out of equilibrium without introducing an obvious bias into one direction of motion.[1]

A particularly clear example of such a ratchet is provided by an overdamped particle in a time-dependent on–off spatially sawtooth potential subject to some random noise.[2] With the potential switched on and off regularly, diffusion and relaxation combine to yield a non-zero average particle drift. Since the dynamics of Brownian motors are significantly influenced by microscopic fluctuations, these motors are usually rather tiny. If only one individual such system is observed in the experiment, the experimental observation requires a quite sophisticated technology. However, it has been shown[3] that ferrofluids enable one to observe fluctuation-driven *macroscopic* transport.

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Ferrofluids are colloidal suspensions of ferromagnetic nano-grains.[4] Owing to their small size, of approximately 10 nm, the nanoparticles in the ferrofluid are significantly influenced by thermal fluctuations.[5] A potential for the orientation of the particles can be easily realized by an external magnetic field. If the nano-grains are set into rotation by rectifying thermal fluctuations with the aid of a time-dependent magnetic field, they cooperate in the transfer of the angular momentum they get from the magnetic field to the surrounding carrier fluid, resulting in a macroscopic torque acting on the entire ferrofluid.[3]

In previous work[3] the time- and ensemble-averaged torque exerted by the magnetic field on a ferromagnetic nanoparticle was theoretically predicted from the solution of the Fokker–Planck equation for the rotation of the particle in the presence of a magnetic field. In this paper we derive the Langevin equation for the same problem and investigate its connection with the Fokker–Planck equation. In addition, we solve the Langevin equation numerically in order to confirm the prediction of thermal ratchet behaviour.

Our paper is organized as follows. We first derive the Fokker–Planck equation and the Langevin equation for dissipative motion on arbitrary curved surfaces in the presence of thermal fluctuations and a deterministic force exerted by a potential U. The results can be used to analyse the rotational Brownian motion of a ferromagnetic particle in a viscous fluid. Next, we present an alternative derivation of the Langevin equation that is based on the balance of the torques acting on the magnetic nanoparticle. We then solve the Langevin equation numerically for a magnetic particle in a time-dependent magnetic field and calculate the resulting torque exerted by the magnetic field. The results of our study are then summarized.

DIFFUSION ON A CURVED SURFACE

Mathematical preliminaries

In this section we consider the motion of a particle on a two-dimensional surface in three-dimensional space. The surface can be parametrized by two parameters u_1 and u_2, i.e. it is given by an $\mathbb{R}^3$-valued function $\boldsymbol{x}(u_1, u_2)$. The geometric properties of the surface are comprised in the metric tensor

$$g_{ij} = \frac{\partial \boldsymbol{x}}{\partial u_i} \cdot \frac{\partial \boldsymbol{x}}{\partial u_j} \quad i, j \in \{1, 2\} \tag{1}$$

The determinant of (g_{ij}) is usually denoted by g and g^{ij} represents the components of the inverse matrix of (g_{ij}).

There are many different parameterizations of the same surface possible, e.g. one can map the pair of parameters (u_1, u_2) onto different parameters (u_1', u_2') given by the transformation

$$u_i' = u_i'(u_1, u_2) \tag{2}$$

Using these new coordinates the surface can be described by a new function $\boldsymbol{x}'(u_1', u_2')$ that fulfils the condition

$$\boldsymbol{x}'(u_1'(u_1, u_2), u_2'(u_1, u_2)) = \boldsymbol{x}(u_1, u_2) \tag{3}$$

A function φ that is defined on the surface is called a scalar, or reparameterization invariant function, if it does not change its values at any point under a reparameterization $u_i \to u_i'$ of the surface. It is known that if φ and ψ are two scalar functions, then the expression

$$\frac{1}{\sqrt{g}} \partial_i (\varphi \sqrt{g} g^{ij} \partial_j \psi) \tag{4}$$

is also a scalar function.[6,7] That means for two different parameterizations u_i and u_i' the equation

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial u_i} \left(\varphi \sqrt{g} g^{ij} \frac{\partial \psi}{\partial u_j} \right) = \frac{1}{\sqrt{g'}} \frac{\partial}{\partial u_i'} \left(\varphi' \sqrt{g'} g'^{ij} \frac{\partial \psi'}{\partial u_j'} \right) \tag{5}$$

holds at every point of the surface. Here, the functions φ' and ψ' are given by the two relations

$$\varphi'(u_1'(u_1, u_2), u_2'(u_1, u_2)) = \varphi(u_1, u_2) \tag{6}$$

$$\psi'(u_1'(u_1, u_2), u_2'(u_1, u_2)) = \psi(u_1, u_2) \tag{7}$$

The expression in Eqn (4) represents a generalization of the derivative $\nabla \cdot (\varphi \nabla \psi)$ in flat space. For $\varphi = 1$ the operator $1/\sqrt{g}\partial_i \sqrt{g} g^{ij} \partial_j$ acting on ψ is called the Laplace–Beltrami operator.

The Fokker–Planck equation

We now turn to the physical problem of a particle moving along the surface under a deterministic force given by the gradient of a potential $U(u_1, u_2)$ and additional random forces due to thermal fluctuations. We will only consider the overdamped case in which inertial terms in the equation of motion may be neglected. The stochastic dynamics of the particle can be described by the probability density $\rho(u_1, u_2, t)$ of its position on the surface at time t. In the case of a flat surface the Fokker–Planck equation for the time evolution of this probability function reads

$$\partial_t \rho = \nabla \cdot (\rho \nabla U) + D \Delta \rho \tag{8}$$

We derive a generalization of this equation by equating scalar functions which are generalizations of the different terms appearing in Eqn (8) (for an alternative derivation of the covariant form of the Fokker–Planck equation see also Ref. 8 (section 4.10)). The expression $\rho \mathrm{d}u_1 \, \mathrm{d}u_2$ represents the probability that the diffusing particle is in the interval $[u_1, u_1 + \mathrm{d}u_1] \times [u_2, u_2 + \mathrm{d}u_2]$. Therefore, it is a scalar quantity. $\sqrt{g}\, \mathrm{d}u_1 \, \mathrm{d}u_2$ is the area of the surface corresponding to the aforementioned interval and is, therefore, also a scalar quantity. Dividing two scalar quantities we infer that $\rho/\sqrt{g}$ and $\partial_t \rho/\sqrt{g}$ are scalar functions as well. Scalar functions that are generalizations of the terms on the right-hand side (r.h.s.) of Eqn (8) are easily obtained, if one identifies $\varphi = \rho/\sqrt{g}$ and $\psi = U$ for the first term and $\varphi = 1$ and $\psi = \rho/\sqrt{g}$ for the second term on the r.h.s. of Eqn (8) and uses Eqn (4). The resulting Fokker–Planck equation for the diffusion on a curved surface then reads

$$\frac{\partial_t \rho}{\sqrt{g}} = \frac{1}{\sqrt{g}} \partial_i (\rho g^{ij} \partial_j U) + \frac{D}{\sqrt{g}} \partial_i \left[\sqrt{g} g^{ij} \partial_j \left(\frac{\rho}{\sqrt{g}} \right) \right] \tag{9}$$

which can be simplified to

$$\partial_t \rho = \partial_i (\rho g^{ij} \partial_j U) + D \partial_i \left[\sqrt{g} g^{ij} \partial_j \left(\frac{\rho}{\sqrt{g}} \right) \right] \tag{10}$$

Insertion of the equilibrium distribution at a given temperature T

$$\rho = \sqrt{g}\, \mathrm{e}^{-U/k_\mathrm{B} T} \tag{11}$$

into Eqn (10) yields the noise strength $D = k_\mathrm{B} T$.

The Langevin equation

The Langevin equation[8–10] for the diffusion of a particle on a curved surface has the form

$$\mathrm{d}u_i = b^i \, \mathrm{d}t + c_l^i \, \mathrm{d}w_l \quad i = 1, 2 \quad l = 1, 2, 3, \ldots \tag{12}$$

where the $w_l(t)$ represent independent standard Brownian motions. That means they are independent, scalar random functions whose increases $w_l(t_1) - w_l(t_2)$ and $w_l(t_3) - w_l(t_4)$ during two separated time intervals $[t_2, t_1]$ and $[t_4, t_3]$ ($t_1 \geq t_2 \geq t_3 \geq t_4$) are independent, normally distributed random numbers that possess the ensemble averaged means

$$\langle w_l(t_1) - w_l(t_2) \rangle = 0 \tag{13}$$

$$\langle (w_l(t_1) - w_l(t_2))^2 \rangle = t_1 - t_2 \tag{14}$$

$$\langle (w_l(t_1) - w_l(t_2))(w_l(t_3) - w_l(t_4)) \rangle = 0 \tag{15}$$

The starting point of $w_l(t)$ is usually fixed to $w_l(0) = 0$. Equation (12) is meant in the Ito calculus (as all Langevin equations are in this paper, if nothing different is notified). The first task is to determine the specific form of the coefficients b^i and c_l^i. This can be accomplished by exploiting the transformation properties. After a transformation into a different coordinate system we obtain with the help of the Ito lemma the Langevin equation

$$\mathrm{d}u'_k = b'^k \,\mathrm{d}t + c'^k_l \,\mathrm{d}w_l \tag{16}$$

$$= \frac{\partial u'_k}{\partial u_i}(b^i \,\mathrm{d}t + c_l^i \,\mathrm{d}w_l) + \frac{1}{2}\frac{\partial^2 u'_k}{\partial u_i \partial u_j} c_l^i c_l^j \,\mathrm{d}t \tag{17}$$

This yields the transformation rules

$$c'^k_l = \frac{\partial u'_k}{\partial u_i} c_l^i \tag{18}$$

$$b'^k = \frac{\partial u'_k}{\partial u_i} b^i + \frac{1}{2}\frac{\partial^2 u'^k}{\partial u_i \partial u_j} c_l^i c_l^j \tag{19}$$

In the local Euclidean coordinate system of the surface, i.e. for $g_{ij} = \delta_{ij}$ and $\partial g_{ij}/\partial u_k = 0$, the coefficients of Eqn (12) are the same as on a flat surface:

$$c_l^i c_l^j = \delta_{ij} 2D \tag{20}$$

$$b^i = -\partial_i U \tag{21}$$

From the transformation rules of Eqns (18) and (19) we hence obtain the following general expressions for the coefficients:

$$c_l^i c_l^j = g^{ij} 2D \tag{22}$$

$$b^i = -g^{ij}\partial_j U + \frac{1}{\sqrt{g}}\frac{\partial \sqrt{g} g^{ij}}{\partial u_j} D \tag{23}$$

Using these coefficients one can verify that the Fokker–Planck equation corresponding to the Langevin equation, Eqn (12)

$$\partial_t \rho = -\partial_i (b^i \rho) + \frac{1}{2}\partial_i \partial_j (c_l^i c_l^j \rho) \tag{24}$$

agrees with Eqn (10) (for the connection between Fokker–Planck equations and stochastic differential equations, e.g. see Ref. 9 (section 4.3.4)).

It can also be seen from Eqn (24) that the stochastic properties of a process represented by a Langevin equation of the form in Eqn (12) are completely determined by the expressions b^i and $c_l^i c_l^j$. This means that it is not necessary to fix the coefficients c_l^i themselves, nor even the number of independent Brownian motions (i.e. the maximum of the index l) in Eqn (12), except for concrete numerical simulations.

The Langevin equation in Cartesian coordinates

In this section we derive the Langevin equation for the diffusion on a surface in terms of the Cartesian coordinates of the embedding three-dimensional space. This means our aim is a representation of the motion that is independent of the parameterization of the surface. For that purpose we apply Ito's formula to the function $\boldsymbol{x}(u_1(t), u_2(t))$ and obtain the stochastic differential equation

$$\mathrm{d}\boldsymbol{x} = \frac{\partial \boldsymbol{x}}{\partial u_i}(b^i \,\mathrm{d}t + c_l^i \,\mathrm{d}w_l) + \frac{1}{2}\frac{\partial^2 \boldsymbol{x}}{\partial u_i \partial u_j} c_l^i c_l^j \,\mathrm{d}t \tag{25}$$

Some calculations reveal that this equation is equivalent to the Langevin equation

$$\mathrm{d}\boldsymbol{x} = (E_3 - \boldsymbol{n}\boldsymbol{n}) \cdot (-(\nabla U)\,\mathrm{d}t + \sqrt{2D}\,\mathrm{d}\boldsymbol{w}) - D\kappa \boldsymbol{n}\,\mathrm{d}t \tag{26}$$

where E_3 is the 3×3 unit matrix, $\boldsymbol{n}$ is the unit vector perpendicular to the surface, $\kappa = \nabla \cdot \boldsymbol{n}$ is the mean curvature of the surface, and $\boldsymbol{w}(t)$ represents a three-dimensional standard Brownian motion (i.e. each component of $\boldsymbol{w}(t)$ is an independent standard Brownian motion). The first term in Eqn (26) is just the expected projection of the three-dimensional dynamics on the surface perpendicular to $\boldsymbol{n}$. The second term is specific to a stochastic dynamics. If a particle moves a short distance $\mathrm{d}l$ tangential to a curved surface, it moves a small distance $\mathrm{d}l_\perp$ of order $\mathrm{d}l^2$ away from the surface, as shown in Fig. 1. In a deterministic motion this effect can be neglected, since $\mathrm{d}l^2 \sim \mathrm{d}t^2$. For a stochastic motion, however, the second moment of the tangential distance $\mathrm{d}l$ traversed by the particle during a small time interval $\mathrm{d}t$ increases *linearly* with time, i.e. $\langle \mathrm{d}l^2 \rangle \sim \mathrm{d}t$. To stay on the surface this effect must, hence, be explicitly compensated in the Langevin equation.

Example: diffusion on the unit sphere

The rotation of a ferromagnetic mono-domain particle can be described by the unit vector parallel to the direction of the

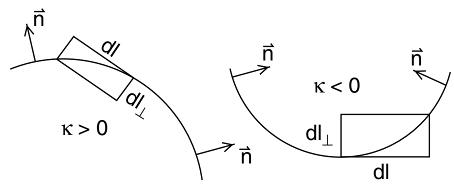

Figure 1. Sketch of the diffusion on a curved surface and explanation for the last term in Eqn (26).

magnetic dipole moment. This unit vector can be expressed with the help of two angles Θ and φ by

$$n = \begin{pmatrix} \sin\Theta\cos\varphi \\ \sin\Theta\sin\varphi \\ \cos\Theta \end{pmatrix} \quad (27)$$

For this parameterization, $u_1 = \Theta$ and $u_2 = \varphi$, of the unit sphere the metric tensor is given by

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\Theta \end{pmatrix} \quad (28)$$

Using the Eqns (22) and (23) we find for the coefficients in Eqn (12)

$$b^1 = -\partial_\Theta U + D\cot\Theta \quad (29)$$

$$b^2 = -\frac{1}{\sin^2\Theta}\partial_\varphi U \quad (30)$$

$$c_1^1 = \sqrt{2D} \quad (31)$$

$$c_2^1 = 0 \quad (32)$$

$$c_1^2 = 0 \quad (33)$$

$$c_2^2 = \frac{\sqrt{2D}}{\sin\Theta} \quad (34)$$

The Langevin equations for the diffusion on a unit sphere in spherical coordinates then take the form

$$d\Theta = (-\partial_\Theta U + D\cot\Theta)\,dt + \sqrt{2D}\,dw_\Theta \quad (35)$$

$$d\varphi = \frac{1}{\sin^2\Theta}(-\partial_\varphi U)\,dt + \frac{\sqrt{2D}}{\sin\Theta}\,dw_\varphi \quad (36)$$

where $w_\Theta(t)$ and $w_\varphi(t)$ are independent standard Brownian motions.

Note that even in the absence of a deterministic force ($U = 0$) the Langevin equation for Θ (Eqn (35)) contains a deterministic part $D\cot\Theta$. This can be understood intuitively as follows. The Brownian motion of the particle on the unit sphere has two independent components, one goes along the meridian (φ = const, represented by dw_Θ) and the other one goes along the great circle that intersects the meridian at a right angle (represented by dw_φ). The situation is sketched in Fig. 2. This great circle moves away from the parallel of latitude and hence the stochastic motion shifts the particle nearer to the equator ($\Theta = \pi/2$). This effect is quantitatively expressed by the $D\cot\Theta$ term.

For the special case of a magnetic field $\boldsymbol{H}$ acting on the particle with magnetic dipole moment $\boldsymbol{m} = m\boldsymbol{n}$, the potential is given by $U = -\boldsymbol{m}\cdot\boldsymbol{H} = -m\boldsymbol{n}\cdot\boldsymbol{H}$.

The Fokker–Planck equation for the Brownian motion on the unit sphere reads

$$\partial_t\rho = \partial_\Theta(\rho\partial_\Theta U) + \frac{1}{\sin^2\Theta}\partial_\varphi(\rho\partial_\varphi U) + D\partial_\Theta \times \left[\sin\Theta\,\partial_\Theta\left(\frac{\rho}{\sin\Theta}\right)\right] + \frac{D}{\sin^2\Theta}\partial_\varphi^2\rho \quad (37)$$

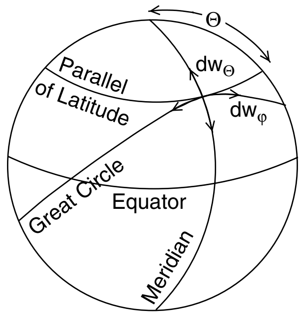

Figure 2. Sketch of the Brownian motion on a unit sphere.

or, after rescaling to $\tilde{\rho} = \rho/\sin\Theta$:

$$\partial_t\tilde{\rho} = \frac{1}{\sin\Theta}\partial_\Theta(\tilde{\rho}\sin\Theta\,\partial_\Theta U) + \frac{1}{\sin^2\Theta}\partial_\varphi(\tilde{\rho}\partial_\varphi U) + \frac{D}{\sin\Theta}\partial_\Theta(\sin\Theta\,\partial_\Theta\tilde{\rho}) + \frac{D}{\sin^2\Theta}\partial_\varphi^2\tilde{\rho} \quad (38)$$

The numerical integration of Eqns (35) and (36) inflicts some difficulties, if the particle comes close to the poles ($\Theta = 0$ and $\Theta = \pi$). Therefore, it is easier to solve the Langevin equation in Cartesian coordinates. For the surface of the unit sphere we have $\boldsymbol{x} = \boldsymbol{n}$ and $\kappa = 2$ and Eqn (26) yields the stochastic differential equation

$$d\boldsymbol{n} = (E_3 - \boldsymbol{nn})\cdot(-(\nabla U)\,dt + \sqrt{2D}\,d\boldsymbol{w}) - 2D\boldsymbol{n}\,dt \quad (39)$$

which has the property that $|\boldsymbol{n}| = 1$ remains constant, as it should.

ROTATION OF A FERROMAGNETIC PARTICLE IN A MAGNETIC FIELD

Derivation of the Langevin equation

In this section we consider a ferromagnetic particle that carries a magnetic dipole moment $\boldsymbol{m} = m\boldsymbol{n}$ (m = const) and rotates in a time-dependent magnetic field $\boldsymbol{H}(t)$. We assume that the particle has a spherical shape (typical radius is about 10 nm) of volume V and is suspended in a fluid of viscosity η. Since the direction of the dipole moment is fixed to the orientation of the particle, the rate of change of the unit vector $\boldsymbol{n}$ is given by

$$\dot{\boldsymbol{n}} = \boldsymbol{\Omega}\times\boldsymbol{n} \quad (40)$$

where $\boldsymbol{\Omega}$ represents the axis and angular frequency of the rotation of the particle. The time dependence of $\boldsymbol{\Omega}$ can be determined from the condition that the rate of change of the angular momentum $I\boldsymbol{\Omega}$ equals the sum of all torques acting on the particle. Here, I is the moment of inertia of the rotating particle. If a sphere of volume V rotates in a fluid of viscosity η at a small Reynolds number then the velocity field in the fluid can be obtained from the solution of the Stokes equation. One can then calculate the torque that slows down the rotation of the sphere due to energy dissipation in the fluid, yielding $-6\eta V\boldsymbol{\Omega}$.[11] The magnetic field contributes a torque of the form $\boldsymbol{m} \times \boldsymbol{H} = m\boldsymbol{n} \times \boldsymbol{H}$ that acts on the particle. Finally, the molecules of the surrounding fluid permanently collide with the suspended nanoparticle and transfer angular momentum to it. Since these collisions cannot be predicted due to the lack of information about the trajectories of the molecules, this contribution will be expressed by a random term $\sqrt{2D}\boldsymbol{\xi}(t)$. $\boldsymbol{\xi}(t) = \mathrm{d}_t\boldsymbol{w}$ represents the time derivative of a three-dimensional standard Brownian motion. Hence, its components fulfil the conditions

$$\langle \xi_i(t) \rangle = 0 \tag{41}$$

$$\langle \xi_i(t)\xi_j(t') \rangle = \delta(t-t')\delta_{ij} \quad i,\ j \in \{1,\ 2,\ 3\} \tag{42}$$

where $\langle \ldots \rangle$ represents the ensemble average.

The balance equation for the angular momentum of the ferromagnetic particle takes the form

$$I\dot{\boldsymbol{\Omega}} = -6\eta V\boldsymbol{\Omega} + m\boldsymbol{n} \times \boldsymbol{H} + \sqrt{2D}\boldsymbol{\xi}(t) \tag{43}$$

Eqns (40) and (43) represent a complete description of the rotation of the particle. They can be interpreted both in the Stratonovitch and Ito calculus, since they contain only additive noise. The noise strength at temperature T is given by $D = 6\eta V k_{\mathrm{B}} T$.

As above, we focus on the overdamped case where the inertia term $I\dot{\boldsymbol{\Omega}}$ on the left-hand side of Eqn (43) can be neglected. We then obtain

$$\boldsymbol{\Omega} = \frac{1}{6\eta V}(m\boldsymbol{n} \times \boldsymbol{H} + \sqrt{2D}\boldsymbol{\xi}) \tag{44}$$

Insertion of this expression into Eqn (40) and definition of a time unit such that $6\eta V = 1$ yields the Langevin equation

$$\mathrm{d}\boldsymbol{n} = (m\boldsymbol{n} \times \boldsymbol{H}\,\mathrm{d}t + \sqrt{2D}\,\mathrm{d}\boldsymbol{w}) \times \boldsymbol{n} \tag{45}$$

Note that this equation only conserves the length of the unit vector $\boldsymbol{n}$, if it is interpreted in the Stratonovitch calculus. A transformation from the Stratonovitch to the Ito calculus then yields the Langevin equation

$$\mathrm{d}\boldsymbol{n} = (m\boldsymbol{n} \times \boldsymbol{H}\,\mathrm{d}t + \sqrt{2D}\,\mathrm{d}\boldsymbol{w}) \times \boldsymbol{n} - 2D\boldsymbol{n}\,\mathrm{d}t \tag{46}$$

It can be shown that this equation is equivalent to Eqn (39), since $U = -m\boldsymbol{n} \cdot \boldsymbol{H}$ is the potential energy of a magnetic dipole moment $\boldsymbol{m} = m\boldsymbol{n}$ in a magnetic field $\boldsymbol{H}$.

Thermal ratchet behaviour

We consider a time-dependent magnetic field consisting of a constant part in the x-direction and an oscillatory part along the y-direction (as in Engel *et al.*[3]):

$$\boldsymbol{H}(t) = (\alpha_x/m,\ \alpha_y f(t)/m,\ 0) \tag{47}$$

with

$$f(t) = \cos(\omega t) + a\ \sin(2\omega t + \beta) \tag{48}$$

We solved Eqn (46) numerically and extracted the z-component of the averaged torque $\overline{\langle \boldsymbol{m} \rangle \times \boldsymbol{H}}$ exerted by the magnetic field on the particle. That means we calculated

$$\overline{\langle N_z \rangle} = (\overline{\langle \boldsymbol{m} \rangle \times \boldsymbol{H}})_z = \overline{\langle n_x \rangle \alpha_y f(t)} - \overline{\langle n_y \rangle \alpha_x} \tag{49}$$

as functions of the parameters α_x, α_y, a, β, ω, and D. Here $\overline{(\ldots)}$ denotes the time average over one period of the external field and $\langle \ldots \rangle$ represents the ensemble average. It turned out that $\overline{\langle N_z \rangle}$ is usually not zero, although the magnetic field $\boldsymbol{H}$ as defined above contains no net rotating component.[3]

Figure 3 shows the torque $\overline{\langle N_z \rangle}$ resulting from the Langevin equation, Eqn (46), and also, for comparison, resulting from the Fokker–Planck equation, Eqn (38), using the parameters $\alpha_x = 0.3$, $\alpha_y = 1$, $a = 1$, $\omega = 1$, and different values of β and D. Of course, the predictions from the Langevin equation and from the Fokker–Planck equation are in agreement. $\overline{\langle N_z \rangle}$ is a 2π-periodic function of β just as $\boldsymbol{H}(t)$. The thermal ratchet effect is strongest for intermediate noise strengths D because at larger thermal fluctuations the correlation between the orientation of the unit vector and the magnetic field becomes weaker. Figure 4 shows the torque $\overline{\langle N_z \rangle}$ resulting from Eqn (46) and also from Eqn (38) using the parameters $\alpha_x = 0.3$, $\alpha_y = 1$, $a = 1$, $\beta = 0$, $D = 0.2$, and different values of ω. The dependence of the average torque $\overline{\langle N_z \rangle}$ on ω clearly reveals that the ratchet effect is a kind of a resonance effect. The time scales of the deterministic driving and of the stochastic transitions caused by the noise have to match in order to get a noticeable result. For $\omega \to 0$ the torque vanishes because the thermal ratchet behaviour disappears for a constant magnetic

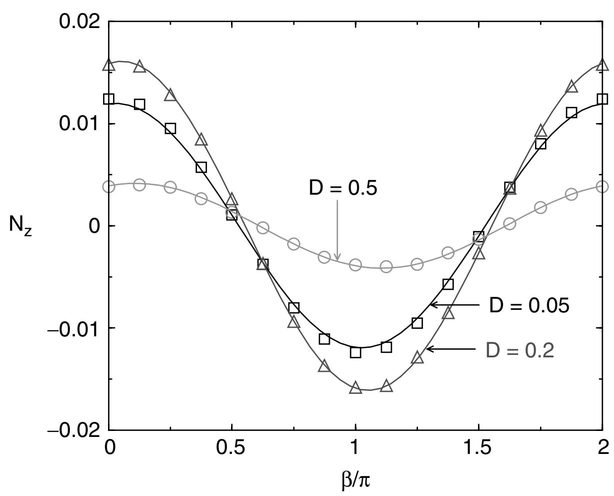

Figure 3. Torque $\overline{\langle N_z \rangle}$ resulting from Eqn (46) (squares, triangles, and circles) and resulting from Eqn (38) (lines) using the parameters $\alpha_x = 0.3$, $\alpha_y = 1$, $a = 1$, $\omega = 1$, and different values of D as a function of β.

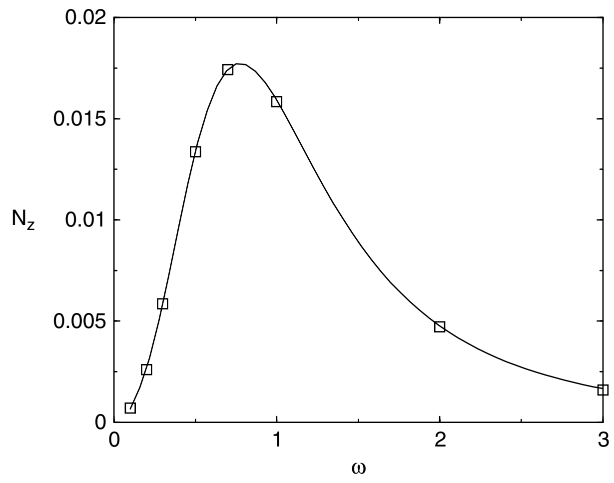

Figure 4. Torque $\overline{\langle N_z \rangle}$ resulting from the Eqns (46) (squares) and (38) (line) using the parameters $\alpha_x = 0.3$, $\alpha_y = 1$, $a = 1$, $\beta = 0$, and $D = 0.2$ as a function of ω.

field. In the opposite limit, $\omega \to \infty$, the torque $\overline{\langle N_z \rangle}$ also goes to zero because those terms in the integral version of Eqn (46) that stem from the oscillatory part of $\boldsymbol{H}(t)$ vanish. Therefore, we effectively have the same situation as in case of a time-independent magnetic field.

CONCLUSIONS

In this study, we have obtained the covariant Fokker–Planck equation and Langevin equation for the diffusion on a curved surface in two independent ways and ascertained their agreement. Then we investigated the specific example of diffusion on a unit sphere modelling the orientation of the magnetic dipole moment of a ferromagnetic particle. The resulting Langevin equation could also be obtained from a balance equation for the angular momentum transferred to the particle. Finally, we solved the Langevin equation for the rotation of a magnetic particle in the presence of a magnetic field with one constant and one oscillatory component. As a result we observed thermal ratchet behaviour due to the interaction of deterministic and random forces acting on the particle. That means the magnetic field transferred angular momentum to the particle and, therefore, to the entire ferrofluid, although it contained no net rotating component.

Acknowledgments

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REFERENCES

1. Reimann P. *Phys. Rep.* 2002; **361**: 57.
2. Ajdari A, Prost J. *C. R. Acad. Sci. Paris II* 1992; **315**: 1635.
3. Engel A, Müller HW, Reimann P, Jung A. *Phys. Rev. Lett.* 2003; **91**: 60 602.
4. Rosensweig RE. *Ferrohydrodynamics*. Cambridge University Press: Cambridge, 1985.
5. Shliomis MI. *Sov. Phys. Usp.* 1974; **17**: 153.
6. Einstein A. *Grundzüge der Relativitätstheorie*. Vieweg: Braunschweig, 1969.
7. Grosche G, Ziegler V. *Ergänzende Kapitel zu I.N. Bronstein and K.A. Semendjajew, Taschenbuch der Mathematik*. Verlag Harri Deutsch: Thun, 1980.
8. Risken H. *The Fokker–Planck Equation*. Springer-Verlag: Berlin, 1989.
9. Gardiner CW. *Handbook of Stochastic Methods*. Springer-Verlag: Berlin, 1985.
10. Van Kampen NG. *Stochastic Processes in Physics and Chemistry*. North-Holland: Amsterdam, 1992.
11. Guyon E, Hulin J-P, Petit L. *Hydrodynamik*. Vieweg: Braunschweig, 1997.