Combinatorica **27** (3) (2007) 329–365
DOI: 10.1007/s00493-007-2010-5

COMBINATORICA
Bolyai Society – Springer-Verlag

K_4-FREE SUBGRAPHS OF RANDOM GRAPHS REVISITED

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Received October 10, 2002
Revised August 30, 2005

In Combinatorica 17(2), 1997, Kohayakawa, Łuczak and Rödl state a conjecture which has several implications for random graphs. If the conjecture is true, then, for example, an application of a version of Szemerédi's regularity lemma for sparse graphs yields an estimation of the maximal number of edges in an H-free subgraph of a random graph $G_{n,p}$. In fact, the conjecture may be seen as a probabilistic embedding lemma for partitions guaranteed by a version of Szemerédi's regularity lemma for sparse graphs. In this paper we verify the conjecture for $H=K_4$, thereby providing a conceptually simple proof for the main result in the paper cited above.

1. Introduction

In this paper we consider a conjecture by Kohayakawa, Łuczak, and Rödl [12] which, if true, has several applications in random graph theory, see Conjecture 1.3, Theorem 1.4 and the discussion thereafter. In fact, this conjecture can be seen as a probabilistic embedding lemma for a partition guaranteed by a version of Szemerédi's regularity lemma for sparse graphs. To state the conjecture we need the following definitions.

Definition 1.1. A bipartite graph $B=(U\cup W,E)$ with $|U|=|W|=n$ and $|E|=m$ is called (ε,n,m)-regular if for all $U'\subseteq U$ and $W'\subseteq W$ with $|U'|\geq\varepsilon n$ and $|W'|\geq\varepsilon n$,

$$\left|\frac{|E(U',W')|}{|U'|\cdot|W'|}-\frac{m}{n^2}\right|\leq\varepsilon\frac{m}{n^2},$$

Mathematics Subject Classification (2000): 05C80, 05C35

* Supported by DFG-Grant Ste 464/3-1.

where $E(U',W')$ denotes the set of edges with one endpoint in U' and one endpoint in W'.

If the number of edges and vertices is clear from the context, we sometimes call an (ε,n,m)-regular graph simply ε-regular.

Definition 1.2. Let $H=(V(H),E(H))$ be a graph with $V(H)=[\ell]$. An ℓ-partite graph $G=(V_1\cup\ldots\cup V_\ell,E)$ is called *$(H,n,m;\varepsilon)$-regular* if for all $\{i,j\}\in E(H)$, the graph $G[V_i,V_j]$ induced by V_i and V_j is (ε,n,m)-regular, and for all $\{i,j\}\notin E(H)$, the graph $G[V_i,V_j]$ is empty.

Note, that an $(H,n,m;\varepsilon)$-regular graph satisfies that $|V_1|=|V_2|=\ldots=|V_\ell|=n$ and that $|E(V_i,V_j)|\in\{0,m\}$.

Let $\mathcal{S}(H,n,m;\varepsilon)$ denote the set of $(H,n,m;\varepsilon)$-regular ℓ-partite graphs, and let

$$\mathcal{F}(H,n,m;\varepsilon):=\{G\in\mathcal{S}(H,n,m;\varepsilon): H\not\subseteq G\}.$$

The following conjecture was first stated in [12]. Here, we cite a simplified version from [13].

Conjecture 1.3 ([13]). Let $H=(V(H),E(H))$ be a fixed graph. For any $\beta>0$, there exist constants $\varepsilon_0>0, C>0, n_0>0$ such that

$$|\mathcal{F}(H,n,m;\varepsilon)|\le\beta^m\binom{n^2}{m}^{|E(H)|}$$

for all $n\ge n_0$, $0<\varepsilon\le\varepsilon_0$ and all $m\ge Cn^{2-1/d_2(H)}$, where

$$d_2(H):=\max\left\{\frac{|E(F)|-1}{|V(F)|-2}: F=(V(F),E(F))\subseteq H, |V(F)|\ge 3\right\}.$$

Since it is easily shown that the number of graphs in $\mathcal{S}(H,n,m;\varepsilon)$ is $(1-o(1))\binom{n^2}{m}^{|E(H)|}$, Conjecture 1.3 states that all but a β^m fraction of these graphs contain a copy of H, and that one can choose $\beta>0$ arbitrarily small provided ε is sufficiently small and $n,m(n)$ are sufficiently large.

One might hope that all graphs in $\mathcal{S}(H,n,m;\varepsilon)$ contain a copy of H. However, Łuczak showed that there are graphs in $\mathcal{S}(K_3,n,m;\varepsilon)$ with $m\gg n^{2-1/d_2(K_3)}$ not containing a copy of K_3, see [13] where Łuczak is quoted. Let us also remark that the statement of Conjecture 1.3 is not true if $m\ll n^{2-1/d_2(H)}$. Furthermore, it is essential to consider graphs in $\mathcal{S}(H,n,m;\varepsilon)$ and not the class of graphs where the graphs $G[V_i,V_j]$ induced by V_i and V_j for $\{i,j\}\in E(H)$ are just arbitrary graphs with m edges (not necessarily (ε,n,m)-regular). For example, it is easy to show that if $H=K_3$ and $m\ll$

$n/2$, then the number of graphs that do not contain a triangle is bigger than $c^m \binom{n^2}{m}^3$ for a constant $c \geq 1/(4e)^3$ which is obviously not arbitrarily small.

It is known that Conjecture 1.3 is true when H is a triangle [17]. In this paper we prove the conjecture when $H = K_4$. In the years since the submission of this paper, the ideas presented here have been further developed. Now Conjecture 1.3 is also known to be true if H is a cycle [1,3] of arbitrary length or a complete graph K_5 on five vertices [6]. If m/n^2 is approximately the square-root of the conjectured value then the conjecture is true for all complete graphs [4].

It was shown that the conjecture has the following implications for the random graph $G_{n,p}$, that is, the graph on n vertices where each of the possible $\binom{n}{2}$ edges is present with probability p. The first theorem shows that Conjecture 1.3 implies one of the long-standing open questions in the theory of random graphs, namely the analogue of the Erdős–Stone theorem from extremal graph theory.

Theorem 1.4 **([12]).** *Let* $\mathrm{ex}(G,H)$ *be the maximal number of edges a subgraph of* G *may have while avoiding the graph* H. *If Conjecture 1.3 is true, then with probability tending to one as* n *tends to infinity*

$$\mathrm{ex}(G_{n,p},H) = \left(1 - \frac{1}{\chi(H)-1} + o(1)\right) |E(G_{n,p})|, \tag{1}$$

whenever $pn^{-d_2(H)} \to \infty$ *as* $n \to \infty$, *where* $d_2(H)$ *is defined as in Conjecture 1.3.*

A proof of Theorem 1.4 can for example be found in [5]. Equation (1) has been shown to be true for some special graphs H without using Conjecture 1.3. It was shown to hold when H is a cycle [2,7,8,11,9,10,14,16] and when $H = K_4$ [12]. In the latter paper it was also suggested to prove the result by means of Conjecture 1.3. Proofs for the case when H is a complete graph and p is approximately the square-root of the conjectured value can be found in [15] and [18].

In [17] Łuczak showed that for any graph H for which Conjecture 1.3 is true, almost all H-free graphs can be made $(\chi(H)-1)$-partite by removing only a tiny fraction of the edges. More precisely, for every $\delta > 0$, there exists $c = c(\delta, H)$ such that the probability that a graph chosen uniformly at random from the family of all H-free labelled graphs on n vertices and $m \geq cn^{2-1/d_2(H)}$ edges can be made $(\chi(H)-1)$-partite by removing δm edges tends to one as n tends to infinity.

In the same paper Łuczak showed that for any graph H for which Conjecture 1.3 is true, for all sufficiently large n, and for all $cn^{2-1/d_2(H)} \leq m \leq n^2/c$

(with appropriately chosen constant c), the probability that a graph $G(n,m)$ drawn uniformly at random from all labelled graphs on n vertices and m edges does not contain H is $((\chi(H)-2)/(\chi(H)-1)+\delta)^m$ for a small error δ.

As mentioned above the main result of this paper is the proof of Conjecture 1.3 in the case when $H=K_4$ is the complete graph K_4 on four vertices. We state this result in the next theorem.

Theorem 1.5. *For any $\beta>0$, there exist constants $\varepsilon_0>0, C>0, n_0>0$ such that*

$$|\mathcal{F}(K_4,n,m;\varepsilon)| \le \beta^m \binom{n^2}{m}^6$$

for all $m \ge Cn^{8/5} = Cn^{2-d_2(K_4)}$, $n \ge n_0$ and $0<\varepsilon\le\varepsilon_0$.

Since we concentrate on the case when H is the complete graph K_4, we write $\mathcal{S}(n,m;\varepsilon) := \mathcal{S}(K_4,n,m;\varepsilon)$ and $\mathcal{F}(n,m;\varepsilon) := \mathcal{F}(K_4,n,m;\varepsilon)$.

Outline of the paper. In Section 2 we give an outline of our proof strategy. In Section 3 we introduce some notation and prove a few technical lemmas. Section 4 presents the main concepts of the paper namely covers, multicovers and triangle covers, and Section 5 discusses a counting lemma which is used repeatedly thereafter. Section 6 contains the proof of the main theorem.

2. Outline of the Proof

The main idea to show that $\mathcal{F}(n,m;\varepsilon)$ is only a tiny fraction of $\mathcal{S}(n,m;\varepsilon)$ is counting. In fact we will define appropriate classes $\mathcal{B}_i(\beta) \subseteq \mathcal{S}(n,m;\varepsilon)$ that satisfy $\mathcal{F}(n,m;\varepsilon) \subseteq \bigcup_i \mathcal{B}_i(\beta)$ and $\sum_i |\mathcal{B}_i(\beta)| \le \beta^m |\mathcal{S}(n,m;\varepsilon)|$. This will prove Theorem 1.5.

To define the sets $\mathcal{B}_i(\beta)$ we use that the graphs in $\mathcal{S}(n,m;\varepsilon)$ consist of six (ε,n,m)-regular graphs, one for each edge in K_4. We do not consider all six of these graphs at once but consider them one by one. First we prove some properties of a single such (ε,n,m)-regular graph. Then we consider a second (ε,n,m)-regular graph and prove some properties about the union of both graphs while using the properties we showed for the first one. We then continue by considering the third (ε,n,m)-regular graph and so on. To be more precise, consider a subgraph H of K_4. Clearly, for every graph in $\mathcal{S}(n,m;\varepsilon)$ we can just keep the (ε,n,m)-regular graphs that correspond to edges in H to obtain a graph in $\mathcal{S}(H,n,m;\varepsilon)$. Vice versa, every graph in $\mathcal{S}(H,n,m;\varepsilon)$ can be extended in at most $\binom{n^2}{m}^{|E(K_4)|-|E(H)|}$ ways to a graph in $\mathcal{S}(n,m;\varepsilon)$. Assume now that we can define a set $\mathcal{G}(H,\beta) \subseteq \mathcal{S}(H,n,m;\varepsilon)$

in such a way that $|\mathcal{G}(H,\beta)| \le (\beta^m/k)\binom{n^2}{m}^{|E(H)|}$ where k is the number of sets $\mathcal{B}_i(\beta)$ we shall construct. Then we can define the set $\mathcal{B}_i(\beta)$ as the set of all extensions of graphs in $\mathcal{G}(H,\beta)$, and proceed by considering only those graphs G in $\mathcal{S}(n,m;\varepsilon)$ with the property that the induced graph in $\mathcal{S}(H,n,m;\varepsilon)$ does not belong to $\mathcal{G}(H,\beta)$, i.e. those graphs for which the induced subgraph has some 'nice' properties. We can repeat this process, by considering a subgraph H' of K_4 such that $H \subseteq H' \subseteq K_4$. That is, we define a set $\mathcal{G}(H',\beta)$ such that $|\mathcal{G}(H',\beta)| \le (\beta^m/k)\binom{n^2}{m}^{|E(H')|}$, where for counting the elements of $\mathcal{G}(H',\beta)$ we use the fact that the induced subgraph of G that is an element of $\mathcal{S}(H,n,m;\varepsilon)$ does not belong to $\mathcal{G}(H,\beta)$, and proceed in this way for larger and larger subgraphs H' until we finally reach K_4. By then we shall have collected enough structural information to show that none of the remaining graphs belongs to $\mathcal{F}(n,m;\varepsilon)$.

Next we give some more intuition for the 'nice' structures we are looking for.

Random graphs as a guide. Let $G(n,m,4)$ be a chosen uniformly at random from the set of all subgraphs of the complete 4-partite graph $K_{n,n,n,n}$ with n vertices in each partition class and with m edges between each pair of partition classes. As an alternative we may consider a binomial random subgraph $G(n,m/n^2,4)$ of $K_{n,n,n,n}$ in which edges are present independently with probability m/n^2. It is not hard to see that for $m=\omega(n)$ such random graphs are ε-regular with high probability. Thus, as ε-regular graphs in some sense approximate the uniform distribution of random graphs of the same density, random graphs are a guiding example to gain some intuition on the structure of the latter. Note, however, that as remarked in the introduction, it does not suffice to consider the class $G(n,m/n^2,4)$ instead of $\mathcal{S}(n,m;\varepsilon)$. While random graphs $G(n,m/n^2,4)$ do exhibit similar phenomena as those in $\mathcal{S}(n,m;\varepsilon)$, the probabilities with which certain events occur differ in both classes. In particular, if one wants to show that only a β^m fraction (for arbitrary small $\beta>0$) of all graphs in $\mathcal{S}(n,m;\varepsilon)$ does not contain a K_4, one really has to use the regularity of the induced bipartite graphs. Nevertheless, to get some intuition, it helps to look at random graphs first.

In a random graph $G(n,m/n^2,4)$, the neighbourhood of a vertex $v\in V_i$ in any one of the other partition classes V_j, $j\neq i$, has expected size $q:=m/n$, since $(m/n^2)n=q$. We therefore expect that a set $C\subset V_i$ of size $|C|=n^2/m$ has a neighbourhood of size roughly $n^2/m\cdot m/n=n$ in each of the other partition classes. We call a set $C\subset V_i$ such that $|\Gamma(C)\cap V_j|\approx n$ a cover for V_j. Note that the above observation implies that in a random graph

$G(n,m/n^2,4)$ we expect that almost every set $C \subset V_i$ of size $|C| = n^2/m$ is a cover for V_j.

Next consider a set $Q \subset V_i$ of size $|Q| = m/n$. We can partition Q into m^2/n^3 sets of size n^2/m, each of which we expect to be a cover for V_j. Let us call such a set $Q \subset V_i$ a multicover for V_j. Note that almost all vertices of V_j have roughly m^2/n^3 neighbours in a multicover Q for V_j. The above remarks imply that we expect that in a random graph $G(n,m/n^2,4)$ almost every set $Q \subset V_i$ of size $|Q| = m/n$ is a multicover for V_j.

Finally, let us call a set $T \subset V_i$ a triangle cover for V_j via V_k, if almost all vertices $w \in V_j$ have the property that there exist n^2/m vertices in V_k such that each of these vertices lies in a triangle containing w and a point in T. How large do we have to choose $t = |T|$ in order to expect that in a random graph $G(n,m/n^2,4)$ almost every set $T \subset V_i$ of size $|T| = t$ is a triangle cover for V_j via V_k? Elementary calculations show that $t = n^7/m^4$ suffices.

Now let us see what this gives us. A typical vertex $v \in V_1$ has neighbourhoods $Q_i := \Gamma(v) \cap V_i$ of size roughly m/n in V_2, V_3, V_4, which we may assume to be multicovers. Observe that for $m \geq Cn^{8/5}$ we have $n^7/m^4 \leq m/n$, so we may also assume that, for example, Q_3 is a triangle cover for V_2 via V_4, which in turn implies that we may assume that most of the vertices in Q_2 are 'good' vertices in V_2, i.e. vertices which have n^2/m neighbours in V_4 that are contained in a triangle containing w and a vertex from Q_3. In a random graph we expect that, for different vertices $w \in V_2$, the corresponding n^2/m neighbours in V_4 are almost disjoint. Note that this implies that we find roughly $|Q_2| \cdot (n^2/m) = n$ vertices in V_4 that lie in a triangle containing a vertex from Q_2 and a vertex from Q_3. Note that this implies that if Q_4 contains any of these vertices then there exists a K_4. That is, in order to avoid a K_4, Q_4 has to be contained within a tiny set (defined by v and Q_2 and Q_3). As Q_4 is a random set, this is highly unlikely, implying that for $m \geq Cn^{8/5}$ the random graph $G(n,m/n^2,4)$ contains with high probability a K_4. Of course, for the random graph there are more elegant ways to prove this fact. The above approach, however, has the advantage that it can also be used for graphs in $\mathcal{S}(n,m;\varepsilon)$.

(ε,n,m)-regular graphs. More precisely we shall show that all but a β^p fraction of sets of size $p \approx n^2/m$ in an (ε,n,m)-regular graph have a neighbourhood of size $\Theta(n)$ (and $\beta \to 0$ as $\varepsilon \to 0$), see

Graphs in $\mathcal{S}(n,m;\varepsilon)$ are not random. In the previous paragraph we considered random graphs as a guide for ε-regular graphs. Now we want to hint at some difficulties that occur when considering graphs in $\mathcal{S}(n,m;\varepsilon)$ instead of random 4-partite graphs. One of the main differences is that in the random graph the disjointness of certain sets comes naturally whereas in ε-regular graphs one has to work for it. For example, consider a fixed set $Q \subseteq V_j$. In a random graph we expect that most vertices see $|Q| \cdot m/n^2$ vertices in the set Q and this is independent of the choice of Q. In an ε-regular graph this is different. There might be some small sets Q in V_j that see no vertex in V_i since the definition of ε-regular graphs concerns only sets of linear size.

We circumvent this difficulty by choosing our sets Q with foresight. For example, we do not only insist that a cover in V_i sees roughly n of the vertices in V_j but also demand that most vertices in a cover have neighbourhoods that are pairwise rather disjoint. Also we introduce forbidden sets X which allow us to select the vertices one after the other. This works as follows. We show, for example, that for each fixed forbidden set $X \subset V_4$ there are linearly many vertices in V_j which behave as expected. If we now choose the neighbourhood in V_j of a vertex $v \in V_i$ one by one then it is very likely that we first select a vertex w that behaves as expected. Now we fix the neighbourhood of w in V_4 and add it to the forbidden set X to obtain a new forbidden set. Still there are linearly many vertices in V_j that behave as expected with respect to this new forbidden set, and hence it is likely that we pick a good vertex when choosing the next vertex of the neighbourhood of v in V_j.[1]

Counting bad graphs. Let us return to the problem of counting the elements in $\mathcal{F}(n,m;\varepsilon)$, or better the sets $\mathcal{B}_i(\beta)$. Our main tool to count these graphs is Lemma 5.3 which states, roughly speaking, that for $\pi \ll \beta$, if only a $\pi^{\tilde{q}}$ fraction of sets of size $\tilde{q} \approx q$ does not satisfy a property A then all but a β^m fraction of graphs in $\mathcal{S}(n,m;\varepsilon)$ contain linearly many vertices that contain a set satisfying property A in their neighbourhood. Thus, for example, if one has shown that all but a $\pi^{\tilde{q}}$ fraction of sets of size $\tilde{q}$ are triangle covers then it follows that almost all graphs in $\mathcal{S}(n,m;\varepsilon)$ have linearly many vertices in V_1 that have a triangle cover in their neighbourhood. Hence we have to show that all but a $\pi^{\tilde{q}}$ fraction of sets of size $\tilde{q}$ satisfy the properties we desire.

Things get slightly more complicated when considering neighbourhoods of vertices in a set Q with $|Q| \ll n$ because the expected size of such a neighbourhood has size $r = |Q|m/n^2 \ll q$. Often we are able to show that at most a π^r fraction of all sets of size r in Q have an undesired property but this is not enough to apply Lemma 5.3. To resolve this difficulty we consider

[1] Since this paper was submitted (ε,n,m)-regular graphs have been far better understood. Using the results of [3] (which are based on the ideas presented here) one can show more directly (with the methods described in Section 7) that the desired structures exist without using forbidden sets.

$n/|Q|$ disjoint sets of size $|Q|$ at once and show that in most of these sets most vertices have a neighbourhood of size r satisfying the desired property. (We use this technique for example in Lemma 4.20.) This idea has been further developed and simplified in [4].

3. Preliminaries

3.1. Conventions and notation

In order to increase the clarity of the presentation we make use of the following conventions: All constants that are denoted by greek letters are tacitly assumed to be smaller than, say, 10^{-3}. Furthermore, ε will always be smaller than all other constants.

We will not introduce floors and ceilings when we are talking about integral terms, e.g., cardinalities of sets. Since we are only interested in the asymptotic behaviour of those quantities, this would merely introduce lower order error terms which complicate the exposition. However, it would be a standard but laborious task to modify the proofs such that the integrality of all terms is respected.

The neighbourhood of a vertex v is denoted by $\Gamma(v)$. We will use the abbreviations $\Gamma_i(v) := \Gamma(v) \cap V_i$ and $d_i(v) := |\Gamma_i(v)|$.

For technical reasons, i.e., as to gain control over the size of certain sets, we will often need to deterministically fix a subset of a given cardinality in a larger set of vertices. In order to do so we assume that the vertices in V_i are ordered in an arbitrary but unique way (e.g. we may assume that $V_i = \{1, \ldots, n\}$). By $[A]_x$ we denote the set $B \subseteq A$ of size $|B| = x$ that contains the x smallest elements in A. If $|A| < x$, we define $[A]_x := A$.

3.2. Simple lemmas

This section contains a collection of auxiliary results with a rather technical flavour. Hence, it may be skipped at first reading and then be consulted on demand.

Lemma 3.1 (Technical inequalities for binomial coefficients).

(i) *If $0 \le a \le b \le n$, then*

$$\binom{n}{a}\binom{n}{b-a} \le \binom{n}{b} 4^b.$$

(ii) *If* $0 \le x \le 1$ *then*

$$\binom{xa}{b} \le \binom{a}{b} \cdot x^b.$$

Proof. Case (i) can be seen as follows: Instead of choosing a elements first and then $b-a$ elements, we can choose b elements and then mark each of the b elements to belong either to the first set of a elements, or to the second set of $b-a$ elements, or to none of these sets, or to both of these sets. Alternatively, one calculates:

$$\begin{aligned}\binom{n}{a}\binom{n}{b-a} &= \frac{n!}{(n-a)!a!}\frac{n!}{(b-a)!(n-(b-a))!} \\ &= \frac{b!}{a!(b-a)!}\frac{n!}{(n-b)!b!}\frac{n!(n-b)!}{(n-a)!(n-b+a)!}.\end{aligned}$$

Now for $1 \le a \le b \le n$ and $0 \le c \le a-1$, one easily verifies that $(n-c)/(n-b+a-c) \le (b-c)/(a-c)$, and hence the last factor of the product above is smaller than $b!/((b-a)!a!)$. Thus

$$\binom{n}{a}\binom{n}{b-a} \le \binom{b}{a}\binom{b}{b-a}\binom{n}{b} \le 4^b\binom{n}{b}.$$

For the proof of (ii) we simply observe that

$$\binom{xa}{b} = x^b \cdot \frac{a(a-\frac{1}{x})\cdots(a-\frac{b-1}{x})}{b!} \le x^b\binom{a}{b}. \qquad \blacksquare$$

The following lemma is inspired by a very simple observation: If in a bipartite graph $G=(U \cup W, E)$ all vertices in U have large degree, then there must be many vertices in W with large degree too. This is proved by an easy counting argument.

Lemma 3.2 (Overlap lemma). *Let* $\alpha > 0$, *and let* $G = (U \cup W, E)$ *be a bipartite graph with* $d(u) \ge \alpha|W|$ *for all* $u \in U$. *Then for all* $\beta > 0$,

$$|\{w \in W : d(w) \ge \beta|U|\}| \ge \frac{\alpha-\beta}{1-\beta}|W|.$$

Proof. Assume for a contradiction that there are less than $\frac{\alpha-\beta}{1-\beta}|W|$ vertices in W with degree at least $\beta|U|$. It follows that

$$|E| < \frac{\alpha-\beta}{1-\beta}|W| \cdot |U| + \left(1 - \frac{\alpha-\beta}{1-\beta}\right)|W| \cdot \beta|U| = \alpha|W| \cdot |U|.$$

On the other hand it is clear by the lower bound on the degree of the vertices in U that $|E| \geq \alpha|W| \cdot |U|$ and we get a contradiction. ∎

Although the proof of Lemma 3.2 is quite simple, the following corollary will be vital in the proof of our main result.

Corollary 3.3. *Let $\varepsilon > 0$. If a bipartite graph $G = (U \cup W, E)$ satisfies $d(u) \geq (1-\varepsilon)|W|$ for all $u \in U$, then*

$$|\{w \in W : d(w) \geq (1-\sqrt{\varepsilon})|U|\}| \geq (1-\sqrt{\varepsilon})|W|.$$

Proof. Set $\alpha = 1-\varepsilon$ and $\beta = 1-\sqrt{\varepsilon}$ in Lemma 3.2. ∎

The following lemma is a simple consequence of the definition of (ε, n, m)-regularity.

Lemma 3.4 (Degree lemma). *Consider an (ε, n, m)-regular graph $B = (U \cup W, E)$ and a set $W' \subseteq W$ with $|W'| \geq \varepsilon n$. For $q := m/n$ we define*

$$X^< := \left\{ u \in U : \ |\Gamma(u) \cap W'| < (1-\varepsilon) q \cdot \frac{|W'|}{n} \right\}.$$

and $X^>$ analogously. Then $\max\{|X^<|, |X^>|\} < \varepsilon n$.

Proof. We only consider the case that $|X^<| \geq \varepsilon n$, since the case $|X^>| \geq \varepsilon n$ is almost identical. By the definition of $X^<$ we know that

$$|E(X^<, W')| < (1-\varepsilon) q \cdot \frac{|W'|}{n} \cdot |X^<| = (1-\varepsilon) \frac{|X^<| \cdot |W'|}{n^2} \cdot m.$$

This obviously contradicts the (ε, n, m)-regularity of B. ∎

4. Covering sets

4.1. Abbreviations

In this section we will need quite a few abbreviations, which we collect here for easier reference. The following quantities are defined as functions of n, m and a parameter $\lambda > 0$. This parameter will later be replaced by one of several constants, see Equations (12)–(20), and Equation (21) for the order of these constants.

$$q := \frac{m}{n}, \tag{2}$$

$$p_\lambda := \frac{2}{\lambda} \frac{n^2}{m}, \tag{3}$$

$$q_\lambda := (1-\lambda)\lambda q, \tag{4}$$

$$t_\lambda := \frac{1}{\lambda}\frac{n^2}{m}, \tag{5}$$

$$r_\lambda(x) := \frac{x}{2p_\lambda}. \tag{6}$$

The following quantities are also functions of one or two parameters $\nu,\mu>0$. In order to avoid clumsy notation we will often drop these dependencies when it will be clear from the context where the values for ν and μ come from.

$$\tilde{r}_1 = \tilde{r}_1(\nu,\mu) := r_\nu(\mu^2 q) = \frac{\mu^2\nu}{4}\frac{m^2}{n^3}, \qquad \tilde{r}_2 = \tilde{r}_2(\mu) := r_\mu(\mu^2 q) = \frac{\mu^3}{4}\frac{m^2}{n^3}, \tag{7}$$

$$\tilde{p} = \tilde{p}(\mu) := \frac{p_\mu}{4} = \frac{1}{2\mu}\frac{n^2}{m} \tag{8}$$

Note the redundancy in this notation, as for example $p_\lambda = 2t_\lambda$. However, this is just a coincidence as we will use these two abbreviations for completely different purposes. This redundant notation is intended to remind the reader where the various quantities come from. Usually, sets of size q_λ are neighbourhoods, and sets of size $\tilde{r}_1, \tilde{r}_2$ are common neighbourhoods of two vertices. Sets of size p_λ will be essential because we expect that the neighbourhood of a set of this size covers most of any other partition class. At the threshold $m = Cn^{8/5}$ and for fixed λ, ν, μ, we have that $\tilde{r}_i \ll p_\lambda \ll q_\lambda$.

In the following three subsections we shall describe precisely the concepts we sketched in Section 2, namely covers, multicovers and triangle covers.

4.2. Simple covers

In a random bipartite graph with two vertex sets of size n and edge probability m/n^2, we expect that a set of size n^2/m has approximately $(n^2/m)(m/n^2)n = n$ neighbours. Hence, we may expect that a set P of size $\Theta(n^2/m)$ in partition class V_i covers a partition class V_j for $j \neq i$, i.e., $|\Gamma_j(P)| \approx V_j$. Such covering sets are important in our proof.

For technical reasons it is beneficial to concentrate on covers with a special structure. Informally speaking, we say that $P = \{v_1,\dots,v_p\}$ covers V_j if we can find disjoint sets $W_1,\dots,W_{p/2} \subseteq V_j$ of equal size such that $W_i \subseteq \Gamma(v_i)$ and $\bigcup_{i=1}^{p} W_i \approx V_j$. The concept of a cover is illustrated in Figure 1. The following definition provides a formalisation of this approach.

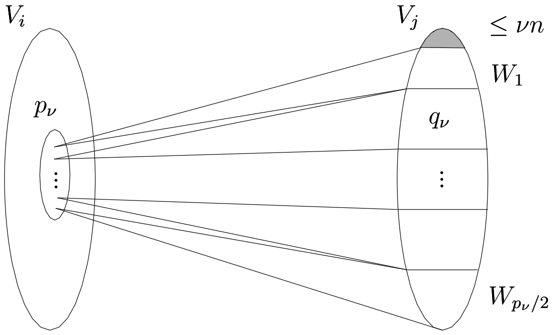

Figure 1. A ν-cover

Definition 4.1 (Covers). Consider a graph $G \in \mathcal{S}(n,m;\varepsilon)$, $\{i,j\} \subseteq \{1,\ldots,4\}$ and $\nu > 0$. A set $P \subseteq V_i$ is called a *ν-cover* of V_j if there exists a subset $P^* \subseteq P$ of size $|P^*| \geq |P|/2$ such that there are pairwise disjoint sets $(W_v \subseteq \mathit{\Gamma}_j(v))_{v\in P^*}$ with $|W_v| = q_\nu$ for all $v \in P^*$. The sets $(W_v)_{v\in P^*}$ are called *covering neighbourhoods.*

Mostly we will be interested in ν-covers with cardinality $|P| = p_\nu$. Note that then

$$|\mathit{\Gamma}_j(P)| \geq \left|\bigcup_{v\in P^*} W_v\right| \geq (p_\nu/2)\cdot q_\nu = (1-\nu)n. \tag{9}$$

Definition 4.2 (Sets of covers). Consider a graph $G \in \mathcal{S}(n,m;\varepsilon)$ and $\{i,j\} \subseteq \{1,\ldots,4\}$. Let $\mathcal{P}_{i,j}(G;\nu) \subseteq \{P \subset V_i : |P| = p_\nu\}$ denote the family of sets in V_i of cardinality p_ν that form a ν-cover of V_j. Furthermore, let $\bar{\mathcal{P}}_{i,j}(G;\nu) := \{P \subset V_i : |P| = p_\nu\} \setminus \mathcal{P}_{i,j}(G;\nu)$.

The following lemma shows that almost all sets of size p_ν (up to a superexponentially small fraction) are ν-covers. This lemma closely resembles Lemma 11 in [11].

Lemma 4.3 ($\bar{\mathcal{P}}$ is small). *For $\pi,\nu > 0$, there exists a constant $\varepsilon^{\mathrm{Cov}} = \varepsilon^{\mathrm{Cov}}(\pi,\nu)$ such that for any pair $\{i,j\} \subseteq \{1,\ldots,4\}$, any graph $G \in \mathcal{S}(n,m;\varepsilon)$ with $\varepsilon \leq \varepsilon^{\mathrm{Cov}}$, and any $V' \subseteq V_i$ with $|V'| \geq \sqrt{\varepsilon} n$, we have*

$$|\bar{\mathcal{P}}_{i,j}(G;\nu) \cap \{P \subset V' : |P| = p_\nu\}| \leq \pi^{p_\nu}\binom{|V'|}{p_\nu},$$

for sufficiently large n.

Proof. Consider a set $P \subset V_i'$ that is not a ν-cover of V_j, and let $P' \subseteq P$ be a set of maximal size such that there exists a family of pairwise disjoint sets $(W_u')_{u \in P'}$ with $W_u' \subset \Gamma_j(u)$ and $|W_u'| = q_\nu$ for all $u \in P'$. Since P is not a ν-cover, $|P'| < p_\nu/2$ and hence $|\bigcup_{u \in P'} W_u'| < (p_\nu/2) \cdot q_\nu = (1-\nu)n$. Also, by the maximality of P' all vertices v in $P \setminus P'$ satisfy

$$\left|\Gamma(v) \cap \left(V_j \setminus \bigcup_{u \in P'} W_u'\right)\right| < q_\nu = (1-\nu)\nu q \leq (1-\varepsilon)q \frac{|(V_j \setminus \bigcup_{u \in P'} W_u')|}{n}$$

provided $\varepsilon \leq \nu$. By Lemma 3.4 there are at most $\varepsilon|V_i| \leq \sqrt{\varepsilon}|V_i'|$ such vertices in V_i and thus in V_i'. Hence the number of sets that are not ν-covers is bounded from above by

$$\sum_{0 \leq \bar{p} \leq p_\nu/2} \binom{|V'|}{\bar{p}} \binom{\sqrt{\varepsilon}|V'|}{p_\nu - \bar{p}} \overset{3.1(\text{ii})}{\leq} \sum_{0 \leq \bar{p} \leq p_\nu/2} \binom{|V'|}{\bar{p}} \binom{|V'|}{p_\nu - \bar{p}} (\sqrt{\varepsilon})^{p_\nu - \bar{p}}$$

$$\overset{3.1(\text{i})}{\leq} \frac{p_\nu}{2} 4^{p_\nu} \binom{|V'|}{p_\nu} \varepsilon^{p_\nu/4} \leq \pi^{p_\nu} \binom{|V'|}{p_\nu},$$

when ε is sufficiently small and n and thus p_ν is sufficiently large. ∎

4.3. Multicovers

In the previous section we introduced simple covers of size $p_\nu = \Theta(n^2/m)$, whereas in our proof we will often be concerned with sets of size $\Theta(q) = \Theta(m/n)$, i.e., neighbourhoods of vertices. The following definition therefore transfers the notion of a cover to such (larger) sets.

Definition 4.4 (Multicovers). Consider a graph $G \in \mathcal{S}(n,m;\varepsilon)$, $\{i,j\} \subseteq \{1,\ldots,4\}$ and $\nu > 0$. We call a set $Q \subseteq V_i$ a *ν-multicover* of V_j if there exist pairwise disjoint subsets $P_1,\ldots,P_r \subseteq Q$ such that

(i) $r = r_\nu(|Q|) := \frac{|Q|}{2p_\nu}$,
(ii) $|P_i| = p_\nu$ for all $i = 1,\ldots,r$, and
(iii) P_i is a ν-cover of V_j for all $i = 1,\ldots,r$.

For each ν-multicover Q we consider an arbitrary but fixed partition into ν-covers and set $Q^*(\nu) := \bigcup_{k=1}^r P_k^*$, where the sets P_k^* are defined as in Definition 4.1.

Consider the case when Q is a multicover of size $\Theta(m/n)$. Observe that for a vertex $w \in V_j$ the average number of neighbours within Q is $\Theta((m/n) \cdot (m/n^2)) = \Theta(m^2/n^3) = \Theta(r_\nu(|Q|))$. In the following definition we consider the set $C_j(Q;\nu)$ of vertices in V_j that (up to an appropriately chosen multiplicative constant) have at least as many neighbours in Q as expected.

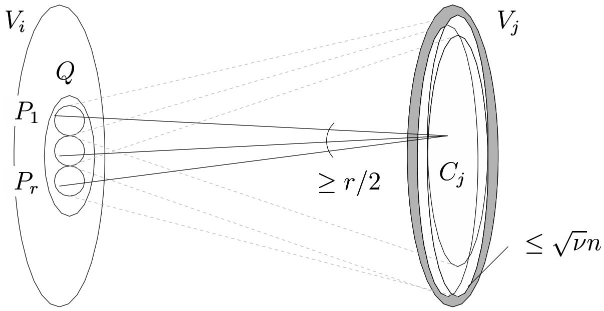

Figure 2. ν-multicover

Definition 4.5 (Covered neighbourhood of Q-sets). Consider a graph $G \in \mathcal{S}(n,m;\varepsilon)$, $\{i,j\} \subseteq \{1,\ldots,4\}$ and $\nu > 0$. For a ν-multicover $Q \subseteq V_i$ of V_j, we define

$$C_j(Q;\nu) := \{w \in V_j : |\Gamma_i(w) \cap Q| \geq r_\nu(|Q|)/2\}.$$

Figure 2 illustrates the structure of a ν-multicover and of the sets $C_j(Q;\nu)$.

Lemma 4.6 (Large covered neighbourhoods). *Given a graph $G \in \mathcal{S}(n,m;\varepsilon)$ let $Q \subseteq V_i$ be a ν-multicover of V_j for some $\nu > 0$. Then*

$$|C_j(Q;\nu)| \geq (1-\sqrt{\nu})n.$$

Proof. Let $P_1,\ldots,P_r$ with $r := r_\nu(|Q|)$ denote pairwise disjoint sets of size p_ν in Q that are ν-covers of V_j. The following represents a typical application of Corollary 3.3. Consider the auxiliary bipartite graph $B = (\{P_1,\ldots,P_r\} \cup V_j, E_B)$, where $\{P_k, w\} \in E_B$ if and only if $w \in \Gamma_j(P_k)$ in G. The sets $P_1,\ldots,P_r$ can be interpreted as 'super-vertices' in G that are obtained by merging the vertices in the sets P_i into a single vertex. By (9) we have $|\Gamma_j(P_k)| \geq (1-\nu)n$, hence all vertices P_k have degree at least $(1-\nu)n$ in B. Thus, we can apply Corollary 3.3 to obtain that

$$Z := \{z \in V_j : d_B(z) \geq (1-\sqrt{\nu})r\} \tag{10}$$

satisfies $|Z| \geq (1-\sqrt{\nu})n$. If $z \in Z$, then

$$|\{k = 1,\ldots,r : z \in \Gamma_j(P_k)\}| \geq (1-\sqrt{\nu})r \geq r/2 = r_\nu(|Q|)/2,$$

and thus $z \in C_j(Q;\nu)$. Hence $Z \subseteq C_j(Q;\nu)$, which concludes the proof of the lemma. ∎

Let $p_\nu \leq s \leq q$. Note that $p_\nu \leq q$ for $m \geq \sqrt{(2/\nu)n^3} \geq \nu^{-1}n^{3/2}$. In the following we show that, analogously to Lemma 4.3, almost all sets of size s in V_i form a ν-multicover of V_j.

Definition 4.7 (Sets of multicovers). Let $s \in \mathbb{N}$ and $\nu > 0$, and consider $G \in \mathcal{S}(n,m;\varepsilon)$ and $\{i,j\} \subseteq \{1,\dots,4\}$. Let $\mathcal{Q}_{i,j}(G;s,\nu)$ denote the family of sets of size s in V_i that form a ν-multicover of V_j. Furthermore, we define $\bar{\mathcal{Q}}_{i,j}(G;s,\nu) := \{S \subseteq V_i : |S| = s\} \setminus \mathcal{Q}_{i,j}(G;s,\nu)$.

Lemma 4.8 ($\bar{\mathcal{Q}}$ is small). *For $\pi,\nu > 0$, there exists a constant $\varepsilon^{\mathrm{Mul}} = \varepsilon^{\mathrm{Mul}}(\pi,\nu)$ such that for any pair $\{i,j\} \subseteq \{1,\dots,4\}$, and any graph $G \in \mathcal{S}(n,m;\varepsilon)$ with $\varepsilon \le \varepsilon^{\mathrm{Mul}}$, $\nu^{-1} n^{3/2} \le m \le n^2/4$, and any s with $p_\nu \le s \le q$,*

$$|\bar{\mathcal{Q}}_{i,j}(G;s,\nu)| \le \pi^s \binom{n}{s},$$

for n sufficiently large.

Proof. We first count the number of ways to choose a family of $r' := \lfloor s/p_\nu \rfloor$ pairwise disjoint sets $P_1,\dots,P_r$ of size p_ν in such a way that at least $r'/2$ of the sets P_k belong to $\bar{\mathcal{P}}_{i,j}(G;\nu)$, and an additional set of size $s - r'p_\nu$. By Lemma 4.3 with $\bar{\pi} > 0$ and $\varepsilon < \varepsilon^{\mathrm{Cov}}(\bar{\pi},\nu)$, and since $n - r'p_\nu \ge \sqrt{\varepsilon} n$ for sufficiently large n, we have at most

$$\binom{r'}{r'/2} (\bar{\pi}^{p_\nu})^{r'/2} \left(\prod_{i=0}^{r'-1} \binom{n - ip_\nu}{p_\nu} \right) \binom{n - r'p_\nu}{s - r'p_\nu} \le 2^{r'} \bar{\pi}^{s/2} \frac{s!}{(p_\nu!)^{r'} (s - r'p_\nu)!} \binom{n}{s}$$

ways to do so.

Now, consider a set $Q \in \bar{\mathcal{Q}}_{i,j}(G;s,\nu)$. There are $s!/((p_\nu!)^{r'}(s-p_\nu)!)$ ways to partition Q into r' ordered disjoint sets of size p_ν and one additional set of size $s - r'p_\nu$, and since $Q \in \bar{\mathcal{Q}}_{i,j}(G;s,\nu)$ all these partitions contain at least $r'/2$ sets in $\bar{\mathcal{P}}_{i,j}(G;\nu)$. In addition, two different sets $Q' \ne Q \in \bar{\mathcal{Q}}_{i,j}(G;s,\nu)$ never yield the same partition. It follows that there are at most

$$2^{r'} \bar{\pi}^{s/2} \binom{n}{s}$$

sets in $\mathcal{Q}_{i,j}(G;s,\nu)$ and by choosing $\bar{\pi}$ sufficiently small the claim follows. ∎

Before discussing further properties and applications of multicovers we need the following simple definition which is used to control the overlap between a family of sets.

Definition 4.9 (Quasidisjoint sets). A family of sets $A_1,\dots,A_b \subseteq V$ is called s-*quasidisjoint* if each $v \in V$ belongs to at most s of these sets, that is, if

$$|\{i \in \{1,\dots,b\} : x \in A_i\}| \le s \quad \text{for all } x \in V.$$

Note that sets that belong to a family of 1-quasidisjoint sets are disjoint. Consider a ν-multicover Q with $Q^* = \bigcup_{k=1}^r P_k^*$ and $r = r_\nu(|Q|)$. For a fixed P_k^*, all neighbourhoods W_u for $u \in P_k^*$ are disjoint (see Definition 4.1). Since there are only r sets P_k^*, we conclude that the sets W_v, $v \in Q^*$ are r-quasidisjoint.

In our subsequent proofs we will often exclude a subset X of vertices in a partition class. Later we want to choose (rather) disjoint sets and X denotes the set of already chosen sets. Then we want to find the next set in $V \setminus X$. The following definition and lemma show that the structure of a ν-multicover remains intact (up to constants) if this set X is smaller than $(1-2\sqrt{\nu})n$ (thus X can be quite large).

Definition 4.10 (Resistant multicovers). A ν-multicover $Q \subseteq V_i$ of V_j is called *X-resistant* for $X \subseteq V_j$ if there exists a subset $Q^{**} \subseteq Q$ with $|Q^{**}| = \nu^2 q$ such that there exist $r_\nu(|Q|)$-quasidisjoint sets $(W'_v \subseteq \Gamma_j(v) \setminus X)_{v \in Q^{**}}$ with $|W'_v| = \nu^2 q$. We call the sets W'_v *covering neighbourhoods.*

Lemma 4.11 (All multicovers are resistant). *Let $\nu > 0$ be a sufficiently small constant, and consider a graph $G \in \mathcal{S}(n,m;\varepsilon)$, $\{i,j\} \subseteq \{1,\dots,4\}$, $X \subseteq V_j$ with $|X| \leq (1-2\sqrt{\nu})n$. Then any ν-multicover $Q \subseteq V_i$ of V_j with $|Q| = q_\nu$ is also X-resistant.*

Proof. We count the number of occurrences of vertices $x \in V_j$ in covering neighbourhoods W_u of the ν-multicover Q. More precisely, if a vertex x belongs to a covering neighbourhood W_u, such that u belongs to one of the ν-covers P that make up the ν-multicover, then this corresponds to one such occurrence.

Let us go back to the proof of Lemma 4.6. We showed there that $C_j(Q;\nu)$ is large by examining the set $Z \subseteq C_j(Q;\nu)$ defined in (10). Now we consider this set Z again. Every vertex $z \in Z$ corresponds to at least $(1-\sqrt{\nu})r_\nu(|Q|) \geq r_\nu(|Q|)/2$ occurrences of z in covering neighbourhoods.

Since $|Z| \geq (1-\sqrt{\nu})n$, it follows that $|Z \setminus X| \geq \sqrt{\nu}n$. We conclude that at least

$$\sum_{u \in Q^*} |W_u \setminus X| \geq \sqrt{\nu}n \cdot r_\nu(|Q|)/2 \geq \frac{1}{8}\nu^{3/2}\frac{m}{n}|Q| \tag{11}$$

occurrences remain if we restrict the sets W_u to vertices in $V_j \setminus X$.

Let $Q' := \{u \in Q^*(\nu) : |W_u \setminus X| \geq \nu^2 q\}$ where $Q^*(\nu)$ is defined as in Definition 4.4, and assume that $|Q'| \leq \nu^2 q$. Then the number of these occurrences is at most

$$\sum_{u \in Q^*} |W_u \setminus X| \leq |Q \setminus Q'| \cdot \nu^2 q + |Q'| \cdot q_\nu \leq |Q|\nu^2 q + \nu^2 q|Q| \leq 2\nu^2 \frac{m}{n}|Q|.$$

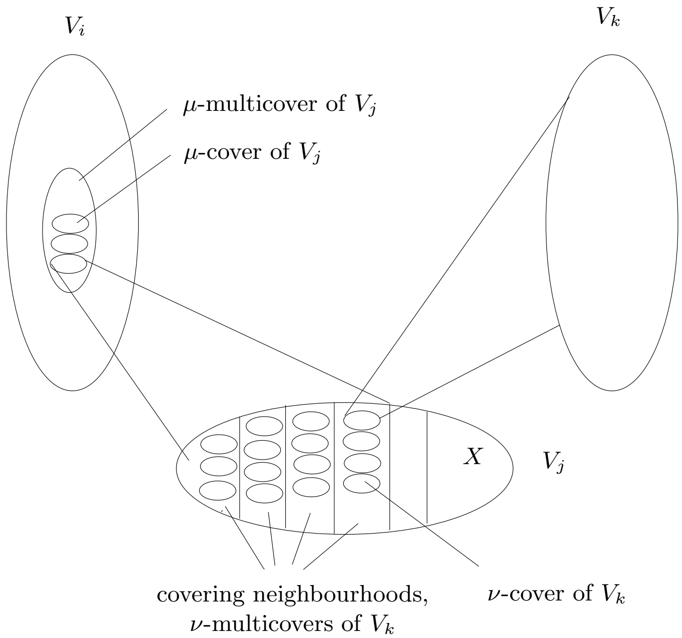

Figure 3. Qualified multicovers

This obviously contradicts (11) for ν sufficiently small.

Finally observe that the sets W'_u are $r_\nu(|Q|)$-quasidisjoint since they are subsets of the covering neighbourhoods W_u which are already $r_\nu(|Q|)$-quasi-disjoint. ∎

The concepts introduced so far – covers and (resistant) multicovers – are only concerned with edges between *two* classes V_i and V_j. In order to get closer to finding clique candidates we have to consider structures between *three* classes, too. The following definition introduces multicovers where the covering neighbourhoods are themselves multicovers, see Figure 3.

Definition 4.12 (Qualified resistant multicovers). An X-resistant μ-multicover $Q \subseteq V_i$ of V_j is called *ν-qualified for V_k* if the $r_\mu(|Q|)$-quasidisjoint covering neighbourhoods $W'_1, \dots, W'_{\mu^2 q} \subseteq V_j$ of cardinality $\mu^2 q$ are ν-multicovers of V_k.

4.4. Triangle covers

Informally speaking, a *triangle cover* is a set $Q \subseteq V_3$ for which there are many vertices $w \in V_2$ such that each w spans many triangles between itself, Q and $V_4 \setminus X$ for some forbidden set X (see Figure 4).

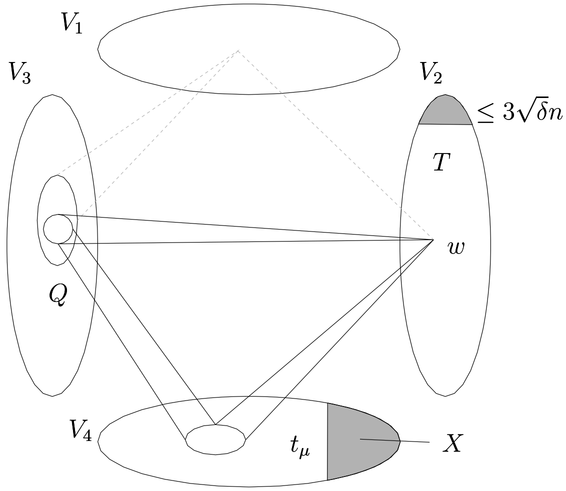

Figure 4. (μ,δ)-triangle cover

Definition 4.13 (Resistant triangle covers). Let $\delta,\mu>0$. Consider $G\in \mathcal{S}(n,m;\varepsilon)$, and $X\subseteq V_4$. We call a set $Q\subseteq V_3$ an *X-resistant (μ,δ)-triangle cover* of V_2 via V_4 if there exists a set $T=T(Q)\subseteq V_2$ with $|T(Q)|\geq(1-3\sqrt{\delta})n$ such that for all $w\in T$

$$|TRI_4(w,Q)\setminus X|\geq t_\mu,$$

where $t_\mu:=\mu^{-1}\frac{n^2}{m}$ (as in (5)) and

$$TRI_4(w,Q):=\Gamma_4(w)\cap\Gamma_4(\Gamma_3(w)\cap Q).$$

Figure 4 illustrates the structure of triangle covers. Later in the proof we will consider triangle covers Q with $Q\subseteq\Gamma_3(v)$ for a vertex $v\in V_1$, and we will be interested in finding vertices $w\in T\cap\Gamma_2(v)$.

Unfortunately, we are not able to prove directly that there exist many triangle covers. Instead we are taking a little detour via so-called *triangle candidate covers* Q, defined below (see also Figure 5). Given a set $X\subseteq V_4$, we fix a set $\tilde{Q}\subseteq Q$ with the following property: Almost all vertices $u\in\tilde{Q}$ have a large common neighbourhood in $V_4\setminus X$ with almost all vertices w in V_2. Observe that a priori nothing is required about the edges between Q and w.

Definition 4.14 (Resistant triangle candidate covers). Consider a graph $G\in\mathcal{S}(n,m;\varepsilon)$, and $X\subseteq V_4$. A set $Q\subseteq V_3$ with $|Q|=q_\mu$ is called an *X-resistant (ν,μ)-triangle candidate cover* of V_2 via V_4 if there exist

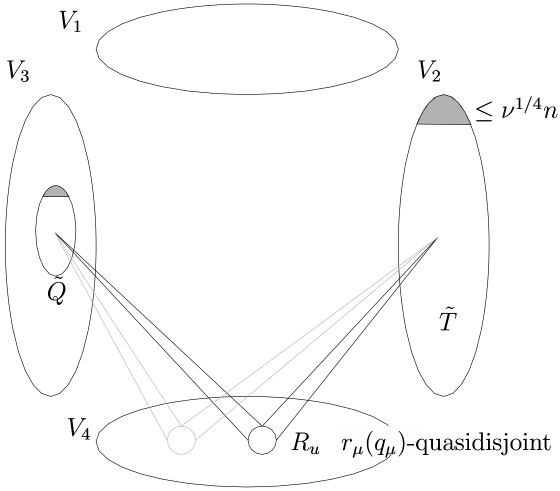

Figure 5. (ν, μ)-triangle candidate cover

sets $\tilde{Q} \subseteq Q$ with $|\tilde{Q}| = \mu^2 q$ and $\tilde{T} = \tilde{T}(Q) \subseteq V_2$ with $|\tilde{T}| \geq (1 - \nu^{1/4})n$ such that the following condition is satisfied: For all $w \in \tilde{T}$, there exists a set $\tilde{Q}^*(w) \subseteq \tilde{Q}$, $|\tilde{Q}^*(w)| \geq (1 - \nu^{1/4})\mu^2 q$ and an $r_\mu(q_\mu)$-quasidisjoint family of sets $(R_u \subseteq \Big(\Gamma_4(u) \cap \Gamma_4(w)\Big) \setminus X)_{u \in Q^*(w)}$ with $|R_u| \geq \tilde{r}_1/2$, where $\tilde{r}_1 := r_\nu(\mu^2 q) = \frac{1}{4}\mu^2 \nu \frac{m^2}{n^3}$ as in (7).

While the above definition might lead the reader to believe that a triangle candidate cover is a very sophisticated structure, the following lemma shows that we have already encountered them.

Lemma 4.15 (Qualified multicovers are triangle candidate covers). *Consider a graph $G \in \mathcal{S}(n, m; \varepsilon)$, and $X \subseteq V_4$ with $|X| \leq (1 - 2\sqrt{\mu})n$. Let $Q \subseteq V_3$ with $|Q| = q_\mu$ be an X-resistant μ-multicover of V_4 that is ν-qualified for V_2. Then Q is also an X-resistant (ν, μ)-triangle candidate cover of V_2 via V_4.*

Proof. We will show that Q^{**}, as given by Definition 4.10, satisfies the properties of $\tilde{Q}$ in Definition 4.14. There exist at least $\mu^2 q$ $r_\mu(q_\mu)$-quasidisjoint covering neighbourhoods $(W'_v \subseteq V_4 \setminus X)_{v \in Q^{**}}$ with cardinality $\mu^2 q$. As Q is ν-qualified for V_2 these sets are ν-multicovers of V_2 and by Lemma 4.6 we have $|C_2(W'_v; \nu)| \geq (1 - \sqrt{\nu})n$ for all $v \in Q^{**}$. By Corollary 3.3 we conclude that there is a set $\tilde{T} \subseteq V_2$ with $|\tilde{T}| \geq (1 - \nu^{1/4})n$ such that every $w \in \tilde{T}$ belongs to at least $(1 - \nu^{1/4})\mu^2 q$ of the sets $(C_2(W'_v; \nu))_{1 \leq v \leq \mu^2 q}$.

In other words, for every $w \in \tilde{T}$ the set $\tilde{Q}^*(w) := \{v \in Q^{**} : w \in C_2(W'_v;\nu)\} \subseteq \tilde{Q} = Q^{**}$ satisfies $|\tilde{Q}^*(w)| \geq (1-\nu^{1/4})\mu^2 q$. Now for every $u \in \tilde{Q}^*(w)$ define $R_u := \Gamma_4(w) \cap W'_u$. As $R_u \subseteq W'_u$, the family $(R_u)_{u\in\tilde{Q}^*(w)}$ is $r_\mu(q_\mu)$-quasidisjoint. Observe that by definition of $\tilde{Q}^*(w)$, we have $w \in C_2(W'_u;\nu)$, hence, by definition of the latter, $|R_u| = |\Gamma_4(w) \cap W'_u| \geq r_\nu(\mu^2 q)/2 = \tilde{r}_1/2$. On the other hand, the definition of W'_u implies that $W'_u \subseteq \Gamma_4(u) \setminus X$, and therefore $R_u \subseteq (\Gamma_4(u) \cap \Gamma_4(w)) \setminus X$, which proves the claim. ∎

Consider a triangle candidate cover Q and a vertex $w \in \tilde{T}(Q)$. We expect that w has a neighbourhood R of size $\Theta(m^2/n^3)$ in $\tilde{Q}^*(w)$. Since every vertex $u \in \tilde{Q}^*(w)$ has $\tilde{r}_1/2 = \Theta(m^2/n^3)$ common neighbours with w in V_4, we expect that R and w complete $\Theta(m^2/n^3 \cdot m^2/n^3) = \Theta(m^4/n^6)$ triangles in V_4 assuming that all neighbourhoods are disjoint. Furthermore, note that $m^4/n^6 = \Omega(n^2/m)$ for $m = \Omega(n^{8/5})$. This leads to the following definition, where we introduce bad sets R that are involved in significantly less than the expected number of triangles. Note that the quasidisjointness of the sets R_u in Definition 4.14 gives us the necessary control on the overlap of the sets R_u for $u \in R$.

Definition 4.16 **(Bad R-sets).** Consider a graph $G \in \mathcal{S}(n,m;\varepsilon)$, $X \subseteq V_4$, $w \in V_2$ and a set $Q \subseteq V_3$ that is an X-resistant (ν,μ)-triangle candidate cover of V_2 via V_4. We define

$$\bar{\mathcal{R}}(w,Q;X,\lambda) := \{R \subset \tilde{Q} : |R| = \tilde{r}_2/2,\ |(\Gamma_4(R) \cap \Gamma_4(w)) \setminus X| < t_\lambda\},$$

where $t_\lambda := \frac{1}{\lambda} \cdot \frac{n^2}{m}$ and $\tilde{r}_2 := r_\mu(\mu^2 q) = \frac{\mu^3}{4}\frac{m^2}{n^3}$.

The following lemma shows that such bad R-sets indeed occur very rarely.

Lemma 4.17 **(Few bad R-sets).** *Consider a graph $G \in \mathcal{S}(n,m;\varepsilon)$ with $m \geq 2^8(\lambda\mu^5\nu)^{-1}n^{8/5} =: m^{\mathrm{Tri}}(\nu,\mu,\lambda)$ and n sufficiently large. Let $X \subseteq V_4$, and let $Q \subseteq V_3$ with $|Q| = q_\mu$ be an X-resistant (ν,μ)-triangle candidate cover of V_2 via V_4. For any $w \in \tilde{T}(Q)$, we have*

$$|\bar{\mathcal{R}}(w,Q;X,\lambda)| \leq \left(33\nu^{1/4}\right)^{\tilde{r}_2/4} \binom{\mu^2 q}{\tilde{r}_2/2} \leq \kappa^{\tilde{r}_2} \binom{\mu^2 q}{\tilde{r}_2/2},$$

where $\kappa := 3\nu^{1/16}$.

Proof. Consider a set $R \in \bar{\mathcal{R}}(w,Q;X,\lambda)$, and let $R' \subset R$ be a set of maximal size such that there exist pairwise disjoint sets $(T_u \subset (\Gamma_4(u) \cap \Gamma_4(w)) \setminus X)_{u\in R'}$ of size $|T_u| = \tilde{r}_1/4$. Clearly,

$$|R'| < \frac{16}{\lambda\mu^2\nu}\frac{n^5}{m^3} =: k_0,$$

since otherwise

$$|(\Gamma_4(R)\cap\Gamma_4(w))\setminus X| \geq |(\Gamma_4(R')\cap\Gamma_4(w))\setminus X| \\ \geq k_0\cdot\frac{\tilde r_1}{4} = \frac{16}{\lambda\mu^2\nu}\frac{n^5}{m^3}\cdot\frac{1}{16}\mu^2\nu\frac{m^2}{n^3} = \lambda^{-1}\cdot\frac{n^2}{m} = t_\lambda.$$

Consider the set $\tilde Q^*(w)$ as defined in Definition 4.14 and the sets $(R_u)_{u\in\tilde Q^*(w)}$. Since the sets $(R_u)_{u\in\tilde Q^*(w)}$ are $r_\mu(q_\mu)$-quasidisjoint, at most

$$\frac{|\bigcup_{u\in R'}T_u|\cdot r_\mu(q_\mu)}{\tilde r_1/4} \leq \frac{t_\lambda\cdot r_\mu(q_\mu)}{\tilde r_1/4} \leq k_0\cdot r_\mu(q_\mu) = \Theta\left(\frac{n^5}{m^3}\cdot\frac{m^2}{n^3}\right) = \Theta(n^2/m)$$

vertices in $\tilde Q^*(w)$ have less than $\tilde r_1/4$ neighbours in $\Gamma_4(w)\setminus(X\cup\bigcup_{u\in R'}T_u)$. Observe that $n^2/m=o(q)$ for $m=\Omega(n^{8/5})$. Also, $|\tilde Q\setminus\tilde Q^*(w)|\leq\nu^{1/4}|\tilde Q|$, and thus for sufficiently large n, there are at most $2\nu^{1/4}|\tilde Q|$ vertices in $\tilde Q\setminus R'$ that have a neighbourhood in $\Gamma(w)\setminus(X\cup\bigcup_{u\in R'}T_u)$ of size less than $\tilde r_1/4$. Note also that

$$\tilde r_2/4 = \frac{1}{16}\mu^3\frac{m^2}{n^3} = k_0\cdot 2^{-8}\mu^5\nu\lambda\frac{m^5}{n^8} \geq k_0,$$

by choice of m, and hence $\tilde r_2/2-k_0\geq\tilde r_2/4$. Now

$$\begin{aligned} |\bar{\mathcal{R}}(w,Q;X,\lambda)| &\leq \sum_{k\leq k_0}\binom{|\tilde Q|}{k}\binom{2\nu^{\frac14}|\tilde Q|}{\tilde r_2/2-k} \\ &\overset{3.1(\mathrm{ii})}{\leq} \sum_{k\leq k_0}\binom{|\tilde Q|}{k}\binom{|\tilde Q|}{\tilde r_2/2-k}\left(2\nu^{\frac14}\right)^{\tilde r_2/2-k} \\ &\overset{3.1(\mathrm{i})}{\leq} \sum_{k\leq k_0}4^{\tilde r_2/2}\binom{|\tilde Q|}{\tilde r_2/2}\left(2\nu^{\frac14}\right)^{\tilde r_2/2-k} \\ &\leq k_0 4^{\tilde r_2/2}\binom{|\tilde Q|}{\tilde r_2/2}\left(2\nu^{\frac14}\right)^{\frac{\tilde r_2}{4}} \leq \left(33\nu^{\frac14}\right)^{\tilde r_2/4}\binom{|\tilde Q|}{\tilde r_2/2} \end{aligned}$$

for sufficiently large n. ∎

Although we have already defined triangle covers we have not yet proved their occurrence in typical graphs. Instead we have considered triangle candidate covers $Q\subseteq V_3$ and showed that most sets of size $\Theta(m^2/n^3)$ inside Q close $\Theta(n^2/m)$ triangle candidates. Observe that a typical vertex $w\in V_2$ has $\Theta(m^2/n^3)$ neighbours in Q. If these neighbours indeed close $\Theta(n^2/m)$ triangle candidates and this is true for most vertices in V_2, this suffices to show that Q is a triangle cover. Unfortunately, by considering one triangle candidate cover alone we do not obtain a sufficiently small probability for bad graphs. This technical difficulty is overcome by considering a partition of V_3 into $\Theta(n^2/m)$ triangle candidate covers Q.

Definition 4.18 (Resistant cover family). Let $\nu,\mu>0$ and $\tilde{p}:=p_\mu/4$ as in (8). A (ν,μ)-*cover family* that is resistant to $(X_i\subseteq V_4)_{i=1,\dots,\tilde{p}}$ consists of pairwise disjoint sets $Q_1,\dots,Q_{\tilde{p}}\subseteq V_3$ such that the following conditions are satisfied for all $i=1,\dots,\tilde{p}$:

(i) $|Q_i|=q_\mu$,
(ii) Q_i is an X_i-resistant (ν,μ)-triangle candidate cover of V_2 via V_4,
(iii) $\tilde{Q}_i$, which is given by Definition 4.14, is a μ-multicover of V_2.

Moreover, we say that a set $P^*=\{v_1,\dots,v_{\tilde{p}}\}\subseteq V_1$ *induces* a (ν,μ)-cover family $(Q_i)_{1\le i\le\tilde{p}}$ that is resistant to $(X_i)_{1\le i\le\tilde{p}}$ if for $i=1,\dots,\tilde{p}$, we have $Q_i\subseteq\Gamma_3(v_i)$.

Our final aim of this section is to show that cover families contain many triangle covers. A cover family consists of triangle candidate covers which we would like to turn into triangle covers by showing that the necessary edges between V_3 and V_2 do exist, i.e., that the sets $\tilde{T}(Q_i)$ from Definition 4.14 can be turned into the sets $T(Q_i)$ from Definition 4.13. The following lemma introduces certain 'bad' vertices in V_2. If we can show that there are only few such vertices, then the cover family behaves as desired and we can prove that there exist many triangle covers. Before we can state the lemma, we need one more definition.

Definition 4.19 (Non-spreading vertices). Let $\delta,\nu,\mu>0$, and let $Q_1,\dots,Q_{\tilde{p}}\subseteq V_3$ be a (ν,μ)-cover family that is resistant to $X_1,\dots,X_{\tilde{p}}\subseteq V_4$. A vertex $w\in V_2$ is called (δ)-*non-spreading* if there exist at least $\delta\tilde{p}$ sets Q_i such that

$$w\in\tilde{T}(Q_i)\quad\text{and}\quad \exists R\subseteq\Gamma_3(w)\cap\tilde{Q}_i,\ |R|=\tilde{r}_2/2:\ R\in\bar{\mathcal{R}}(w,Q_i;X_i,\mu^2).$$

Lemma 4.20 (Cover families contain many triangle covers). *Let $0<\sqrt{\nu}\le\mu$ and $2\mu^{1/4}\le\delta$. Consider a (ν,μ)-cover family $Q_1,\dots,Q_{\tilde{p}}\subseteq V_3$ that is resistant to $X_1,\dots,X_{\tilde{p}}\subseteq V_4$. If there are at most δn (δ)-non-spreading vertices, then*

$$|\{i\in\{1,\dots,\tilde{p}\}: Q_i \text{ is an } X_i\text{-resistant } (\mu^2,\delta)\text{-triangle cover}\}|\ge(1-2\sqrt{\delta})\tilde{p}.$$

Proof. For $i=1,\dots,\tilde{p}$ consider the sets

$$\tilde{C}_i:=\tilde{T}(Q_i)\cap C_2(\tilde{Q}_i;\mu)\subseteq V_2,$$

where $\tilde{T}(Q_i)$ is defined according to Definition 4.14 and $C_2(\tilde{Q}_i;\mu)$ according to Definition 4.5. The idea is that the vertices in $\tilde{T}(Q_i)$ are guaranteed to share many common neighbours (inside V_4) with the set $\tilde{Q}_i$ while the set $C_2(\tilde{Q}_i;\mu)$ sees many vertices in $\tilde{Q}_i$, hence their intersection is a good way of getting closer to the set $T(Q_i)$ from Definition 4.13. By Lemma 4.6, Definition 4.14 and Definition 4.18(iii) we conclude that

$$|\tilde{C}_i| \geq |\tilde{T}(Q_i)| - |V_2 \setminus C_2(\tilde{Q}_i;\mu)| \geq (1-\nu^{1/4}-\sqrt{\mu})n \geq (1-2\sqrt{\mu})n.$$

Hence by Corollary 3.3 there are at least

$$\left(1-\sqrt{2\sqrt{\mu}}\right)n \geq (1-2\mu^{1/4})n \geq (1-\delta)n$$

vertices $w \in V_2$ such that w belongs to at least $(1-2\mu^{1/4})\tilde{p} \geq (1-\delta)\tilde{p}$ sets $\tilde{C}_i$. Let S denote the set of these vertices. If $w \in S$ satisfies $w \in \tilde{C}_i$, we say that Q_i is *apt* for w. Recall that in this case we know that $w \in C_2(\tilde{Q}_i;\mu)$, hence by Definition 4.5 we have that

$$|\tilde{Q}_i \cap \mathit{\Gamma}_3(w)| \geq r_\mu(\mu^2 q)/2 = \tilde{r}_2/2.$$

So we fix a set $R_i(w) \subseteq \tilde{Q}_i \cap \mathit{\Gamma}_3(w)$ with $|R_i(w)| = \tilde{r}_2/2$ for every set Q_i that is apt for w. Now we remove all (δ)-non-spreading vertices from S, and call the resulting set S'. Note that $|S'| \geq |S| - \delta n \geq (1-2\delta)n$.

For any vertex $w \in S'$ there are at least $(1-\delta)\tilde{p}$ apt sets Q_i. Recall that since $w \in S'$ we have that $w \in \tilde{T}(Q_i)$ but also that w is (δ)-spreading. These two conditions together imply that there exist at least $(1-\delta-\delta)\tilde{p} = (1-2\delta)\tilde{p}$ sets Q_i for which

$$\forall R \subseteq \mathit{\Gamma}_3(w) \cap \tilde{Q}_i,\ |R| = \tilde{r}_2/2 :\ R \notin \bar{\mathcal{R}}(w, Q_i; X_i, \mu^2).$$

Letting $R := R_i(w)$ this shows that by Definition 4.16 of $\bar{\mathcal{R}}$, we have

$$|(\mathit{\Gamma}_4(R_i(w)) \cap \mathit{\Gamma}_4(w)) \setminus X_i| \geq t_{\mu^2}.$$

Now recall that

$$TRI_4(w, Q_i) := \mathit{\Gamma}_4(\mathit{\Gamma}_3(w) \cap Q_i) \cap \mathit{\Gamma}_4(w) \supseteq \mathit{\Gamma}_4(R_i(w)) \cap \mathit{\Gamma}_4(w)$$

to obtain that $|TRI_4(w,Q_i) \setminus X_i| \geq t_{\mu^2}$. Let $T(Q_i) := \{w \in S' : |TRI_4(w,Q_i) \setminus X_i| \geq t_{\mu^2}\}$. It remains to show that $|T(Q_i)| \geq (1-3\sqrt{\delta})n$. To do so, note that for any vertex $w \in S'$ we have $w \in T(Q_i)$ for at least $(1-2\delta)\tilde{p}$ sets Q_i. We apply Corollary 3.3 one more time and obtain that there are at least $(1-\sqrt{2\delta})\tilde{p}$ sets Q_i such that

$$|T(Q_i)| \geq (1-\sqrt{2\delta})|S'| \geq (1-\sqrt{2\delta})(1-2\delta)n \geq (1-3\sqrt{\delta})n.$$

Hence these sets Q_i are X_i-resistant (μ^2,δ)-triangle covers and the proof is complete. ∎

5. General counting lemma

Recall from Section 2 that in order to prove our main theorem we want to define bad classes $\mathcal{B}_i(\beta)$. In Section 4 we showed that in a graph belonging to $\mathcal{S}(n,m;\varepsilon)$ there are only very few sets that are not covers, multicovers or triangle candidate covers. Thus if we for example fix the (ε,n,m)-regular graph between V_2 and V_3 and we want to construct an (ε,n,m)-regular graph between V_1 and V_2 then there are only very few possibilities to construct graphs that have many vertices in V_1 that have no mulitcover in their neighbourhood as there are only very few sets in V_2 that are no multicovers. The aim of this section is to provide a rather general counting lemma to count sets $\mathcal{B}_i(\beta)$ that consists of graphs with a linear number of vertices that have an unlikely neighbourhood.

We will construct and thus count the number of atypical graphs $B_i(\beta)$ by first fixing their edges up to $E[B,V_j]$, where $B \subseteq V_i$ for $i \neq j$ denotes a set of atypical vertices that have an unlikely neighbourhood. We will require that $|B| = \Theta(n)$. If there are only very few 'bad' sets in V_j, then one can construct only very few graphs such that all the vertices in B have a bad set as their neighbourhood in V_j. Of course it will be important that we can identify the bad sets in V_j without looking at the edges between B and V_j. The next definitions formalise this idea.

Definition 5.1 (Neighbourhood function). Consider a 4-partite graph $G = (V_1 \cup \cdots \cup V_4, E)$ and a set $B \subseteq V_i$. We denote by $G \setminus B_j$ the graph G without the edges between B and V_j. Given the set $B \subset V_i$, a graph $G \setminus B_j$, a value d_v and a vertex $v \in B$ a *neighbourhood function* $\mathcal{N}(B, G \setminus B_j, d_v, v)$ returns a set of sets $S \subseteq V_j$ of size d_v. We often simply write $\mathcal{N}(v)$ if it is clear which set B, which graph and which value d_v are relevant.

In other words, all of the graph G except for the edges between B and V_j is given to $\mathcal{N}$, even the degree of the vertex v is specified. The function $\mathcal{N}$ then proposes a list of possible neighbourhoods for v. We say that there exists a *bad neighbourhood function* $\mathcal{N}$ for $\mathcal{B}_i(\beta)$ if $\mathcal{N}$ proposes only a very limited number of neighbourhoods for the vertices of a set B of linear size and still guarantees that every graph in $\mathcal{B}_i(\beta)$ cannot choose a neighbourhood for the vertices in B outside the proposed list.

Definition 5.2 (Bad neighbourhood function). Let $\delta, \pi > 0$ and let $\mathcal{G} \subseteq \mathcal{S}(n,m;\varepsilon)$. A neighbourhood function $\mathcal{N}$ is called a *bad neighbourhood function* for the set $\mathcal{G}$ and the parameters δ, π if the following condition holds:

For each $G=(V,E)\in\mathcal{G}$, there exist $1\le i,j\le 4$, $i\ne j$, and set $B\subseteq V_i$ with $|B|\ge\delta n$ and $d_v:=d_j(v)\ge q/2$ for all $v\in B$, such that for each $v\in B$

$$\Gamma_j(v)\in\mathcal{N}(B,G\setminus B_j,d_v,v) \quad \text{and} \quad |\mathcal{N}(B,G\setminus B_j,d_v,v)|\le\pi^{d_v}\binom{n}{d_v}.$$

Lemma 5.3 **(Counting bad graphs).** *Let $\beta,\delta>0$, and let $\pi:=\pi(\beta,\delta)>0$ be sufficiently small such that $8\cdot\pi^{\delta/2}\le\beta$. If $\mathcal{N}$ is a bad neighbourhood function for $\mathcal{G}\subseteq\mathcal{S}(n,m;\varepsilon)$ and parameters δ,π, then*

$$|\mathcal{G}|\le\frac{\beta^m}{4}\binom{n^2}{m}^6$$

for n sufficiently large and any $m=\omega(n\log n)$ (that is $m/(n\log n)\to\infty$ as $n\to\infty$).

Proof. We construct all graphs in $\mathcal{G}$ as follows: We start by choosing i,j with $i\ne j$, the set $B\subseteq V_i$ and the degree values $d_v:=|\Gamma_j(v)|$ for $v\in V_i$. Observe that there are at most $4\cdot 3\cdot n!\cdot n^n\le 2^m$ possibilities to do that, with lots of room to spare. Then we fix the edges in $E\setminus E(V_i,V_j)$ (at most $\binom{n^2}{m}^5$ possibilities). For the vertices $v\in V_i\setminus B$ we have at most $\binom{n}{d_v}$ choices to fix their remaining neighbourhood $\Gamma_j(v)$. For each vertex $v\in B$ we choose a set from $\mathcal{N}(B,G\setminus B_j,d_v,v)$ as its neighbourhood $\Gamma_j(v)$. From the assumption that $\mathcal{N}$ is a bad neighbourhood function it follows that there are at most $\pi^{d_v}\binom{n}{d_v}$ possibilities to do so. Hence, we obtain that the total number of possibilities for choosing the edges between V_i and V_j is bounded by

$$\begin{aligned}\left(\prod_{v\in V_i\setminus B}\binom{n}{d_v}\right)\cdot\left(\prod_{v\in B}\pi^{d_v}\binom{n}{d_v}\right) &\le \pi^{|B|\cdot\min_{v\in B}\{d_v\}}\prod_{v\in V_i}\binom{n}{d_v}\\ &\le \pi^{\delta n\cdot q/2}\binom{n^2}{m} = \pi^{\delta m/2}\binom{n^2}{m}.\end{aligned}$$

Thus it follows that

$$|\mathcal{G}|\le 2^m\cdot\binom{n^2}{m}^5\cdot\pi^{\delta m/2}\binom{n^2}{m}\le\frac{\beta^m}{4}\cdot\binom{n^2}{m}^6,$$

which proves the claim. ∎

A relatively easy application of Lemma 5.3 can be seen in the proof of Lemma 6.4.

6. Proof of the Main Theorem

6.1. Constants

For the proof of Theorem 1.5 we will need the following constants which depend on β:

$$\delta, \delta_r, \delta_c, \nu_q, \mu_q, \varepsilon_q > 0.$$

The constant ε_0 which we need for Theorem 1.5 will be much smaller than all other constants. We characterise these constants by means of the following inequalities (The function $\pi(\beta,\delta)$ is given in Lemma 5.3.) Here, we do not collect the inequalities that involve ε_0 as there are quite a few and as ε_0 always needs to be small with respect to all other constants one can choose it last.

$$2\mu_q^{1/2} \le \pi(\beta, \delta_c), \tag{12}$$

$$\varepsilon_q \le \mu_q^2, \tag{13}$$

$$\nu_q^{1/2} \le \mu_q, \tag{14}$$

$$2\mu_q^{1/4} \le \delta_r, \tag{15}$$

$$200\delta \le \delta_c, \tag{16}$$

$$200\delta \le \mu_q, \tag{17}$$

$$4\delta_r^{1/2} \le \delta_c, \tag{18}$$

$$60\delta_r^{1/4} \le \pi(\beta, \delta_c), \tag{19}$$

$$(16\nu_q)^{\delta_r \mu_q^2/160} \le \pi(\beta, \delta_r/2) \tag{20}$$

It can be seen that suitable values for the constants exist by fixing them in the following order:

$$\delta_c \gg \delta_r \gg \mu_q \gg \varepsilon_q \gg \delta \gg \nu_q \gg \varepsilon_0. \tag{21}$$

Furthermore, we fix a sufficiently large constant $C \ge 1$ such that the conditions of Lemma 4.17 are satisfied, i.e., $Cn^{8/5} \ge m^{\mathrm{Tri}}(\nu_q, \mu_q, \mu_q^2)$. In the remainder we will assume that $Cn^{8/5} \le m \le n^2/4$ and that n is sufficiently large. Furthermore $0 < \varepsilon \le \varepsilon_0$.

6.2. Counting bad graphs

As mentioned before we want to investigate properties of a typical graph in $\mathcal{S}(n,m;\varepsilon)$. In this subsection we show that for nearly all graphs in $\mathcal{S}(n,m;\varepsilon)$ the neighbourhoods of most vertices have roughly their expected sizes and are multicovers. The next step will then be to prove that in these graphs only very few vertices are non-spreading, which by Lemma 4.20 gives rise to many triangle covers. Finally, this implies that a typical graph in $\mathcal{S}(n,m;\varepsilon)$ contains a linear number of vertices in V_1 which have many clique candidates.

6.2.1. Vertices with atypical degree. We say that a vertex $v\in V_i$ satisfies the *degree property* if the following condition (D) is met:

$$(D) \qquad \forall j \neq i: \ (1-\varepsilon)q \leq d_j(v) \leq (1+\varepsilon)q.$$

Definition 6.1 (Graphs with vertices of wrong degree).

$$\mathcal{B}^D(n,m;\varepsilon) := \{G \in \mathcal{S}(n,m;\varepsilon) : \\ \exists i \text{ s.t. at least } 6\varepsilon n \text{ vertices in } V_i \text{ do not satisfy } (D)\}.$$

Lemma 6.2 ($\mathcal{B}^D$ is very small indeed).

$$\mathcal{B}^D(n,m;\varepsilon) = \emptyset.$$

Proof. It follows from Lemma 3.4 that $|\{v\in V_i : |d_j(v)-q| > \varepsilon q\}| < 2\varepsilon n$ for all $i\neq j$. ∎

6.2.2. Vertices without multicovers in their neighbourhood. By Lemma 4.8 we know that almost all sets of size αq for some $0<\alpha<1$ are ν-multicovers. Using Lemma 5.3 we conclude that almost all vertices have multicovers in their neighbourhood for an overwhelming proportion of graphs. The following definition and lemma formalise this intuitive argument.

We say that a vertex $v\in V_i$ satisfies the *multicover property* if the following condition (MC_ν) is met:

$$(MC_\nu)\ \forall j \neq i\ \forall Q \subseteq \Gamma_j(v) \text{ s.t. } \varepsilon_q q \leq |Q| \leq q\ \forall k \neq i,j: \ Q \in \mathcal{Q}_{j,k}(G;|Q|,\nu),$$

where $\mathcal{Q}_{j,k}$ was defined in Definition 4.7.

Definition 6.3 (Vertices with a bad Q-set).

$$\mathcal{B}^{MC}(n,m;\varepsilon,\nu) := \{G \in \mathcal{S}(n,m;\varepsilon) : \exists i \text{ s.t. at least} \\ 3\delta n \text{ vertices in } V_i \text{ do satisfy } (D) \text{ but not } (MC_\nu)\}.$$

Lemma 6.4 **($\mathcal{B}^{MC}$ is small).** *There is a constant $\varepsilon^{MC} = \varepsilon^{MC}(\varepsilon_q, \beta, \delta, \nu) > 0$ such that*

$$|\mathcal{B}^{MC}(n, m; \varepsilon, \nu)| \leq \frac{\beta^m}{4} \binom{n^2}{m}^6$$

for n sufficiently large and $0 < \varepsilon \leq \varepsilon^{MC}$.

Proof. By Lemma 5.3 it suffices to show that there exists an appropriate bad neighbourhood function $\mathcal{N}$. In order to define $\mathcal{N}$, observe first that for each graph $G \in \mathcal{B}^{MC}(n, m; \varepsilon, \nu)$ there exist i and j such that the set $B \subseteq V_i$ of those vertices that satisfy (D) but fail (MC_ν) for the given value of j and at least one set $Q \subseteq \Gamma_j(v)$ has cardinality $|B| \geq \delta n$.

Therefore, we let

$$\mathcal{N}(v) := \Big\{ U \in \tbinom{V_j}{d_v} : \exists Q \subseteq U \text{ s.t. } Q \in \bar{\mathcal{Q}}_{j,k}(G; t, \nu) \text{ for some } k \neq i, j \text{ and } \varepsilon_q q \leq t \leq q \Big\}$$

and apply it to graphs $G \in \mathcal{B}^{MC}(n, m; \varepsilon, \nu)$ with i, j, B as indicated above and $d_v := |\Gamma_j(v)|$ for $v \in B$. Note that it is not necessary to consider the edges between V_i and V_j in order to determine the sets $\bar{\mathcal{Q}}_{j,k}(G; t, \nu)$.

Let $\pi = \pi(\beta, \delta)$ be as in Lemma 5.3, choose $\gamma > 0$ sufficiently small such that $5\gamma^{\varepsilon_q/2} \leq \pi$ and assume that $\varepsilon < \varepsilon^{\mathrm{Mul}}(\gamma, \nu)$, where the latter is as in Lemma 4.8. By Lemma 4.8 and Lemma 3.1 (i) we have for sufficiently large n,

$$\begin{aligned} |\mathcal{N}(v)| &\overset{4.8}{\leq} 2 \sum_{\varepsilon_q q \leq t \leq q} \left[\gamma^t \binom{n}{t} \cdot \binom{n-t}{d_v - t} \right] \overset{3.1(i)}{\leq} 2q \cdot \gamma^{\varepsilon_q q} 4^{d_v} \binom{n}{d_v} \\ &\overset{(D)}{\leq} 2 \frac{d_v}{1-\varepsilon} \gamma^{\varepsilon_q \frac{d_v}{1+\varepsilon}} 4^{d_v} \binom{n}{d_v} \leq (5\gamma^{\varepsilon_q/(1+\varepsilon)})^{d_v} \binom{n}{d_v} \leq \pi^{d_v} \binom{n}{d_v}. \end{aligned}$$

This concludes the proof of the lemma. ∎

6.2.3. Vertices without 'good' neighbourhoods. In the preceding lemmas we have shown that most vertices in a typical graph $G \in \mathcal{S}(n, m; \varepsilon)$ satisfy properties (D) and (MC). For the remainder of this paper we concentrate our attention on *good vertices*, which are denoted by

$$GV_i := \{ v \in V_i : v \text{ satisfies } (D), (MC_{\nu_q}), \text{ and } (MC_{\mu_q}) \}.$$

Let

$$\mathcal{S}^{DM}(n, m; \varepsilon) := \mathcal{S}(n, m; \varepsilon) \setminus \Big[\mathcal{B}^{D}(n, m; \varepsilon) \cup \bigcup_{\nu \in \{\nu_q, \mu_q\}} \mathcal{B}^{MC}(n, m; \varepsilon, \nu) \Big].$$

The notation for the 'bad' sets $\mathcal{B}$ and the 'good' sets $\mathcal{S}$ follows the convention that the superscript indicates the property respectively properties which characterise this set of graphs. For bad sets the given property is violated by a significant number of vertices, whereas for good sets the properties indicated by the superscript are satisfied by almost all vertices.

Consider a graph $G \in \mathcal{S}^{DM}(n,m;\varepsilon)$. From Lemma 6.2, Lemma 6.4 we can immediately conclude that, say,

$$|GV_i| \geq (1-99\delta)n \qquad \text{for all } i = 1,\dots,4$$

with lots of room to spare. We say that a vertex $v \in V_i$ satisfies the *good degree property* if the following condition (D') is met:

$$(D') \quad \forall j: \ |\Gamma_j(v) \setminus GV_j| \leq 100\delta q.$$

The subset of GV_i which consists of all vertices satisfying (D') is denoted by GV_i', i.e.,

$$GV_i' := \{v \in GV_i : v \text{ satisfies } (D')\}.$$

Lemma 6.5 **(GV_i' is big).** *There exists $\varepsilon^G = \varepsilon^G(\delta) > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ and any $G \in \mathcal{S}^{DM}(n,m;\varepsilon)$, we have for all $i = 1,\dots,4$,*

$$|GV_i'| \geq (1-100\delta)n.$$

Proof. Let $j \in \{1,\dots,4\}$, $j \neq i$, and consider any set $F \subseteq V_j$ with $F \supseteq V_j \setminus GV_j$ and $|F| = 99\delta n$. By Lemma 3.4 we obtain that $|\Gamma_j(v) \cap F| \leq (1+\varepsilon)99\delta q \leq 100\delta q$ for all but at most εn vertices in v_i. As $\varepsilon \leq 100\delta$, the lemma immediately follows. ∎

6.2.4. Vertices without triangle covers. Due to Lemma 4.20 we find the desired triangle covers if we are able to prove that there are few non-spreading vertices. The following definition and lemma show that this is indeed the case for most graphs.

Definition 6.6 **(Too many non-spreading vertices).**

$$\begin{aligned}\mathcal{B}^R(n,m;\varepsilon) := \{G \in \mathcal{S}(n,m;\varepsilon) : {} & \exists X_1,\dots,X_{\tilde{p}} \subseteq V_4 \text{ where} \\ & \tilde{p} = \tilde{p}(\mu_q) = n^2/(2\mu_q m)\ \exists(\nu_q,\mu_q)\text{-cover family} \\ & Q_1,\dots,Q_{\tilde{p}} \subseteq V_3 \text{ resistant to } (X_l)_{l=1,\dots,\tilde{p}} : \\ & \text{there exist at least } \delta_r n\ (\delta_r)\text{-non-spreading vertices } w \in V_2\}\end{aligned}$$

Lemma 6.7 **($\mathcal{B}^R$ is small).** *There exist $\varepsilon^R = \varepsilon^R(\nu_q, \mu_q, \delta_r, \beta) > 0$ such that*

$$|\mathcal{B}^R(n,m;\varepsilon)| \le \frac{\beta^m}{4}\binom{n^2}{m}^6,$$

whenever $0 < \varepsilon < \varepsilon^R$, $m \ge m^{\mathrm{Tri}}(\nu_q, \mu_q, \mu_q^2)$ (where m^{Tri} is defined as in Lemma 4.17) and n sufficiently large.

Proof. There are at most $2^{n \cdot \tilde{p}} \le 2^m$ choices for the sets X_l and at most $(\tilde{p}+1)^n$ choices for the cover family $Q_1, \ldots, Q_{\tilde{p}}$. Hence we have at most, say, 3^m possibilities for fixing the (ν_q, μ_q)-cover family with the forbidden sets X_l, and, consequently, it suffices to count the number of graphs which are bad with respect to a specific cover family.

To this end we apply Lemma 5.3, where the set of bad vertices B corresponds to $\delta_r n/2$ (δ_r)-non-spreading vertices which satisfy (D). Observe that there are at least $\delta_r n - 6\varepsilon n \ge \delta_r n/2$ such vertices. We define the neighbourhood function

$$\mathcal{N}(w) := \Big\{ X \in \binom{V_3}{d_w} : \exists I \subseteq \{1, \ldots, \tilde{p}\},\ |I| = \delta_r \tilde{p}$$
$$\forall i \in I\ \ \exists R \subseteq X \cap \tilde{Q}_i,\ |R| = \tilde{r}_2/2 :\ R \in \bar{\mathcal{R}}(w, Q_i; X_i, \mu_q^2) \Big\}.$$

Now let us estimate $|\mathcal{N}(w)|$. First we choose the set I (at most $2^{\tilde{p}}$ possibilities). Then we fix the $\delta_r \tilde{p}$ bad R-sets. Recall from Definition 4.19 that $w \in \tilde{T}(Q_i)$. Thus, by Lemma 4.17 there are at most $\kappa^{\tilde{r}_2}\binom{\mu_q^2 q}{\tilde{r}_2/2}$ possibilities for each bad R-set. Observe that $\delta_r \tilde{p} \cdot \mu_q^2 q = \delta_r \mu_q n/2$ and that $\delta_r \tilde{p} \cdot r_2/2 = \frac{1}{16}\mu_q^2 \delta_r q$. This leads to the bound

$$\begin{aligned}
|\mathcal{N}(w)| &\le 2^{\tilde{p}} \cdot \left(\kappa^{\tilde{r}_2}\binom{\mu_q^2 q}{\tilde{r}_2/2}\right)^{\delta_r \tilde{p}} \cdot \binom{n - \tilde{r}_2/2 \cdot \delta_r \tilde{p}}{d_v - \tilde{r}_2/2 \cdot \delta_r \tilde{p}} \\
&\le 2^{\tilde{p}} \cdot \kappa^{\tilde{r}_2 \cdot \delta_r \tilde{p}} \cdot \binom{\delta_r \mu_q n/2}{\tilde{r}_2 \delta_r \tilde{p}/2} \cdot \binom{n - \tilde{r}_2 \delta_r \tilde{p}/2}{d_v - \tilde{r}_2 \delta_r \tilde{p}/2} \\
&\le (2\kappa)^{\tilde{r}_2 \cdot \delta_r \tilde{p}} \cdot \binom{n}{\tilde{r}_2 \delta_r \tilde{p}/2} \cdot \binom{n - \tilde{r}_2 \delta_r \tilde{p}/2}{d_v - \tilde{r}_2 \delta_r \tilde{p}/2} \overset{3.1(i)}{\le} (2\kappa)^{\delta_r \mu_q^2 q/8} \cdot 4^{d_v}\binom{n}{d_v} \\
&\overset{(D)}{\le} (2\kappa)^{\delta_r \mu_q^2 d_v/10} \cdot 4^{d_v}\binom{n}{d_v} \overset{4.17}{=} \left(6\nu_q^{1/16}\right)^{\delta_r \mu_q^2 d_v/10} \cdot 4^{d_v}\binom{n}{d_v} \\
&\le \left(4^{20/(\delta_r \mu_q^2)} \nu_q^{1/16}\right)^{\delta_r \mu_q^2 d_v/10}\binom{n}{d_v} \overset{(20)}{\le} \pi(\beta, \delta_r/2)^{d_v}\binom{n}{d_v}
\end{aligned}$$

Hence we can apply Lemma 5.3. ∎

Let

$$\mathcal{S}^{DMR}(n,m;\varepsilon) := \mathcal{S}^{DM}(n,m;\varepsilon) \setminus \mathcal{B}^{R}(n,m;\varepsilon).$$

For graphs in $\mathcal{S}^{DMR}$ we deduce from Lemma 4.20 that cover families contain many triangle covers. In order to find many cover families we will consider $\tilde{p}$ vertices in V_1. The following lemma shows that such vertices induce a cover family which was defined in Definition 4.18.

Lemma 6.8 **(Induced cover families).** *Let $P \subseteq GV_1'$ be a $2\mu_q$-cover of V_3 with $P^* =: \{v_1,\ldots,v_{\tilde{p}}\}$, and for $i=1,\ldots,\tilde{p}$, let $X_1,\ldots,X_{\tilde{p}} \subseteq V_4$ be sets with $|X_i| \le (1-2\sqrt{\mu_q})n$. Then P^* induces a (ν_q,μ_q)-cover family $Q_1,\ldots,Q_{\tilde{p}} \subseteq V_3$ that is resistant to $X_1,\ldots,X_{\tilde{p}}$.*

Proof. Denote the covering neighbourhoods of the vertices in P^* by $W_1,\ldots,W_{\tilde{p}}$, where $|W_i| = q_{2\mu_q} = (1-2\mu_q)2\mu_q q$ for $i=1,\ldots,\tilde{p}$. Recall that we denote by $[A]_x$ the set $B \subseteq A$ of size $|B| = x$ that contains the x smallest elements in A. If $|A| < x$, we define $[A]_x := A$. Let

$$Q_i' := [W_i \cap GV_3]_{q_{\mu_q}}.$$

Note that $|W_i \setminus GV_3| \le 100\delta q$, since $v_i \in GV_1'$. Hence

$$|W_i \cap GV_3| \ge (1-2\mu_q)2\mu_q q - 100\delta q \overset{(17)}{\ge} q_{\mu_q}$$

and $|Q_i'| = q_{\mu_q} \overset{(13)}{\ge} \varepsilon_q q$.

It suffices to show that the sets $Q_1',\ldots,Q_{\tilde{p}}'$ satisfy the requirements of Definition 4.18.

Condition (i) has already been shown.

Since $P^* \subseteq GV_1'$ it follows from (MC_{μ_q}) that Q_i' is a μ_q-multicover of V_4 for $i=1,\ldots,\tilde{p}$. As $|X_i| \le (1-2\sqrt{\mu_q})n$, Lemma 4.11 implies that Q_i' is also X_i-resistant. The X_i-resistant covering neighbourhoods of Q_i' in V_4 have size

$$\mu_q^2 q \overset{(13)}{\ge} \varepsilon_q q.$$

Hence $Q_i' \subseteq GV_3$ implies that the covering neighbourhoods are ν_q-multicovers of V_2. Consequently, Q_i' is ν_q-qualified for V_2. By Lemma 4.15 we conclude that Q_i' is an X_i-resistant (ν_q,μ_q)-triangle candidate cover, which shows (ii).

Finally, consider the sets $\tilde{Q}_i'$ from Definition 4.14. We have $|\tilde{Q}_i'| \ge \mu_q^2 q \overset{(13)}{\ge} \varepsilon_q q$ and $v_i \in GV_1'$. Hence $\tilde{Q}_i'$ is a μ_q-multicover of V_2 and condition (iii) is satisfied. ∎

Now we are finally in a position to construct many triangle covers. To this aim we first characterise the 'good' vertices $v \in V_1$ which we are looking for. A vertex $v \in V_1$ is said to satisfy the *triangle cover property* (R) if

$$(R) \quad \forall X \subseteq V_4, |X| \leq (1-2\sqrt{\mu_q})n \; \exists Q_v \subseteq \Gamma_3(v): \\ Q_v \text{ is an } X\text{-resistant } (\nu_q^2, \delta_r)\text{-triangle cover.}$$

The following lemma shows that most vertices satisfy (R).

Lemma 6.9 (Construct triangle covers). *Let ε be sufficiently small. For a graph $G \in \mathcal{S}^{DMR}(n,m;\varepsilon)$ there exist $(1-2\delta_c)n$ vertices $v \in V_1$ that satisfy property (R).*

Proof. First we will show a similar property, namely,

$$(R') \quad \forall \text{ families } \big(X_v \subseteq V_4, |X_v| \leq (1-2\sqrt{\mu_q})n\big)_{v\in V_1} \\ \exists RV_1 \subseteq V_1, |RV_1| \geq (1-2\delta_c)n \; \forall v \in RV_1 \; \exists Q_v \subseteq \Gamma_3(v): \\ Q_v \text{ is an } X_v\text{-resistant } (\mu_q^2, \delta_r)\text{-triangle cover.}$$

In order to prove (R') we construct $2\mu_q$-covers $P_1, \ldots, P_{\tilde{q}} \subseteq GV_1'$ of size $p_{2\mu_q}$, where $\tilde{q} := (1-\delta_c)n/\tilde{p}(\mu_q)$, such that the sets $P_1^*, \ldots, P_{\tilde{q}}^*$ are disjoint. Note that P_i^* is of size $p_{2\mu_q}/2 = n^2/(2\mu_q m) = \tilde{p}(\mu_q) = \tilde{p}$. We construct the $2\mu_q$-covers inductively. Assume that $P_1, \ldots, P_l$ have already been chosen. Note that by Lemma 6.5

$$|GV_1' \setminus (P_1^* \cup \cdots \cup P_l^*)| \geq (1-100\delta)n - l \cdot \tilde{p} \geq (1-100\delta)n - \tilde{q} \cdot \tilde{p} \\ = (\delta_c - 100\delta)n \overset{(16)}{\geq} \delta_c n/2.$$

Thus Lemma 4.3 implies that there exist $2\mu_q$-covers $P_{l+1} \subseteq GV_1' \setminus (P_1^* \cup \cdots \cup P_l^*)$ as $\delta_c/2 \geq \sqrt{\varepsilon}$, and we can choose one an arbitrary one of these.

Now consider arbitrary sets $\big(X_v \subseteq V_4, |X_v| \leq (1-2\sqrt{\mu_q})n\big)_{v\in V_1}$. Using Lemma 6.8 we conclude that the sets P_l^* induce (ν_q, μ_q)-cover families for $l = 1, \ldots, \tilde{q}$. By Definition 6.6 it follows that for these cover families at most $\delta_r n$ non-spreading vertices $w \in V_2$ exist. Hence, due to Lemma 4.20 there exist at least $(1-2\sqrt{\delta_r})\tilde{p}$ vertices $v \in P_l^*$ such that $\Gamma_3(v)$ contains an X_v-resistant (μ_q^2, δ_r)-triangle cover. Observe that the constants ν_q, μ_q and δ_r satisfy the requirements of Lemma 4.20 due to (14) and (15).

If we combine these vertices for all sets $P_1, \ldots, P_{\tilde{q}}$, we obtain $\tilde{q} \cdot (1-2\sqrt{\delta_r})\tilde{p} \geq (1-2\delta_c)n$ vertices for which a triangle cover exists. This completes the proof of (R').

Now assume that (R') holds but the claim of the lemma is not true. Thus there exist more than $2\delta_c n$ vertices $v \in V_1$ for which (R) is not satisfied. For every such vertex v there exists a set $X_v = X \subseteq V_4$ with $|X_v| \leq (1-2\sqrt{\mu_q})n$ such that $\Gamma_3(v)$ does not contain an X_v-resistant (ν_q^2, δ_r)-triangle cover. But this is a contradiction to (R'). Note that the existence of a triangle cover for a vertex $v \in V_1$ depends only on 'its own' X-set. ∎

6.2.5. Vertices without clique candidates. Now that we have found triangle covers we intend to finally find clique candidates. As described in Section 2 we want to show that for almost every vertex $v \in V_1$ there exist many clique candidates $x \in V_4$ such that adding the edge $\{v,x\}$ would result in a K_4.

Definition 6.10 (Clique candidates). For $v \in GV_1'$ and a set $A \subseteq V_2$ we define the set $CC(v,A) \subseteq V_4$ by

$$CC(v,A) := \{x \in V_4 : \exists w \in A\ \exists u \in \Gamma_3(v) \text{ s.t. } \{u,w\}, \{x,u\}, \{x,w\} \in E\}.$$

Definition 6.10 allows us to construct the set of clique candidates iteratively. Note that $CC(v, \Gamma_2(v))$ contains all possible clique candidates. We will now show how we can construct a set $A \subseteq \Gamma_2(v)$, adding vertex by vertex until $CC(v,A)$ is large enough. The set A will be called the *processed neighbourhood* of v. For the remainder of the paper we restrict our attention to sets with $|A| \leq \mu_q^2 q$.

A vertex $v \in V_1$ is said to satisfy the *clique candidate property* (C) if

$$(C) \quad \begin{array}{l} \forall A \subseteq \Gamma_2(v), |A| \leq \mu_q^2 q \\ \forall Q \subseteq \Gamma_3(v) \text{ s.t. } Q \text{ is a } CC(v,A)\text{-resistant } (\mu_q^2, \delta_r)\text{-triangle cover :} \\ \quad (\Gamma_2(v) \setminus A) \cap T(Q) \neq \emptyset. \end{array}$$

Definition 6.11 (Vertices without clique candidates).

$$\mathcal{B}^{CC}(n,m;\varepsilon) := \{G \in \mathcal{S}^{DMR}(n,m;\varepsilon) : \\ \text{at least } \delta_c n \text{ vertices } v \in V_1 \text{ satisfy } (D) \text{ but not } (C)\}$$

Lemma 6.12 ($\mathcal{B}^{CC}$ is small).

$$|\mathcal{B}^{CC}(n,m;\varepsilon)| \leq \frac{\beta^m}{4}\binom{n^2}{m}^6$$

for sufficiently small ε and sufficiently large n and $Cn^{8/5} \leq m \leq n^2/4$.

Proof. Once again we will use Lemma 5.3 to prove that $\mathcal{B}^{CC}(n,m;\varepsilon)$ is small. For the $\delta_c n$ bad vertices indicated by Definition 6.11 we define the neighbourhood function

$$\mathcal{N}(v) := \Big\{ X \in \binom{V_2}{d_v} : \exists A \subseteq X, |A| \leq \mu_q^2 q : \\ \exists Q \subseteq \Gamma_3(v) \text{ s.t. } Q \text{ is a } CC(v,A)\text{-resistant} \\ (\mu_q^2, \delta_r)\text{-triangle cover and } (X \setminus A) \cap T(Q) = \emptyset \Big\}.$$

In order to estimate $|\mathcal{N}(v)|$ we first choose the set $Q \subseteq \Gamma_3(v)$ (at most 2^{2q} possibilities as v satisfies (D)). Then we fix $a_v := |A|$ for $v \in B$. There are at most $\mu_q^2 q < n$ choices for the value a_v. For a vertex v, denote by d_v the size of the neighbourhood of v in V_2. Note that $d_v \geq (1-\varepsilon)q$ as v satisfies (D). Recall that $|T(Q)| \geq (1-3\sqrt{\delta_r})n$, therefore we choose a_v vertices for the set A, and then $d_v - a_v$ in $V_2 \setminus (A \cup T(Q))$. Thus

$$\begin{aligned} |\mathcal{N}(v)| &\leq n 2^{2q} \binom{n}{a_v} \cdot \binom{3\sqrt{\delta_r} n}{d_v - a_v} \overset{3.1\text{(ii)}}{\leq} 5^{d_v} \binom{n}{a_v} \cdot (3\delta_r^{1/2})^{d_v - a_v} \binom{n}{d_v - a_v} \\ &\leq 5^{d_v} \binom{n}{a_v} \cdot (3\delta_r^{1/2})^{d_v/2} \binom{n}{d_v - a_v} \overset{3.1\text{(i)}}{\leq} 20^{d_v} (3\delta_r^{1/4})^{d_v} \binom{n}{d_v} \\ &\overset{(19)}{\leq} \pi(\beta, \delta_c)^{d_v} \binom{n}{d_v}, \end{aligned}$$

for n sufficiently large. ∎

Let $\mathcal{S}^{DMRC}(n,m;\varepsilon) := \mathcal{S}^{DMR}(n,m;\varepsilon) \setminus \mathcal{B}^{CC}(n,m;\varepsilon)$. Furthermore, we define

$$GV_1'' := \{v \in V_1 : v \text{ satisfies } (R) \text{ and } (C)\}.$$

For a graph $G \in \mathcal{S}^{DMRC}(n,m;\varepsilon)$ observe that $|GV_1''| \geq (1 - 2\delta_c - \delta_c - 6\varepsilon)n = (1-4\delta_c)n$ due to Lemma 6.2, Lemma 6.9 and Definition 6.11.

The following lemma states that it is possible to successively construct clique candidates $CC(v, \Gamma_2(v))$ using a processed neighbourhood $A \subseteq \Gamma_2(v)$.

Lemma 6.13 (Many candidates for a K_4). *For every vertex $v \in GV_1''$ there exists a processed neighbourhood $A \subseteq \Gamma_2(v)$ such that*

$$|CC(v,A)| \geq (1 - 2\sqrt{\mu_q})n. \tag{22}$$

Proof. We show that as long as $|A| \leq \mu_q^2 q$ and $|CC(v,A)| < (1-2\sqrt{\mu_q})n$ we can find a vertex $w \in \Gamma_2(v) \setminus A$ such that $|CC(v, A \cup \{w\}) \setminus CC(v,A)| \geq t_{\mu_q^2}$. This suffices to prove the claim, since $\mu_q^2 q \cdot t_{\mu_q^2} = n \geq (1-2\sqrt{\mu_q})n$ and we may thus iteratively add the vertices w to the set A until $CC(v,A)$ is large enough.

Let $X := CC(v,A)$ and recall that $|X| \le (1-2\sqrt{\mu_q})n$. Thus we conclude by the definition of (R) that there exists an X-resistant (ν_q^2,δ_r)-triangle cover $Q \subseteq \Gamma_3(v)$. Due to property (C) it follows that there exists a vertex $w \in (\Gamma_2(v) \setminus A) \cap T(Q)$. Clearly, this vertex satisfies the necessary properties and may be added to A. ∎

6.2.6. Vertices without a K_4. Let

$$CC_v := CC(v, \Gamma_2(v)).$$

Now that we know that CC_v comprises almost all vertices in V_4, we are basically done. A final (simple) argument shows that $\Gamma_4(v)$ and CC_v must overlap.

We say that a vertex $v \in V_i$ satisfies the *clique property* if the following condition (K) is met:

$$(K) \quad \Gamma_4(v) \cap CC_v \neq \emptyset.$$

Definition 6.14 (Vertices with no K_4).

$$\mathcal{B}^K(n,m;\varepsilon) := \{G \in \mathcal{S}^{DMRC}(n,m;\varepsilon) : \text{at least } \delta_c n \text{ vertices in } GV_1'' \text{ do not satisfy } (K)\}$$

Lemma 6.15 ($\mathcal{B}^K$ is small).

$$|\mathcal{B}^K(n,m;\varepsilon)| \le \frac{\beta^m}{4}\binom{n^2}{m}^6$$

for sufficiently small ε and sufficiently large n and $Cn^{8/5} \le m \le n^2/4$.

Proof. A simple (and final) application of Lemma 5.3 proves the claim. The bad set B with $|B| \ge \delta_c n$ is defined as indicated by Definition 6.14. Let

$$\mathcal{N}(v) := \left\{X \in \binom{V_4}{d_v} : X \cap CC_v = \emptyset\right\}.$$

From Lemma 6.13 it follows that

$$|\mathcal{N}(v)| \le \binom{2\sqrt{\mu_q}n}{d_v} \overset{3.1(\mathrm{ii})}{\le} (4\mu_q)^{d_v/2}\binom{n}{d_v} \overset{(12)}{\le} \pi(\beta,\delta_c)^{d_v}\binom{n}{d_v},$$

which completes the proof. ∎

This also completes the proof of Theorem 1.5: for any graph $G \in \mathcal{S}^{DMRC}(n,m;\varepsilon) \setminus \mathcal{B}^K(n,m;\varepsilon)$ all but $\delta_c n$ vertices $v \in GV_1''$ have a vertex $x \in \Gamma_4(v) \cap CC_v$ that builds a K_4 containing v. Hence

$$\mathcal{F}(n,m;\varepsilon) \subseteq \mathcal{B}^D(n,m;\varepsilon) \cup \mathcal{B}^{MC}(n,m;\varepsilon) \cup \mathcal{B}^R(n,m;\varepsilon) \cup \mathcal{B}^{CC}(n,m;\varepsilon) \cup \mathcal{B}^K(n,m;\varepsilon)$$

and the result follows from the bounds on the sets $\mathcal{B}^*(n,m;\varepsilon)$.

Note that our proof actually yields a somewhat stronger result than Theorem 1.5. The condition (K) could be strengthened to

$$(K') \quad |\Gamma_4(v) \setminus CC_v| \le \varepsilon_n q$$

for a suitable constant $\varepsilon_n > 0$ which is large in comparison to $2\sqrt{\mu_q}$. Then Lemma 6.15 is still satisfied. Using this we obtain that almost all neighbours $x \in \Gamma_4(v)$ of a vertex in $v \in GV_1''$ which satisfies conditions (K') are part of a subgraph K_4 together with v.

7. Acknowledgements

We would like to thank the referees for their careful reading and the many helpful comments.

References

[1] M. Behrisch: *Random graphs without a short cycle*, Master's thesis, Humboldt-Universität zu Berlin, 2002 (in German).

[2] Z. Füredi: Random Ramsey graphs for the four-cycle, *Discrete Mathematics* **126** (1994), 407–410.

[3] S. Gerke, Y. Kohayakawa, V. Rödl and A. Steger: Small subsets inherit sparse ε-regularity, *Journal of Combinatorial Theory B*, accepted. http://dx.doi.org/10.1016/j.jctb.2006.03.004

[4] S. Gerke, M. Marciniszyn and A. Steger: A probabilistic counting lemma for complete graphs, in: *2005 European Conference on Combinatorics, Graph Theory and Applications (EuroComb '05)*, S. Felsner, ed., pp. 309–316, DMTCS Proceedings, 2005 (extended abstract). To appear in *Random Structures & Algorithms*.

[5] S. Gerke and A. Steger: The sparse regularity lemma and its applications, in: *Surveys in Combinatorics 2005, LMS*, B. S. Webb, ed., LNS **327**, pp. 227–258, CUP, 2005.

[6] S. Gerke, T. Schickinger and A. Steger: K_5-free subgraphs of random graphs, *Random Structures Algorithms* **24(2)** (2004), 194–232.

[7] P. E. Haxell, Y. Kohayakawa and T. Łuczak: Turán's extremal problem in random graphs: forbidding even cycles, *Journal of Combinatorial Theory B* **64** (1995), 273–287.

[8] P. E. Haxell, Y. Kohayakawa and T. Łuczak: Turán's extremal problem in random graphs: forbidding odd cycles; *Combinatorica* **16(1)** (1996), 107–122.
[9] Y. Kohayakawa and B. Kreuter: Threshold functions for asymmetric Ramsey properties involving cycles, *Random Structures & Algorithms* **11** (1997), 245–276.
[10] Y. Kohayakawa, B. Kreuter and A. Steger: An extremal problem for random graphs and the number of graphs with large even girth, *Combinatorica* **18(1)** (1998), 101–120.
[11] Y. Kohayakawa, T. Łuczak and V. Rödl: Arithmetic progressions of length three in subsets of a random set, *Acta Arithmetica* **LXXV.2** (1996), 133–163.
[12] Y. Kohayakawa, T. Łuczak and V. Rödl: On K^4-free subgraphs of random graphs, *Combinatorica* **17(2)** (1997), 173–213.
[13] Y. Kohayakawa and V. Rödl: Regular pairs in sparse random graphs I, *Random Structures & Algorithms* **22(4)** (2003), 359–434.
[14] B. Kreuter: *Probabilistic versions of Ramsey's and Turán's theorems.* PhD thesis, Humboldt-Universität zu Berlin, 1997.
[15] Y. Kohayakawa, V. Rödl and M. Schacht: The Turán theorem for random graphs, *Combin. Probab. Comput.* **13(1)** (2004), 61–91.
[16] D. J. Kleitman and K. J. Winston: On the number of graphs without 4-cycles, *Discrete Mathematics* **41** (1982), 167–172.
[17] T. Łuczak: On triangle-free random graphs, *Random Structures & Algorithms* **16(3)** (2000), 260–276.
[18] T. Szabó and V. H. Vu: Turán's theorem in sparse random graphs, *Random Structures & Algorithms* **23(3)** (2003), 225–234.

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