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NOTE

ON A PROBLEM OF R. HÄGGKVIST CONCERNING EDGE-COLOURING OF BIPARTITE GRAPHS

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In [3] R. Häggkvist posed the following problem: Let $Q(n,G)$ be the set of all proper edge-colourings of a graph G with n colours. Let $q \in Q(n,G)$. Define $L(q)$ $(l(q))$ as the maximum (minimum) length of cycle coloured with exactly two colours. Let

$$L(n,G) = \min\{L(q) : q \in Q(n,G)\}, \quad l(n,G) = \max\{l(q) : q \in Q(n,G)\}.$$

Give bounds on $L(n,G)$ and $l(n,G)$ for reasonably defined graphs. Especially: Is $L(n,K_{n,n}) = 2n$? (Here $K_{n,n}$ is the regular complete bipartite graph with $2n$ vertices.)

In this Note we consider only $L(n,K_{n,n})$. Our main result (see Theorem 5) is that $L(n,K_{n,n}) < 2n$ except when $n \in \{2,3,5\}$. In other words, except for these three values of n, there is always an edge-colouring of $K_{n,n}$ such that *no* union of two colour classes is a Hamilton cycle.

Zelinka [7] proved that $L(n,K_{n,n}) = 4$ for $n = 2^k$ and in [6] it is shown that $L(n,K_{n,n}) = 2n$ for $n \in \{2,3,5\}$, $L(n,K_{n,n}) = 6$ for $n = 3^k$ and $6 \le L(n,K_{n,n}) \le 10$ for $n = 5^k$, where k is a positive integer. It has been proved (cf. [5]) that $L(n,K_{n,n}) \le n$ for every even $n \ge 4$ and in [1] the following bound is given:

$$L(n,K_{n,n}) \le 2^{1-m} \prod_{i=1}^{m} L(p_i, K_{p_i,p_i}),$$

where $n = \prod_{i=1}^{m} p^{\alpha_i}$ is a prime factorization and $p_1 < p_2 < \cdots < p_m$.

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Let (X,Y) be the vertex bipartition of $K_{n,n}$, where $X=\{x_i : 1\leq i\leq n\}$ and $Y=\{y_i : 1\leq i\leq n\}$. A proper edge-colouring of $K_{n,n}$ using the colours $c_i, 1\leq i\leq n$, can be specified by an $n\times n$ array M whose entry in cell (i,j) is k when colour c_i is given to the edge $x_j y_k$. In fact, M is well-defined and is a latin square because (1) for given j, the n edges incident at x_j have different colours, so column j of M is a permutation of $\{1,2,\dots,n\}$ and (2) for given i, the edges of colour c_i form a matching between X and Y, so row i of M is a permutation of $\{1,2,\dots,n\}$. When M is bordered by its row and column labels it becomes the multiplication table of a quasigroup.

Let $\rho(i,i')$ denote the permutation that converts row i of M into row i'. This permutation can be written as a product of cycles, of length at least two, on disjoint sets of symbols.

A cycle in $K_{n,n}$ whose edges are coloured alternately a and b will be called an (a,b)-cycle.

Proposition 1. *Let G be a group of order n. Then there exists an edge-colouring of $K_{n,n}$ such that:*

(i) for each (a,b)-cycle of $K_{n,n}$ there is a $g\in G$ such that the length of the cycle is double the order of g.

(ii) for each non-identity element $g\in G$ there is an (a,b)-cycle of $K_{n,n}$ such that its length is double the order of g.

Proof. When M is the Cayley table of a group G, all the permutations $\rho(i,i')$ form, under composition, a group isomorphic to G. A permutation $\rho(i,i')$ that corresponds to an element of order r in G is a product of disjoint cycles all of the same length r. Each such cycle involves $2r$ entries lying in rows i and i' of M and corresponds to a (c_i,c_i')-cycle of length $2r$ in $K_{n,n}$. ∎

Proposition 2. *Let k be a positive integer and let p be an odd prime. Then*

$$6\leq L(n,K_{n,n})\leq 2p \quad \textit{for} \quad n=p^k.$$

Proof. Consider $\rho(i,i')$, where $i\neq i'$. This permutation cannot be a product of disjoint cycles all of length two because the sum of lengths of the cycles is p^k, which is odd. There are no cycles of length one so there is at least one cycle of length ≥ 3. Hence $L(n,K_{n,n})\geq 6$.

The inequality $L(n,K_{n,n})\leq 2p$ follows by applying Proposition 1 to the direct product,

$$G=Z_p\times Z_p\times\cdots\times Z_p,$$

of k copies of the cyclic group Z_p. ∎

Proposition 2 generalizes results of the theorem in [6]. The next theorem gives a new short proof of the principal theorem in [5].

Proposition 3. *For all even* $n \geq 4$, $L(n, K_{n,n}) \leq n$.

Proof. Let $n = 2k$. Consider the edge-colouring of $K_{n,n}$ determined by the Cayley table for the dihedral group D_k of order $2k$. As the order of every element of D_k is at most k, by Proposition 1, $L(n, K_{n,n}) \leq 2k = n$. ∎

Proposition 4. *For all odd* $n \geq 11$, $L(n, K_{n,n}) \leq 2n - 4$.

Proof. Let S be the cyclic latin square of order $n-4$, with entries from the set $\{1, 2, \ldots, n-4\}$, whose entry in cell (i, j) is $s(i, j) = i + j - 1 \pmod{n-4}$. Every left-to-right broken diagonal of S contains every symbol exactly once, because

$$(*) \qquad s(i+1, j+1) = s(i, j) + 2,$$

where all addition is mod $(n-4)$ and because $n-4$ is odd.

Convert S into a latin square M of order n by *prolongations* (see [2, p. 39]) as follows. Project the four left-to-right broken diagonals containing cells $(1,1)$, $(1,2)$, $(1,3)$ and $(1,5)$ horizontally to the right and vertically downwards into four new rows and columns. Replace the entries in the four diagonals by copies of new symbols $n-3, n-2, n-1$ and n, respectively. Fill the 4×4 intersection of the new rows and columns with the latin square H, namely:

$n-3$	n	$n-2$	$n-1$
$n-1$	$n-2$	n	$n-3$
n	$n-3$	$n-1$	$n-2$
$n-2$	$n-1$	$n-3$	n

By $(*)$ the new array M is a latin square.

We claim that there is at least one *intercalate*, that is, 2×2 latin subsquare [2, p. 42], in every pair of rows. For rows chosen from the last four this is obvious by inspection of H. Table 1, in which $4 \leq k \leq (n-3)/2$, gives details of intercalates in pairs of rows including the first row. (An intercalate is denoted by the pair of symbols it contains.) The presence of intercalates in other pairs of rows follows from $(*)$. (Note that 11 is the least odd value of n to which the proof applies, because the first line of the table requires that $6 < n-3$.)

Since every pair of rows of M contains an intercalate, every permutation $\rho(i, i')$ is a product of cycles of length at most $n-2$. Hence

$$L(n, K_{n,n}) \leq 2n - 4. \qquad ∎$$

As an example, Table 2 shows M when $n = 13$. The spaces should make the structure clear.

Other row	*Intercalate*
2	$n, 6$
3	$n-2, 4$
k	$n-1, k+2$
$n-3$	$n-3, 1$
$n-2$	$n-2, 1$
$n-1$	$n-1, 3$
n	$n, 5$

Table 1. Intercalates of M involving row 1

10	11	12	4	13	6	7	8	9	1	2	3	5
2	10	11	12	6	13	8	9	1	3	4	5	7
3	4	10	11	12	8	13	1	2	5	6	7	9
4	5	6	10	11	12	1	13	3	7	8	9	2
5	6	7	8	10	11	12	3	13	9	1	2	4
13	7	8	9	1	10	11	12	5	2	3	4	6
7	13	9	1	2	3	10	11	12	4	5	6	8
12	9	13	2	3	4	5	10	11	6	7	8	1
11	12	2	13	4	5	6	7	10	8	9	1	3
1	3	5	7	9	2	4	6	8	10	13	11	12
9	2	4	6	8	1	3	5	7	12	11	13	10
8	1	3	5	7	9	2	4	6	13	10	12	11
6	8	1	3	5	7	9	2	4	11	12	10	13

Table 2. M when $n=13$

Theorem 5. *$L(n, K_{n,n}) < 2n$ if, and only if, $n=4$ or $n \geq 6$.*

Proof. For $n = 2, 3, 5, 9$ see [6]; for even $n \geq 4$ see Proposition 3; for odd $n \geq 11$ see Proposition 4; for $n=7$ see [3, p. 121, Theorem 3.2]. ∎

References

[1] V. K. Bulitko and J.Ninčák: Group colourings of complete bipartite graphs and bounds of Häggkvist's numbers, *Math. Slovaca* **38(1)** (1988), 11–17 (in Russian).

[2] J. Dénes and A. D. Keedwell: *Latin Squares and their Applications*, Akad. Kiadó, Budapest; EUP, London; AP, New York, 1974.

[3] J. Dénes and A. D. Keedwell: *Latin Squares: New Developments in the Theory and Applications*, North-Holland, Amsterdam, 1991.

[4] R. Häggkvist: Problems, in: *Proc. of the 5th Hungarian colloquium on combinatorics*, Keszthely. Bolyai János Math. Soc., Budapest (1978), 1203–1204.

[5] J. Ninčák: On a problem of R. Häggkvist concerning edge-colourings of bipartite graphs, *BAM 255/84* (XXXIV), Budapest, (1984), 117–126.

[6] J. Ninčák: A note concerning the colourings of bipartite graphs, in: *Proc. of 6th Conference of EF TU, Math. Section*, Košice (1992), 65–69.
[7] B. Zelinka: On a problem of R. Häggkvist concerning edge-colouring of graphs, *Časop. pro pěstov. mat.* **103** (1978), 289–290.

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