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k-NONCROSSING AND k-NONNESTING GRAPHS AND FILLINGS OF FERRERS DIAGRAMS

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We give a correspondence between graphs with a given degree sequence and fillings of Ferrers diagrams by nonnegative integers with prescribed row and column sums. In this setting, k-crossings and k-nestings of the graph become occurrences of the identity and the antiidentity matrices in the filling. We use this to show the equality of the numbers of k-noncrossing and k-nonnesting graphs with a given degree sequence. This generalizes the analogous result for matchings and partition graphs of Chen, Deng, Du, Stanley, and Yan, and extends results of Klazar to $k>2$. Moreover, this correspondence reinforces the links recently discovered by Krattenthaler between fillings of diagrams and the results of Chen et al.

1. Introduction

Let G be a graph on $[n]$; unless otherwise stated, we allow multiple edges and isolated vertices, but no loops. Two edges $\{i,j\}$ and $\{k,l\}$ are a *crossing* if $i<k<j<l$ and they are a *nesting* if $i<k<l<j$. If we draw the vertices of G on a line and represent the corresponding edges by arcs above the line, crossings and nestings have the obvious geometric meaning. A graph without crossings (respectively, nestings) is called *noncrossing* (resp., *nonnesting*). Klazar [10] proves the equality between the numbers of noncrossing and nonnesting simple graphs, counted by order, and also between the numbers of noncrossing and nonnesting graphs without isolated vertices, counted by size. The purpose of this paper is to study analogous results for sets of k pairwise crossing and k pairwise nested edges.

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A *k-crossing* is a set of k edges every two of them being a crossing, that is, edges $\{i_1, j_1\}, \ldots, \{i_k, j_k\}$ such that $i_1 < i_2 < \cdots < i_k < j_1 < \cdots < j_k$. A *$k$-nesting* is a set of k edges pairwise nested, that is, $\{i_1, j_1\}, \ldots, \{i_k, j_k\}$ such that $i_1 < i_2 < \cdots < i_k < j_k < \cdots < j_1$. A graph with no k-crossing is called *k-noncrossing* and a graph with no k-nesting is called *k-nonnesting*. The largest k for which a graph G has a k-crossing (respectively, a k-nesting) is denoted $\mathrm{cross}(G)$ (resp., $\mathrm{nest}(G)$). The aim of this paper is to show that the number of k-noncrossing graphs equals the number of k-nonnesting graphs, counted by order, size, and degree sequence. This problem was originally posed by Martin Klazar and we learned of it at the Homonolo 2005 workshop [2]; the case where the number of vertices of the graph is $2k + 1$ was proved by A. Pór (unpublished). Our main result (Theorem 3.3) states that the numbers of k-noncrossing and k-nonnesting graphs with a given degree sequence are the same.

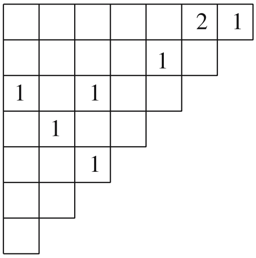
Chen et al. [4] prove the equality of the numbers of k-noncrossing and k-nonnesting graphs for two subclasses of graphs, namely for perfect matchings and for partition graphs, also counted by degree sequences (under a different but equivalent terminology). A perfect matching is a graph where each vertex has degree one, and a partition graph is a graph that is a disjoint union of monotone paths, that is, where each vertex has at most one edge to its right and at most one to its left. The latter correspond in a natural way to set partitions, hence the result can be stated in terms of these. The paper [4] also contains other identities and enumerative results on k-noncrossing and k-nonnesting matchings and partitions. Krattenthaler [12] deduces many of these from his more general results on fillings of Ferrers diagrams. As we will explain further in Section 3, our Theorem 3.3 is implicit in the work of Krattenthaler, but it is not stated there and a careful reading of the proofs is needed to deduce it. Other recent papers that explore fillings and are related to our work are the ones by Backelin, West and Xin [1], Jonsson [7], Jonsson and Welker [8] and Rubey [15]; the last two contain extensions of some of the results of [12] to stack and moon polyominoes, respectively. We delay the discussion of these results until we have introduced the necessary terminology.

Fillings of Ferrers diagrams are also our main tool in studying k-crossings and k-nestings. The idea is to encode graphs by fillings of Ferrers diagrams in such a way that k-crossings and k-nestings are easy to recognize. A k-nonnesting (k-noncrossing) graph becomes a filling of a diagram that avoids the identity (antiidentity) matrix of order k, and the degree sequence of the graph can be recovered from the shape of the diagram and the row and column sums of the filling. Then proving that there are as many k-noncrossing as k-nonnesting graphs is equivalent to showing that the numbers of fillings avoiding these two matrices are the same. The difference between our correspondence and previous ones is that whereas in [12], and also in [7], the results about graphs follow from general theorems by restricting the shape of the diagram, here we show that the results about graphs are in fact equivalent to those about fillings with arbitrary shapes.

The structure of the paper is as follows. In Section 2 we define fillings of Ferrers diagrams and what it means for a filling to avoid a given matrix. We then introduce a first correspondence between graphs and fillings that allows us to rephrase results of Krattenthaler [12] and Jonsson and Welker [8] in terms of k-noncrossing and k-nonnesting graphs. Section 3 introduces a new correspondence between graphs and fillings of diagrams that keeps track of degree sequences. This correspondence allows us to show that the study of fillings of Ferrers diagrams with forbidden matrices is equivalent to the study of graphs avoiding certain subgraphs, in a given sense. In particular, showing that the number of k-noncrossing graphs with a fixed degree sequence equals the number of such k-nonnesting graphs is equivalent to proving a result on fillings of diagrams with restrictions on the row and column sums. Our proof is an adaptation of the one in [1] to allow arbitrary entries in the filling, and this is the content of Section 4. We conclude with some remarks and open questions.

2. Fillings of diagrams

We start by setting some notation on fillings of Ferrers diagrams. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be an integer partition. The *Ferrers diagram of shape* λ (or simply a *diagram*) is the arrangement of square cells, left-justified and from top to bottom, having λ_i cells in row i, for i with $1 \leq i \leq k$. For a Ferrers diagram T of shape λ with rows indexed from top to bottom and columns from left to right, a *filling* L of T consists of assigning a nonnegative integer to each cell of the diagram. We say that a cell is *empty* if it has been assigned the integer 0. Let M be an $s \times t$ $0-1$ matrix. We say that the filling *contains* M if there is a selection of rows $r_1 < \cdots < r_s$ and columns $c_1 < \cdots < c_t$ of T such that if $M_{i,j} = 1$ then the cell (r_i, c_j) of T is nonempty and moreover the cell (r_s, c_t) is in the diagram (in other words, we require that the matrix M is fully contained in T). We say that the filling *avoids* M if there is no such selection of rows and columns. If a filling L contains M, by an *occurrence* of M we mean the set of cells of T that correspond to the 1's in M. We are mainly concerned about diagrams avoiding the identity matrix I_t and the antiidentity matrix J_t; the latter is the matrix with 1's in the main antidiagonal and 0's elsewhere. As an example of these concepts, Figure 1 shows a filling of a diagram of shape $(7,6,5,4,3,2,1)$ that contains the matrices I_3 and J_2 but avoids J_3. (For clarity, we omit the zeros corresponding to the empty cells.)



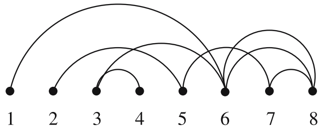


Figure 1. Left: a filling of a diagram that contains I_3 and J_2 but avoids J_3. Right: the graph determined by the filling has a 3-nesting and several 2-crossings, but no 3-crossing.

Studying fillings of diagrams avoiding matrices is a natural generalization of pattern avoiding permutations, as explained in [1,16]. We explore two types of connections between graphs and fillings of diagrams. The first one is straightforward, being essentially the adjacency matrix, and it has been used in [7,12] to derive results on k-noncrossing maximal graphs and k-noncrossing and k-nonnesting matchings and partitions.

Suppose G is a graph on $[n]$ and consider a diagram Δ of shape $(n-1, n-2, \ldots, 2, 1)$. Then if there are $d \geq 0$ edges joining vertices i and j, with $i<j$, fill the cell of column i and row $n-j+1$ with d. Let this filling of the diagram be called $\Delta(G)$. Obviously the sum of the entries of $\Delta(G)$ is the number of edges of G and the number of vertices is just one plus the number of rows of Δ. If G is a simple graph, then $\Delta(G)$ is a $0-1$ filling. If the edges $\{i_1, j_1\}, \ldots, \{i_k, j_k\}$ are a k-nesting of G, then $\Delta(G)$ contains the $k \times k$ identity matrix I_k in columns $i_1, \ldots, i_k$ and rows $n-j_1+1, \ldots, n-j_k+1$. Similarly, if G contains a k-crossing, then $\Delta(G)$ contains the antiidentity matrix J_k (the condition $i_k < j_1$ guarantees that the matrix is indeed contained in the diagram). Figure 1 shows a graph G and its corresponding filling $\Delta(G)$.

Krattenthaler [12] derives many of the results of Chen et al. [4] for matchings and partitions by specializing to $\Delta(G)$ his results on fillings of diagrams avoiding large identity or antiidentity matrices. His Theorem 13 gives a generalization to arbitrary graphs which is implicitly included in the remark after it; we state this generalization in Corollary 2.2 (see also the comment after Theorem 3.3 in the next section). The following is a weak version of [12, Theorem 13].

Theorem 2.1. *For any diagram T and any integer m, consider fillings of T with nonnegative integers adding up to m. Then for each $k>1$, the number* *of such fillings that do not contain the identity matrix I_k equals the number of fillings that do not contain the antiidentity matrix J_k.*

By restricting to $T=\Delta$ we immediately get the following.

Corollary 2.2. *The number of k-noncrossing graphs with n vertices and m edges equals the number of k-nonnesting such graphs.*

Actually, from the statement of [12, Theorem 13] one gets a stronger result. For this we need to introduce weak k-crossings and weak k-nestings. The edges $\{i_1,j_1\},\ldots,\{i_k,j_k\}$ are a *weak k-crossing* if $i_1\le i_2\le\cdots\le i_k<j_1\le\cdots\le j_k$; similarly, they are a *weak k-nesting* if $i_1\le i_2\le\cdots\le i_k< j_k\le\cdots\le j_1$. Let $\mathrm{cross}^*(G)$ (respectively, $\mathrm{nest}^*(G)$) be the largest k for which G has a weak k-crossing (resp., weak k-nesting). Then the following is a corollary of the full version of [12, Theorem 13].

Corollary 2.3. *The number of graphs with n vertices and m edges with $\mathrm{cross}(G) = r$ and $\mathrm{nest}^*(G) = s$ equals the number of such graphs with $\mathrm{cross}^*(G)=s$ and $\mathrm{nest}(G)=r$.*

Ideally, one would like to have an analogous result proving the symmetry of the distribution of $\mathrm{cross}(G)$ and $\mathrm{nest}(G)$ for all graphs. This is known to be true for matchings and partition graphs [4, Theorem 1 and Corollary 4]; actually, for these graphs weak crossings (respectively, weak nestings) are the same as crossings (resp., nestings). For arbitrary graphs, Corollary 2.3 is the best we can hope for, since it has been recently shown [13] that in general the pair $(\mathrm{cross}(G),\mathrm{nest}(G))$ is not symmetrically distributed.

The bijection used to prove [12, Theorem 13] does not preserve the values of the entries of the filling, so we cannot deduce from it the corresponding result for simple graphs. However, this follows from a result of Jonsson and Welker. They deal with fillings not of diagrams, but of stack polyominoes. A *stack polyomino* consists of taking a diagram, reflecting it through the vertical axis, and gluing it to another (unreflected) diagram. The content of a stack polyomino is the multiset of the lengths of its columns. The definitions of fillings and containment of matrices in stack polyominoes are analogous to those for diagrams. The following is Corollary 6.5 of [8]. (The particular case where m below is maximal was proved in [7], and the extension to moon polyominoes in [15].)

Theorem 2.4. *The number of $0-1$ fillings of a stack polyomino with m nonzero entries that avoid the matrix I_k depends only on the content of the polyomino and not on the ordering of the columns.*

By a simple reflection argument we get the following for the triangular diagram Δ of shape $(n-1, n-2, \ldots, 2, 1)$: the number of $0-1$ fillings of Δ with m non-zero entries and that avoid the matrix I_k is the same as those that avoid the matrix J_k. Hence we have the following in terms of graphs.

Corollary 2.5. *The number of k-noncrossing simple graphs on n vertices and m edges equals the number of such k-nonnesting simple graphs.*

In the next section we deal with graphs with a fixed degree sequence. For this we need to consider diagrams of arbitrary shapes, since the correspondence between graphs and fillings is no longer restricted to the triangular diagram Δ.

3. Degree sequences and fillings with prescribed row and column sums

The *left-right degree sequence* of a graph on $[n]$ is the sequence $((l_i, r_i))_{1\le i\le n}$, where l_i (resp., r_i) is the left (resp., right) degree of vertex i; by the left (resp., right) degree of i we mean the number of edges that join i to a vertex j with $j<i$ (resp., $j>i$). Obviously l_i+r_i is the degree of vertex i (loops are not allowed). For instance, if $r_i \le 1$ and $l_i \le 1$ for all i, then the graph is either a matching or a partition graph, perhaps with some isolated vertices. If a graph G has D as its left-right degree sequence, we say that G is a graph *on* D. A useful way of thinking of left-right degree sequences is drawing for each vertex i, l_i half-edges going left and r_i half-edges going right. Then a graph is just a way of matching these half-edges; recall that we allow multiple edges. For completeness we mention here that a sequence $((l_i, r_i))_{1\le i\le n}$ is the left-right degree sequence of some graph on $[n]$ if and only if

$$\sum_{i=1}^{n} l_i = \sum_{i=1}^{n} r_i \text{ and } \sum_{i=1}^{k} l_i \le \sum_{i=1}^{k-1} r_i, \quad \forall k\in[n]. \tag{1}$$

This and the next section are devoted to proving that for each left-right degree sequence D there are as many k-noncrossing graphs on D as k-nonnesting. We stress that the fact that we allow multiple edges is essential, since if we restrict to simple graphs the result does not hold. For instance, one can check that there is one simple nonnesting graph with left-right degree sequence $(0,2),(0,2),(1,1),(2,0),(2,0)$, but no such noncrossing simple graph. However, it turns out that there is a bijection between k-noncrossing and k-nonnesting simple graphs that preserves left degrees (or right degrees, but not both simultaneously). This follows from the following result of Rubey [15, Theorem 4.2] applied to the filling $\Delta(G)$ by noting that the sum of the entries in row $n-j+1$ of $\Delta(G)$ corresponds to the left degree of vertex j. Rubey's result is for moon polyominoes, but we state the version for stack polyominoes. (A weaker version of this result was proved by Jonsson [7, Corollary 26].)

Theorem 3.1. *For any stack polyomino Λ with s rows and for any sequence $(d_1,\ldots,d_s)$ of nonnegative integers, the number of 0–1 fillings of Λ that avoid I_k and have d_i nonzero entries in row i depends only on the content of Λ and not on the ordering of the columns.*

By the same reflection argument as at the end of Section 2 we obtain the following corollary.

Corollary 3.2. *Let $(l_2,\ldots,l_n)$ be a sequence of nonnegative integers. Then the number of k-noncrossing simple graphs on $[n]$ with vertex i having left degree l_i for $2\le i\le n$ is the same as the number of such k-nonnesting simple graphs.*

The main result of this paper says that by allowing multiple edges we can simultaneously fix left and right degrees.

Theorem 3.3. *For any left-right degree sequence D, the number of k-noncrossing graphs on D equals the number of k-nonnesting graphs on D.*

This result generalizes to arbitrary graphs some of the results of [4], which are only for partition graphs but also taking into account degree sequences (with different terminology). To approach Theorem 3.3 we could use again the filling $\Delta(G)$ of the previous section and fix the sums of the entries in each row and column. In this setting Theorem 3.3 is again implicitly included in the remark after [12, Theorem 13] by keeping track of the changes in the partitions involved in the proof of that theorem. (I am grateful to Christian Krattenthaler for this observation.) However, our approach consists of encoding graphs not by the triangular diagram Δ but by an arbitrary diagram whose shape depends on the degree sequence. By doing this we actually show that not only results on k-noncrossing and k-nonnesting graphs can be deduced from results on fillings of Ferrers diagrams avoiding I_k and J_k, but that actually these two families of results are completely equivalent. Moreover, we have an analogous assertion for arbitrary matrices (see Theorem 3.7).

We start with an easy lemma that follows immediately from the fact that the edges in a k-crossing or a k-nesting must be vertex-disjoint.

Lemma 3.4. *The number of k-noncrossing (resp., k-nonnesting) graphs with left-right degree sequence*

$$(l_1,r_1),\dots,(l_n,r_n)$$

is the same as that of those with left-right degree sequence

$$(l_1,r_1),\dots,(l_{i-1},r_{i-1}),(l_i,0),(0,r_i),(l_{i+1},r_{i+1}),\dots,(l_n,r_n),$$

for any i with $1<i<n$.

Hence it is enough to prove Theorem 3.3 for left-right degree sequences whose elements (l_i,r_i) are such that either l_i or r_i is 0. We call these graphs *left-right* graphs; note though that we do not require that the degrees of the vertices alternate between right and left. The case where both left and right degrees are 0 corresponds to an isolated vertex.

We now describe a bijection between left-right graphs and fillings of Ferrers diagrams of arbitrary shape; this bijection has the property that the left-right degree sequence of the graph can be recovered from the shape and filling of the diagram. Let G be a left-right graph. If the degree of vertex i is of the form $(0,r_i)$ we say that i is *opening*, and if it is of the form $(l_i,0)$ we say that i is *closing*. An isolated vertex is both opening and closing. Let $i_1<\cdots<i_c$ be the closing vertices of G and let $j_1<\cdots<j_o$ be the opening ones. For each closing vertex i, let $p(i)$ be the number of vertices j with $j<i$ that are opening. We consider a diagram $T(G)$ of shape $(p(i_c),p(i_{c-1}),\dots,p(i_1))$, and if there are d edges going from the opening vertex j_s to the closing vertex i_r, we fill the cell in column s and row $c-r+1$ with the integer d (see Figure 2). Thus graphs with left degrees $l_1,\dots,l_c$ and right degrees $r_1,\dots,r_o$ correspond to fillings of this diagram with nonnegative entries such that the sum of the entries in row i is l_i and the sum of the entries in column j is r_j. Conversely, any filling of a diagram arises in this way. Indeed, given a filling L of a diagram T, the shape of T gives the ordering of the opening and closing vertices of the graph, the row and column sums give the left and right degrees (it is easy to see that they must satisfy equation (1)), and the entries of the filling give the edges of the graph. Given a graph G, we denote by $L(G)$ the filling of $T(G)$ corresponding to G. Similarly, given a filling L of a diagram, we denote by $G(L)$ the left-right graph corresponding to this filling.

In this setting, it is immediate to check that again k-crossings of G correspond to occurrences of J_k in $L(G)$ and k-nestings to occurrences of I_k.

By a *diagram with prescribed row and column sums* we mean a diagram and two sequences (ρ_i) and (γ_j) of nonnegative integers such that the only fillings allowed for this diagram are those where the row and column sums

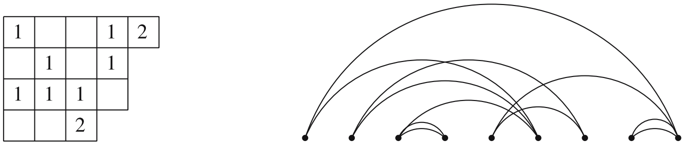


Figure 2. A filling L of a diagram with row sums $4,2,3,2$ and column sums $2,2,3,2,2$, and the corresponding graph $G(L)$.

are given by the sequences (ρ_i) and (γ_j). Given two matrices M and N, we say that they are *equirestrictive* if for all diagrams T with prescribed row and column sums, the number of fillings of T that avoid M equals the number of fillings of T that avoid N. With this notation, Theorem 3.3 is an immediate consequence of the following result, the proof of which is the content of the next section.

Theorem 3.5. *The identity matrix I_k and the antiidentity matrix J_k are equirestrictive.*

Before moving to the proof of Theorem 3.5, let us make some remarks and point out some consequences of the proof. We start by further exploring the bijection between left-right graphs and fillings of diagrams.

Let G and H be graphs on $[n]$ and $[h]$, respectively, with $h \leq n$. For the rest of this section we assume that H is simple (but G can have multiple edges as usual). We say that G *contains* H if there is an order-preserving injection $\sigma : [h] \to [n]$ such that if $\{i,j\}$ is an edge of H then $\{\sigma(i), \sigma(j)\}$ is an edge of G. For instance, a k-noncrossing graph is a graph that does not contain the graph on $[2k]$ with edges $\{1,k+1\},\{2,k+2\},\ldots,\{k,2k\}$.

A $0-1$ matrix M with s rows and t columns can also be viewed as a filling of the diagram of shape $(t,t,\overset{(s)}{\ldots},t)$. By the correspondence between graphs and fillings of diagrams described above, we have that M gives a graph $G(M)$ with t opening vertices and s closing vertices and such that all opening vertices appear before the closing vertices. Let us call such a graph a *split graph*, a particular case being the graph of a k-crossing or a k-nesting. As a consequence of the previous discussion we have that in terms of containment of substructures (matrices or split graphs), fillings of diagrams and graphs are equivalent objects.

Theorem 3.6. *For any split graph H there is a matrix $M(H)$ such that a left-right graph G contains H if and only if the filling $L(G)$ contains $M(H)$. And conversely, for each matrix M there is a split graph $H(M)$ such that a filling L of a diagram T contains M if and only if the graph $G(L)$ contains $H(M)$.*

Observe now that Lemma 3.4 can be generalized by substituting "k-noncrossing graphs" with "graphs that do not contain the split graph H". Hence the following.

Theorem 3.7. *Let H and H' be two split graphs. Then for any left-right degree sequence D there are as many graphs on D avoiding H as graphs on D avoiding H' if and only if for each diagram with prescribed row and column sums there are as many fillings avoiding $M(H)$ as fillings avoiding $M(H')$.*

Following the notation for matrices, we say that two split graphs H and H' are *equirestrictive* if for any left-right degree sequence D, there are as many graphs on D avoiding H as graphs on D avoiding H'. All the split graphs that are known to be equirestrictive are obtained from the graph of a k-crossing or a k-nesting by using Proposition 4.1 from the next section. This proposition states that if M and N are equirestrictive matrices, then for any other matrix A the matrices

$$\begin{pmatrix} M & 0 \\ 0 & A \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} N & 0 \\ 0 & A \end{pmatrix}$$

defined by blocks are also equirestrictive. This has the following implications in terms of graphs. Given a split graph H on $[h]$, a $(k{-}H)$-*crossing* is a graph on $2k{+}h$ such that the graph induced by the vertices $[k]\cup\{k{+}h{+}1,\dots,2k{+}h\}$ is a k-crossing, the graph induced by $\{k+1,\dots,k+h\}$ is H, and there are no other edges. A $(k-H)$-*nesting* is defined similarly. Then by combining Theorems 3.5 and 3.7 and Proposition 4.1 we deduce the following.

Corollary 3.8. *For any split graph H and any nonnegative integer k, $(k-H)$-crossings and $(k-H)$-nestings are equirestrictive.*

Observe that if we take H to be an h-nesting, a $(k{-}H)$-nesting is a $(k{+}h)$-nesting, so it follows that a k-nesting, a k-crossing, and any combination of a t-crossing "over" a $(k-t)$-nesting are equirestrictive. However, it is not true that t-nestings over $(k{-}t)$-crossings are equirestrictive, not even within matchings, as observed in the remark after Theorem 1 of [5]. This implies also that there is no analogous version of Proposition 4.1 where A is the top-left block and M and N are the bottom-right blocks of the matrix.

Finally, we comment on the results known for $0-1$ fillings of diagrams with row and column sums equal to 1. Our correspondence translates these results into results for matchings and partition graphs, as we next explain. In the literature, two permutation matrices M and N are called *shape-Wilf-equivalent* if for each diagram T with row and column sums set to 1, the number of fillings avoiding M equals the number of fillings avoiding N. (In view of this notation, we could have chosen the name graph-Wilf-equivalent instead of equirestrictive.) Let P be a $t{\times}t$ permutation matrix. The split graph corresponding to P is a matching (these are sometimes called *permutation matchings*). Now if two permutation matrices P and P' are shape-Wilf-equivalent, then by straightforward application of Theorem 3.7 we have that for all graphs whose left and right degrees are one, the number of graphs avoiding the matching $H(P)$ equals the number of graphs avoiding the matching $H(P')$. Since graphs with left and right degrees one are exactly partition graphs, it turns out that shape-Wilf-equivalence is equivalent to the matchings $H(P)$ and $H(P')$ being equirestrictive among partition graphs, counted by left-right degree sequences.

There are not many pairs of permutation matrices known to be shape-Wilf-equivalent. Backelin, West, and Xin [1] show that I_k and J_k are shape-Wilf-equivalent; in graph theoretic terms, this gives another alternative proof of the equality between k-noncrossing and k-nonnesting partition graphs from [4]. Let us mention here that Krattenthaler [12] deduces both the result of Chen et al. and that of Backelin, West, and Xin from his Theorem 3, but for the first one he specialises to the diagram Δ and for the second he restricts the number of non-empty cells in the filling (and takes arbitrary shapes). Since these two apparently unrelated results are in fact equivalent, it is not surprising that they follow from the same theorem, but it is interesting that they do in different ways. Another observation is that by Lemma 3.4, and its generalization to split graphs, if we know that two split graphs are equirestrictive within matchings, then they are so within partition graphs. For instance, a bijective proof of the equality of the numbers of k-noncrossing and k-nonnesting matchings would immediately give a bijection for k-noncrossing and k-nonnesting partition graphs.

In addition to the matrices I_t and J_t and the ones that follow from Proposition 4.1, the only other pair of matrices known to be shape-Wilf-equivalent are (see [16])

$$M(213) = \begin{pmatrix} 0\,0\,1 \\ 1\,0\,0 \\ 0\,1\,0 \end{pmatrix} \qquad \text{and} \qquad M(132) = \begin{pmatrix} 0\,1\,0 \\ 0\,0\,1 \\ 1\,0\,0 \end{pmatrix}.$$

The graph theoretic version of this result has been independently proved by Jelínek [5]; in this case it seems to us that the proof in terms of graphs is simpler than the one for fillings of diagrams. It is not known whether $M(231)$ and $M(132)$ are also equirestrictive, or more generally if there is a pair of shape-Wilf-equivalent permutation matrices that are not equirestrictive.

Lastly, let us mention that all the discussion of this section can be carried out with almost no changes to the case where the matrix we want to avoid in the filling can have arbitrary nonnegative entries; this corresponds to avoiding split graphs with multiple edges. The interested reader will have no problems in filling in the details.

4. Proof of Theorem 3.5

This section is devoted to the proof of Theorem 3.5. We show that we can adapt to our setting the proof of [1], which is for shape-Wilf-equivalence, that is, row and column sums equal to 1; we include the details for the sake of completeness. (Actually, [1] contains two proofs of the analogue of our Theorem 3.5 for shape-Wilf-equivalence; the proof we adapt is the first one.) This bijection has been further studied in [3]. Here we show that it extends, in a quite straightforward way, to arbitrary fillings. This gives a result stronger than Theorem 3.5, the consequences of which in graph theoretic terms have already been pointed out at the end of the previous section. Let us also mention that Theorem 3.5 can also be proved using the techniques of [12].

From now on $\bar{T}$ denotes a diagram with prescribed row and column sums. When we say that a cell is above (or below, to the right, to the left) of another cell we always mean strictly. If we say that a cell is weakly above (below, etc.) we mean not below (not above, etc.). Given two cells a and b with a below and to the left of b, the *rectangle determined by a and b* is the set of cells, if any, that are above and to the right of a and below and to the left of b.

If A and B are two matrices, by $[A|B]$ we mean the matrix having A and B as blocks, that is,

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Proposition 4.1. *Let M and N be a pair of equirestrictive matrices and let A be any matrix. Then the matrices $[M|A]$ and $[N|A]$ are also equirestrictive.*

Proof. Let L be a filling of the diagram $\bar{T}$ that avoids $[M|A]$. Let T' be the set of cells (i,j) of T such that the cells to the right and below (i,j) contain the matrix A. T' is a diagram, since if (i,j) is in T' all the cells weakly above and weakly to the left of it are also in T'. Now set the row and column sums of T' according to the restriction of L to T', call it L', giving a diagram $\bar{T}'$. Now L' is a filling of $\bar{T}'$ that avoids M, so by assumption there is a bijection between such fillings and the ones that avoid N. Change the entries of L corresponding to T' to obtain a filling of $\bar{T}$ that avoids $[N|A]$.

The bijection in the other direction goes just in the same way. ∎

Let F_t be the matrix $[J_{t-1}|I_1]$. The proof of the following proposition takes the rest of this section.

Proposition 4.2. *For all t, F_t and J_t are equirestrictive.*

We get as a corollary a stronger version of Theorem 3.5.

Corollary 4.3. *For all t, $[I_t|A]$ and $[J_t|A]$ are equirestrictive.*

Proof. By Proposition 4.1 it is enough to show that I_t and J_t are equirestrictive. The proof is by induction on t; clearly I_1 and J_1 are equirestrictive. By Proposition 4.2, it is enough to show that I_t and F_t are equirestrictive, and this follows by the induction hypothesis combined with Proposition 4.1. ∎

A sketch of the proof of Proposition 4.2 is as follows. We first define two maps between fillings that transform occurrences of J_t into occurrences of F_t, and conversely, and use them to define two algorithms that transform a filling avoiding F_t into a filling avoiding J_t, and conversely. The fact that these two algorithms are inverses of each other follows from a series of lemmas.

For any filling L, given two occurrences G_1 and G_2 of J_t in L, we say that G_1 *precedes* G_2 if the first entry in which they differ, from left to right, is either higher in G_1 or it is at the same height and the one in G_1 is to the left. So two occurrences are either equal or comparable.

The order for the occurrences of F_t goes the other way around, i.e., we look at the first entry in which they differ, from right to left, and the lower entries have preference, and if they are at the same height, the one more to the right goes first.

Let L be a filling with the first occurrence of J_t in rows $r_1 > \cdots > r_t$ and columns $c_1 < \cdots < c_t$. Let $\phi(L)$ be the result of substracting 1 from each cell (r_s, c_s), $1 \le s \le t$ and adding 1 to each cell (r_s, c_{s-1}), $2 \le s \le t$ and to cell (r_1, c_t). Since row and column sums have not been altered, $\phi(L)$ is a filling of $\bar{T}$. So we have changed an occurrence of J_t to an occurrence of F_t. Define ψ as the inverse procedure, that is, ψ takes a filling of the diagram, looks for the first occurrence of F_t, and replaces it by an occurrence of J_t.

We define the algorithms $A1$ and $A2$ in the following way. Algorithm $A1$ starts with a filling avoiding F_t and applies ϕ successively until there is no occurrence of J_t. The result (provided the algorithm finishes) is a filling that avoids J_t. Similarly, algorithm $A2$ starts with a filling avoiding J_t and applies ψ until there are no occurrences of F_t left. We claim that $A1$ and $A2$ are inverse of each other. We prove this through a series of analogous lemmas. It is enough to prove the following claims.

- Both algorithms end. (Lemmas 4.5 and 4.11.)

- If L is a filling that avoids F_t, then $\psi(\phi^n(L)) = \phi^{n-1}(L)$ for all n. (Lemma 4.9.)
- If L is a filling that avoids J_t, then $\phi(\psi^n(L)) = \psi^{n-1}(L)$ for all n. (Lemma 4.15.)

In order to prove these claims, we need to investigate some properties of the maps ϕ and ψ. We start by studying the map ϕ.

Let us first introduce some notation. Let L be a filling of the diagram and let $a_1,\dots,a_t$ be the cells of the first J_t in L, listed from left to right; say they are $(r_1,c_1),\dots,(r_t,c_t)$. So in each cell a_i there is a positive integer, possibly greater than one. Let $b_1,\dots,b_t$ be the cells $(r_2,c_1),(r_3,c_2),\dots,(r_t,c_{t-1})$ and (r_1,c_t); hence, $b_1,\dots,b_t$ are the cells corresponding to the occurrence of F_t that is created after applying ϕ to L. So cell b_i is in the same row as a_{i+1} and in the same column as a_i, for i with $1 \le i \le t-1$.

Consider now the following two paths of cells determined by $a_1,\dots,a_t$ and $b_1,\dots,b_t$ (see Figure 3). The path A starts at the leftmost cell in the row of a_1, continues to the right until it reaches the column of a_2, then takes this column up until it hits cell a_2, then turns right until reaching the column of a_3, goes up until a_3, then turns right again, and so on, until it reaches cell a_t, at which point it continues up until the top of the diagram. The path B is defined in a similar manner. It starts at the leftmost cell of the row of cell b_1, and goes right until it hits b_1. Then it turns up until the row of b_2, where it turns and continues to the right until hitting b_2. Then it goes up until the row of b_3, and then turns to the right until b_3, and so on, until reaching b_{t-1}, at which point it goes up until reaching the top of the diagram. Since $a_1,\dots,a_t$ are the first occurrence of J_t, the cells that are both to the right of B and to the left of A are empty, or, in other words, this region of the diagram avoids J_1. We denote this region by E. The choice of the first J_t also imposes some other less trivial bounds on the longest J_i's that can be found in some other areas determined by E. Note that in the next lemma the area left of E includes the path B.

Lemma 4.4. *For all i with $1 \le i \le t-1$, the fillings L and $\phi(L)$ do not contain any J_i to the left of E and*

(i) below b_i, or
(ii) above and to the right of b_{t-i}, or
(iii) in the rectangle determined by b_j and b_{j+i} for some j with $1 \le j \le t-i-1$.

Proof. The proof is by induction on i. For $i=1$ there is nothing to prove since the area left of E and satisfiying any of (i), (ii) or (iii) is empty. Now suppose that in either L or $\phi(L)$ there is such a J_i for some $i>1$. We show first that J_i does not contain any of the cells b_k for $1 \le k \le t-1$. Indeed,

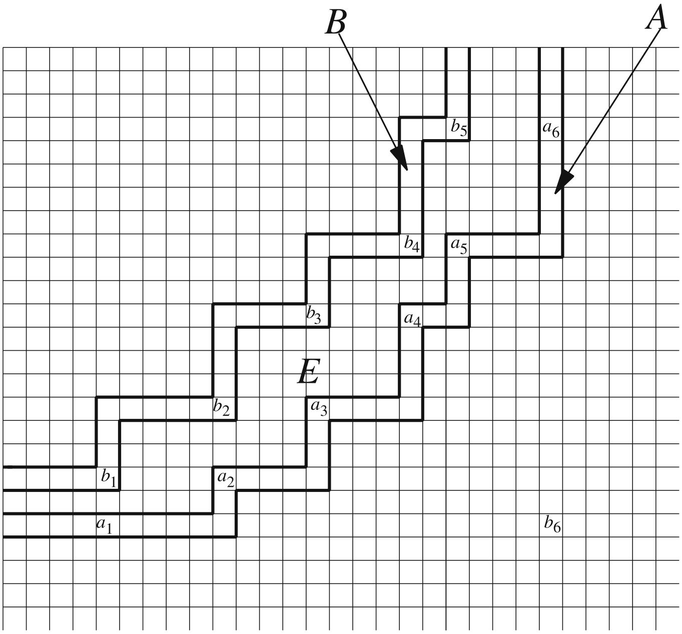


Figure 3. The regions A, B, and E

if this J_i contains such a cell b_k, then we can find a subset of the cells of J_i such that, for some $i' < i$, they are an occurrence of $J_{i'}$ contradicting the induction hypothesis. Hence, J_i consists of cells that are in L, it is left of E and satisfies at least one of (i), (ii) or (iii). If for instance it satisfies (iii), then this J_i together with $a_1, \ldots, a_j$ and $a_{i+j+1}, \ldots, a_t$ would form an occurrence of J_t contradicting the choice of a_{j+1}. In the other two cases we get similar contradictions. ∎

Lemma 4.5. *There is no J_t in $\phi(L)$ in the rows above a_1.*

Proof. We argue by contradiction. Let G be an occurrence of J_t in $\phi(L)$ in the rows above a_1. Since ϕ picked a_1 as the topmost cell being the left-bottom cell of a J_t, G must use at least one of the cells $b_1, \ldots, b_{t-1}$. The idea is to substitute these cells b_i, and possibly others, by some of the cells a_i, to find an occurrence of J_t in L in the rows above a_1, hence contradicting the choice of a_1.

Now for each cell b_i which belongs to G, find the largest integer j such that all cells of G above b_i and weakly below b_{j-1} lie left of E. In this way it is possible to find two sequences $i_1, \ldots, i_s$ and $j_1, \ldots, j_s$ with the following properties:

- $1 \le i_1 < j_1 \le i_2 < j_2 \le \cdots \le i_s < j_s \le t$;
- b_{i_k} is in G for all k;
- if b_l is in G, then $i_k \le l \le j_k - 1$ for some k;
- all cells of G above b_{i_k} and weakly below b_{j_k-1} are to the left of E, and j_k is the largest integer with this property.

Now we show that we can replace the cells of G that fall left of E and are contained in the rectangles determined by b_{i_k} and b_{j_k} by some of the a_i, giving an instance of J_t contained in L and above a_1. We need to distinguish two cases, according to whether $j_s = t$ or not. Assume first that $j_s \neq t$. For each k, consider the rectangles determined by b_{i_k} and b_{j_k}. By Lemma 4.4.(iii), there are at most $j_k - i_k - 1$ elements of G in this rectangle and to the left of E. Replace these cells, together with b_{i_k}, by a (possibly proper) subset of $a_{i_k+1}, \ldots, a_{j_k}$. After doing this for each k, we still have an occurrence of J_t starting above a_1, but now it is contained in the original filling L, contradicting the hypothesis. Now assume that $j_s = t$. For $k < s$, do the same substitutions as in the previous case; for $k = s$, we have by Lemma 4.4.(ii) that there are at most $t - i_k - 1$ cells of G left of E and above b_{i_k}. Replace these cells and b_{i_k} by a subset of $a_{i_k+1}, \ldots, a_t$. Again we obtain an occurrence of J_t in L that starts above a_1, a contradiction. ∎

This lemma alone shows that algorithm $A1$ terminates. Indeed, after one application of ϕ all the cells in the row of a_1 and to the left of a_1 are empty (because of the choice of a_1), and the cell a_1 has decreased its value by one. So the leftmost cell of the first occurrence of J_t in $\phi(L)$ is either a_1, or it is to the right of a_1, or it is below a_1. But since the value in cell a_1 decreases and cells to the left of a_1 stay empty, eventually there will be no occurrence of J_t whose leftmost cell is a_1. So the selection of J_t's goes from top to bottom and from left to right, so for some n the filling $\phi^n(L)$ is free of J_t's.

It is not the case that if we apply ϕ to an arbitrary filling L of $\bar{T}$ we have that $\psi(\phi(L)) = L$. But algorithm $A1$ starts with a filling that avoids F_t and the successive applications of ϕ create occurrences of F_t from top to bottom and from left to right. We need to show that in this situation after each application of ϕ, the first occurrence of F_t is precisely the one created by ϕ. The next lemmas are devoted to proving this.

Lemma 4.6. *If L contains no F_t with at least one cell below a_1, then $\phi(L)$ contains no such F_t.*

Proof. The proof is similar to the one of the previous lemma. Let G be an occurrence of F_t in $\phi(L)$ with at least one cell below a_1. Since L had no such occurrence, G contains at least one of the cells b_i. The bottom-right cell of G is below a_1, and it cannot be to the right of a_{t-1}, otherwise this cell together with $a_1,\dots,a_{t-1}$ would form an F_t in L. By an argument similar to the one in the previous lemma, we change all cells b_i of G, and possibly others, to some of the cells a_i, so that at the end we have an occurrence of J_{t-1} that together with the bottom-right cell of G gives an occurrence of F_t that contradicts the hypothesis.

For each b_i that is in G, look for the smallest j such that all cells in G that are left of b_i and weakly to the right of b_{j+1} are left of E. By doing this we find integers $i_1,\dots,i_s$ and $j_1,\dots,j_s$ with the following properties:

- $0 \le j_s < i_s \le j_{s-1} < i_{s-1} \le \cdots \le j_1 < i_1 \le t-1$;
- b_{i_k} is in G for all k with $1 \le k \le s$;
- j_k is the smallest integer such that all cells of G that are left of b_{i_k} and weakly to the right of b_{j_k+1} are to the left of E;
- if b_l is in G, then $j_k+1 \le l \le i_k$ for some k.

We have to distinguish whether $j_s = 0$ or not. Assume first $j_s \neq 0$. Since by Lemma 4.4.(iii) there are at most $i_k - j_k - 1$ cells of G in the rectangle determined by b_{j_k} and b_{i_k}, these cells, together with b_{i_k}, can be replaced by a (possibly proper) subset of $a_{j_k+1},\dots,a_{i_k}$. By doing this for all k, we have an occurrence of J_{t-1} in L that together with the right-bottom cell of G contradicts the hypothesis. If $j_s = 0$, then we do the same substitutions for all $k \neq s$; for $k = s$, we have by Lemma 4.4.(i) that there are at most $i_s - 1$ cells of G left of E and below b_{i_s}, so we can substitute those and b_{i_s} by $a_1,\dots,a_{i_s}$. After these substitutions, the result is again an occurrence of F_t in L that contains a cell below a_1, contradicting the hypothesis. ∎

The following is easy but we state it for the sake of completeness.

Lemma 4.7. *If L contains no F_t with a cell to the right of a_t and below a_2, then $\phi(L)$ contains no such F_t.*

Proof. Again we argue by contradiction. Suppose G is an F_t in $\phi(L)$ that contains a cell to the right of a_t and below a_2. This cell together with $a_2,\dots,a_t$ gives an occurrence of F_t in L that contradicts the assumption. ∎

Lemma 4.8. *For each k with $1 \le k \le t-1$, there is no J_k in $\phi(L)$ above a_1 and to the left of and below a_{k+1}.*

Proof. Let G be an occurrence of such a J_k. If G contains none of $b_1,\dots,b_{k-1}$, then G followed by $a_{k+1},\dots,a_t$ forms a J_t in L that is above a_1, and this contradicts the choice of a_1. Hence, G uses some b_i for $1 \le i \le k-1$. By an argument analogous to that of the proof of Lemma 4.5, we can substitute the cells b_i that are in G and possibly others by some a_i's so that we get an occurrence of J_k in L that is below a_{k+1} and above a_1. This followed by $a_{k+1},\dots,a_t$, gives an J_t in L that contradicts the choice of a_1. ∎

The following lemma is just a combination of the previous and induction; it implies that the inverse of algorithm $A1$ is $A2$.

Lemma 4.9. *(i) If L does not contain any occurrence of F_t below a_1, then the first occurrence of F_t in $\phi(L)$ is $b_1,\dots,b_t$.*
(ii) If L is a filling that avoids F_t, then $\psi(\phi^n(L))=\phi^{n-1}(L)$.

Proof. For the first statement, let $f_1,\dots,f_t$ be the first occurrence of F_t in $\phi(L)$, with the elements ordered from left to right. Recall that $b_1,\dots,b_t$ is an occurrence of F_t in $\phi(L)$; we need to show that $f_i = b_i$ for all i. By Lemma 4.6, f_t is in the same row as b_t. By Lemma 4.7, f_t cannot be to the right of a_t, hence $f_t=b_t$. Now use induction on $t-i$. Suppose we know $f_{i+1}=b_{i+1},\dots,f_t=b_t$. It is enough now to show that f_i lies in the same row as b_i, since all the cells to the right of b_i but left of b_{i+1} lie in E, which we know contains only empty cells. But now Lemma 4.8 guarantees that there is no J_i below b_i, to the left of b_{i+1}, and above b_t, as required.

For the second statement, it follows by Lemma 4.6 and induction on n that the filling $\phi^n(L)$ contains no F_t whose lowest cell is below the lowest cell of the first occurrence of J_t. Hence the previous statement applied to $\phi^n(L)$ gives immediately that $\psi(\phi^n(L))=\phi^{n-1}(L)$. ∎

So the inverse of algorithm $A1$ is $A2$. Now we only need to prove the converse. The proof follows exactly the same steps and we content ourselves by stating and proving the corresponding lemmas. Actually in this case some proofs are slightly simpler.

We keep the notation as above. Let L be now a filling of $\bar{T}$ that avoids J_t, let $b_1,\dots,b_t$ be the first occurrence of F_t in L and let $a_1,\dots,a_t$ be the occurrence of J_t in $\psi(L)$ created after applying ψ to L. Consider again the region E as defined above. By the choice of $b_1,\dots,b_t$ as the first occurrence of F_t in L and since L contains no J_t, all the cells of E are again empty.

Lemma 4.10. *For all i,j with $1\le i<j\le t$, the rectangle determined by a_i and a_j contains no J_{j-i} to the right of E in either L or $\psi(L)$; that is, there is no J_{j-i} below a_j, above a_i, to the left of a_j, and to the right of E.*

Proof. Suppose there was such a J_{j-i}. Then $b_1,\dots,b_{i-1}$, followed by this J_{j-i} and then followed by $b_j,\dots,b_t$ gives an occurrence of F_t in L that contradicts the choice of b_{j-1}. ∎

Lemma 4.11. *There is no F_t in $\psi(L)$ with at least one cell in a row below a_1.*

Proof. Suppose there is such an F_t. Its right-bottom cell is below a_1 and also weakly to the left of b_{t-1}, since otherwise $b_1,\dots,b_{t-1}$ and this cell would form an F_t contradicting the choice of b_t. Let G be this occurrence of F_t except the right-bottom cell. G must contain some of the cells $a_1,\dots,a_t$. As in the previous lemmas, the idea is to substitute the a_i in G together with other cells by some of the b_i so that we obtain an occurrence of F_t in L contradicting the choice of b_t. Find integers $i_1,\dots,i_s$ and $j_1,\dots,j_s$ with the following properties:

- $1 \le i_1 < j_1 \le i_2 < j_2 \le \cdots \le i_s < j_s \le t$;
- a_{i_k} is in G for all k with $1 \le k \le s$;
- j_k is the largest integer such that all cells of G that are to the right of a_{i_k} and weakly to the left of a_{j_k-1} are to the right of E;
- if a_l is in G, then $i_k \le l \le j_k - 1$ for some k.

Now, by Lemma 4.10, there are at most $j_k - i_k - 1$ elements of G in the rectangle determined by a_{i_k} and a_{j_k}. Together with a_{i_k}, they account for at most $j_k - i_k$ elements of G; substitute them for a subset of $b_{i_k},\dots,b_{j_k-1}$. Doing this for all k, we get an occurrence of F_t in L that contains a cell below a_1, hence contradicting the choice of $b_1,\dots,b_t$ as the first F_t in L. ∎

Lemma 4.12. *If L contains no J_t that is above a_1, then $\psi(L)$ contains no such J_t.*

Proof. Let G be such a J_t in $\psi(L)$; G must contain some of the cells a_i. Find integers $i_1,\dots,i_s$ and $j_1,\dots,j_s$ with the following properties:

- $0 \le j_s < i_s \le j_{s-1} < i_{s-1} \le \cdots \le j_1 < i_1 \le t$;
- a_{i_k} is in G for all k with $1 \le k \le s$;
- j_k is the smallest integer such that all cells of G that are below a_{i_k} and weakly above a_{j_k+1} are to the right of E;
- if a_l is in G, then $j_k + 1 \le l \le i_k$ for some k.

As in the proof of the previous lemma, it is possible to substitute the elements of G contained in the rectangles determined by a_{j_k} and a_{i_k}, plus the cell a_{i_k}, by (a subset of) the elements $b_{j_k},\dots,b_{i_k-1}$. The only case that has to be treated separately is $j_s = 0$, but since G is above a_1, all the cells of G below a_{i_s} are in fact in the rectangle determined by a_1 and a_{i_s}, so we can change these cells for some of $b_1,\dots,b_{i_s-1}$. All these substitutions give a J_t in L that is above a_1, contrary to the hypothesis. ∎

Lemma 4.13. *If L contains no J_t with a cell to the left of a_1 and below a_2, then neither does $\psi(L)$.*

Proof. If this were the case, the leftmost cell of this J_t in $\psi(L)$ together with $b_1,\dots,b_{t-1}$ would give a J_t in L contradicting the hypotheses. ∎

Lemma 4.14. *If L contains no J_t above a_1, there is no J_{t-r} in $\psi(L)$ above a_{r+1} and to the right of a_r.*

Proof. Notice that Lemma 4.12 above is the case $r=0$. Suppose G is an occurrence of such a J_{t-r} in $\psi(L)$. G must contain some of the cells $a_{r+2},\dots,a_t$, otherwise $b_1,\dots,b_r$ followed by G would form a J_t in L contradicting the hypothesis. Find integers $i_1,\dots,i_s$ and $j_1,\dots,j_s$ with the following properties:

- $r \le j_s < i_s \le j_{s-1} < i_{s-1} \le \cdots \le j_1 < i_1 \le t$;
- a_{i_k} is in G for all k with $1 \le k \le s$;
- j_k is the smallest integer such that all cells of G that are below a_{i_k} and weakly above a_{j_k+1} are to the right of E;
- if a_l is in G, then $j_k+1 \le l \le i_k$ for some k.

The rectangle determined by a_{j_k} and a_{i_k} contains at most $i_k - j_k - 1$ cells of G; these cells, together with a_{i_k}, can be replaced by a subset of $b_{j_k},\dots,b_{i_k-1}$. As in the proof of Lemma 4.12, the case $j_s=r$ has to be treated separately, but in an analogous way. After all these substitutions we get an occurrence of J_{t-r} in L that combined with $b_1,\dots,b_r$ gives an occurrence of J_t in L contradicting the hypothesis. ∎

Lemma 4.15. *(i) If L does not contain any occurrence of J_t above b_t, then the first occurrence of J_t in $\phi(L)$ is $a_1,\dots,a_t$.*
(ii) If L is a filling that avoids J_t, then $\phi(\psi^n(L))=\psi^{n-1}(L)$.

Proof. For the first statement, let $d_1,\dots,d_t$ be the first occurrence of J_t in $\psi(L)$, with cells listed from left to right. We want to show that $a_i = d_i$ for all i with $1 \le i \le t$. By Lemma 4.12, d_1 is in the same row as a_1, and by Lemma 4.13 it is weakly to the right of a_1, hence $d_1 = a_1$. Now we proceed by induction on i. Suppose $d_1=a_1,\dots,d_i=a_i$. By Lemma 4.14 we have that the first occurrence of J_{t-i} in $\psi(L)$ above and to the right of a_i is $a_{i+1},\dots,a_t$, hence $d_{i+1}=a_{i+1}$, as needed.

For the second statement, by induction and Lemma 4.12 we get that $\psi^n(L)$ satisfies the hypothesis of part (i), hence it follows that $\phi(\psi^n(L)) = \psi^{n-1}(L)$. ∎

5. Concluding remarks

In his paper [12], Krattenthaler speaks of a "bigger picture" that would englobe several recent results on pattern avoiding fillings of Ferrers diagrams and stack and moon polyominoes. We believe that our correspondence between graphs and fillings of diagrams also belongs to this picture and that it may shed some light in the understanding of it. We have shown that for each statement in pattern avoiding fillings there is a statement about graphs avoiding certain split graphs. So we can claim that in some sense the resources available to attack either problem have doubled. An example of this are the "repeated" results in the literature mentioned at the end of Section 3.

For completeness, we mention here a result by Bousquet-Mélou and Steingrímsson [3] that can be cast in terms of k-noncrossing and k-nonnesting graphs. They restrict to diagrams with self-conjugate shape and row and column sums are set to 1, and they only consider symmetric $0-1$ fillings (that is, symmetric with respect to the main diagonal of the diagram). For these fillings, they show that I_t and J_t are equirestrictive by analysing further properties of the bijection of [1] that we have generalized in Section 4. In terms of matchings, this says that for each left-right degree sequence, the number of k-noncrossing symmetric matchings is the same as the number of k-nonnesting ones, where a matching on $[2n]$ is symmetric if it equals its reflection through the vertical axis that goes between vertices n and $n+1$. Similar results for symmetric graphs can be deduced from [12, Theorem 15].

Let us finish by going back to our initial motivation of studying k-noncrossing and k-nonnesting graphs. Even if our main question has been answered positively, it is fair to say that it has not been solved in the most satisfactory way; ideally we would like to find a bijective proof in graph theoretic terms. Note that due to its roundabout character, our proof of Theorem 3.5 does not give a clear bijection, neither in terms of graphs nor of fillings. Also the proofs of Corollaries 2.2, 2.5, and 3.2 do not provide bijections in graph theoretic terms. A bijective proof of Theorem 3.3 for $k=2$ has recently been found by Jelínek, Klazar, and de Mier [6].

A plausible generalization of Theorem 3.3 one could hope for would be that the number of graphs with r k-crossings and s k-nestings equals the number of graphs with s k-crossings and r k-nestings. The case $k = 2$ is known for matchings [11] and partition graphs [9]. Unfortunately, for $k=3$ this is not true even for matchings; for instance, Marc Noy [14] checked that there are more matchings with six edges and only one 3-crossing than with only one 3-nesting.

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