

A mathematic-geometric model to calculate variation in mandibular arch form

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SUMMARY A mathematic-geometric model was used to evaluate the variation of mandibular dental arch length with respect to the incisor inclination, but without modifying the inter-canine width. In analytical terms, the equations of the curves representing the lower dental arch, before and after incisor inclination of 1 mm and of 1 degree, with controlled and uncontrolled tipping, were studied.

The length of the mandibular dental arch changed in the parabolic arch form by 1.51 mm for each millimetre of incisor inclination with respect to the occlusal functional plane, by 0.54 mm for each degree of controlled tipping and by 0.43 mm for each degree of uncontrolled tipping. In the elliptical arch form ($e = 0.78$), it changed by 1.21, 0.43, and 0.34 mm, respectively, in the hyperbolic form by 1.61, 0.57, and 0.46 mm, in the circular form by 1.21, 0.43, and 0.34 mm, and in the catenary form by 2.07, 0.74, and 0.59 mm.

The results show that by changing the arch form without modifying the dimension of the dental arch, different arch lengths can be gained for each millimetre of proclination. In addition, by controlled tipping an inter-incisive arch one-fifth longer than by uncontrolled tipping can be obtained. It would be advisable in orthodontic treatment planning to evaluate the type of dental arch, since the space available or the space required changes depending on the arch form and on the orthodontic tooth movement.

Introduction

Arch form, dimensions, and variations obtained by orthodontic treatment has been studied for many years, and some authors have tried to identify the geometric curve, which would enable them to accurately define arch form. Bonwill and Hawley (Tweed, 1966) described the alignment of the upper anterior teeth as a circumference arch, whilst MacConaill and Scher (1949) maintained that the dental arch looked like a catenary curve. Izard (1927), in trying to relate the dimension of the dental arch to the facial dimensions, found that the arch form could be accurately represented by an elliptical curve.

Currier (1969) studied the dental casts of 25 patients, comparing the arch form with parabolic and the elliptic curves. By computerized analysis, he discovered that the curve of the incisal edge of the incisors and canines, together with the buccal cusps of premolars and molars, could

be expressed as an ellipse in both arches. Brader (1972), on the other hand, maintained that the teeth were arranged in formation as in the constricted end of a trifocal ellipse. De La Cruz *et al.* (1995) identified the lower arch form as an ellipse in 96 per cent of subjects, having an eccentricity value between 0.49 and 0.99, and as a hyperbola in 4 per cent of cases.

The ellipse is a variable curve and by changing the eccentricity value it appears either as a circumference (wide and round), or as a parabola, or a hyperbola (long and narrow). For ellipse, the eccentricity value is between 0 and 1, a circle is 0, a parabola 1, and a hyperbola higher than 1.

BeGole and Lyew (1998) developed a method, using cubic spline function, to analyse change in dental arch form pre- and post-treatment and post-retention. The spline curves fitted, within acceptable limits, the dental arch form.

During orthodontic treatment, the inter-canine width can be increased, but many authors

(Sillman, 1964; Knott, 1972; Grander, 1976; Little *et al.*, 1981; Uhde *et al.*, 1983; Cozzani and Giannelly, 1985; Felton *et al.*, 1987; De La Cruz *et al.*, 1995) found that any change to the lower canine width was unstable. Therefore, the original width has to be maintained to increase long-term stability.

Walter (1962) and Herberger (1981), in contrast, found that the width was maintained when inter-canine width was increased by orthodontic treatment.

However, the length of the dental arch can be modified by inclining the incisors without varying the inter-canine width. Tweed (1954) stated that the gained or lost space in the inter-incisive arch amounts to approximately 0.8 mm for every degree of buccal or lingual incisor inclination.

Steiner (1960) put forward a new theory that the arch length increases by 2 mm for every millimetre of proclination. Germane *et al.* (1991) evaluated the expansion of the average lower dental arch length using a mathematical model of spline function. They found that the increase in arch length by incisor proclination was slightly greater than by molar and premolar expansion together.

The aim of this investigation was to establish a reliable way of determining the inter-incisive arch length by means of mathematic-geometric analysis, having the incisor inclination evaluated in millimetres and degrees, and without changing the inter-canine distance.

Materials and methods

The arch form can be calculated in geometric curves such as conics and catenary. The ellipse, the hyperbola, the parabola, and the circle are called conics, since they are intersections of the surface of an indefinite round cone with a plane, which does not pass through its vertex (Zwirner, 1988). The catenary is a plane curve, i.e. the curve formed when a length of flexible calibrated thread, which has its extremities fixed at a determined height, is allowed to hang freely (Paolucci, 1971).

In a Cartesian axis system each curve is represented by a specific equation, which describes

the locus of the points of the curve itself. By placing together the generic equation of each curve with the known dimensions of the dental arch into a system, the coefficients of the unknown quantities and the known term, and then the equation that represents the lower dental arch can be calculated (Mutinelli *et al.*, 1999). The known data were drawn from two studies, concerning the lower arch in young men (Knott, 1972; Harris, 1997):

Inter-incisive width (2–2), maximum rectilinear distance between the lower lateral incisors (20.6 mm; Knott, 1972);

Inter-molar width (6–6), maximum rectilinear distance at the buccal surfaces of the molars (52.4 ± 2.74 mm; Harris, 1997);

Arch-cord (6–1), linear distance from the inter-incisal midline at the labial inter-papillary process to the distobuccal aspect of the first molar (41.1 ± 1.79 mm; Harris, 1997).

The calculation sequence was more difficult for the hyperbola, because there was insufficient data to identify the curve inside the Cartesian axis system. The coefficients of the equation were calculated approximately using the definition of the hyperbola, eccentricity value of the hyperbola (De La Cruz *et al.*, 1995), and the depth of inter-incisive width of the other curves.

For the circumference, knowing the inter-canine width (Knott, 1972), the trigonometrical theorems were applied to the Bonwill–Hawley geometric model (Tweed, 1966), in order to calculate the radius length and the equation of the corresponding circle.

It was assumed in this study that by proclining the incisors 1 mm from the initial position of 90 degrees with respect to the functional occlusal plane (Lucchese, 1988) (Figure 1a,b), a new curve similar to the first with each point distanced 1 mm from the corresponding point of the first curve would be obtained (Figures 2–6). The co-ordinates of three new points, N, N', and D (Figures 2–6), positioned 1 mm from the perpendicular of the tangent of the first curve at the points to the lateral incisors (I and I'), and

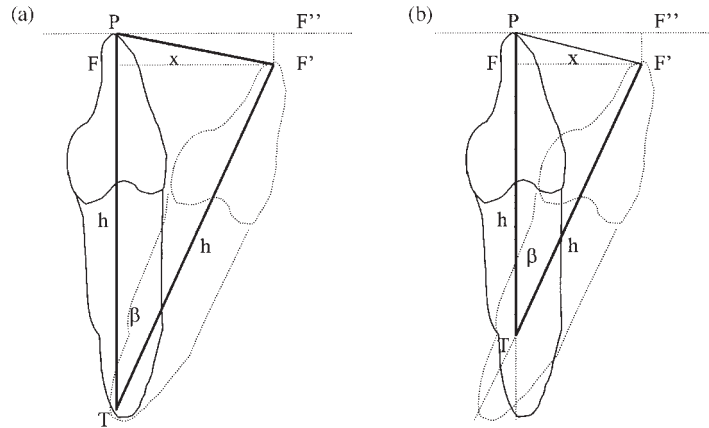


Figure 1 Type of orthodontic movement. (a) Controlled and (b) uncontrolled tipping of lower incisor (Mutinelli *et al.*, 1999).

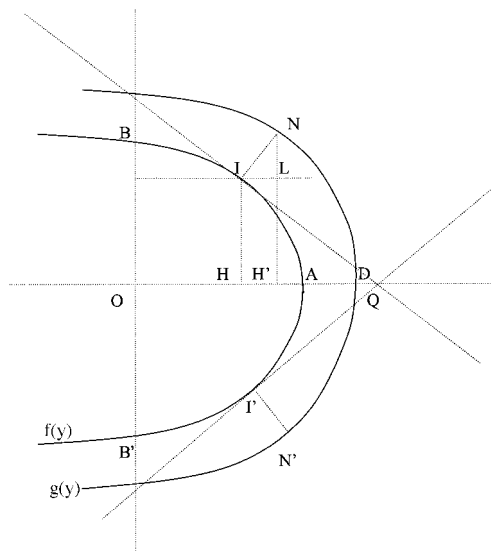


Figure 2 Parabolic arch form before $f(y)$ and after $g(y)$, 1 mm of proclination.

corresponding to the vertex or inter-incisive point (A) were calculated. Using the co-ordinates of N, N', and D, the equations of the new wider curves were then calculated. (Details of all the equations are shown in the Appendix.)

Having calculated the equations of the curves and the co-ordinates of the right lateral incisor

and the left lateral incisor (the extremities of the inter-incisive width) before $f(y)$ and after $g(y)$ 1 mm of proclination, the two inter-incisive arch lengths (II' ; NN') were calculated, solving the definite integral of the square root of the first derivative of square function equation plus 1 between the two extremities (I and I', N and N'):

$$II' = \int_I^{I'} \sqrt{1 + f'(y)^2} \quad (\text{eqn 1.0})$$

$$NN' = \int_N^{N'} \sqrt{1 + g'(y)^2} \quad (\text{eqn 1.1})$$

The increase in the inter-incisive arch was the difference between NN' and II' .

The inclination of the lower incisor, with respect to the occlusal functional plane, in accordance with the orthodontic tooth movement (Figure 1a,b) and lower dental arch expansion was studied (Mutinelli *et al.*, 1999).

The different arch forms were evaluated in a Cartesian axis system, positioned on the Ricketts occlusal functional plane (Lucchese, 1988), which was independent from the incisor position.

During orthodontic treatment, the movement of teeth can occur either by controlled or uncontrolled tipping. In controlled tipping (Marcotte, 1993) the rotation centre is at the root apex, and

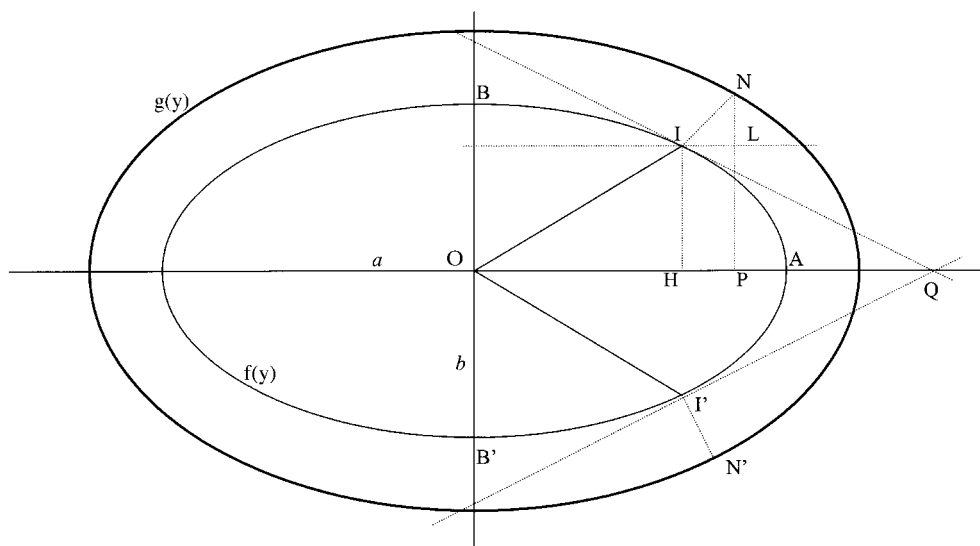


Figure 3 Elliptic arch form before $f(y)$ and after $g(y)$, 1 mm of proclination (Mutinelli *et al.*, 1999).

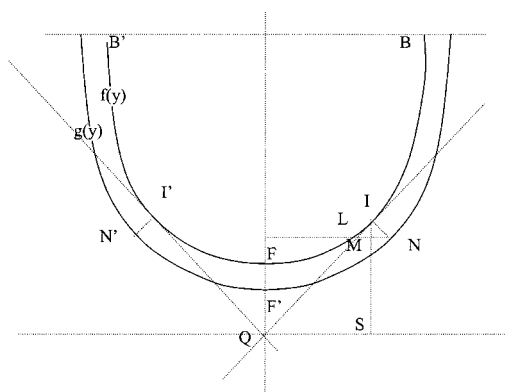


Figure 4 Catenary arch form before $f(y)$ and after $g(y)$, 1 mm of proclination.

the crown and the root move in the same buccal or lingual direction (Figure 1a). In uncontrolled tipping (Cozzani and Giannelly, 1985), the inclination is produced by exerting a single force with the rotation centre between the fourth and fifth apical part of the tooth, i.e. the crown follows the force and the root moves in the opposite direction (Figure 1b).

Applying the trigonometrical theorems, a 1-mm inclination was converted to 1 degree for each type of tipping. The result is the space gained by 1 degree proclination.

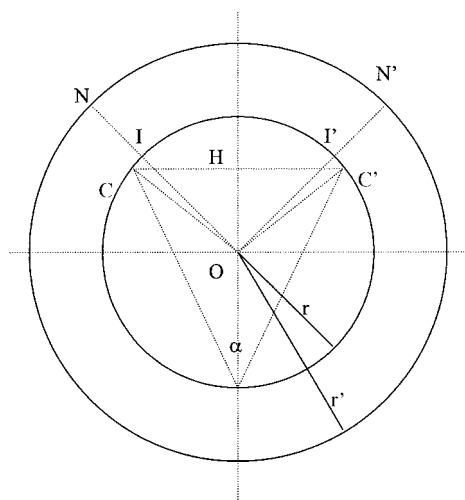


Figure 5 Inter-incisive arch on a circle before (II') and after (NN'), 1 mm of proclination.

Results

The equations of the parabola were $f(y) = -0.046y^2 + 31.67$ before the proclination and $g(y) = -0.43y^2 + 32.67$ after 1 mm of proclination

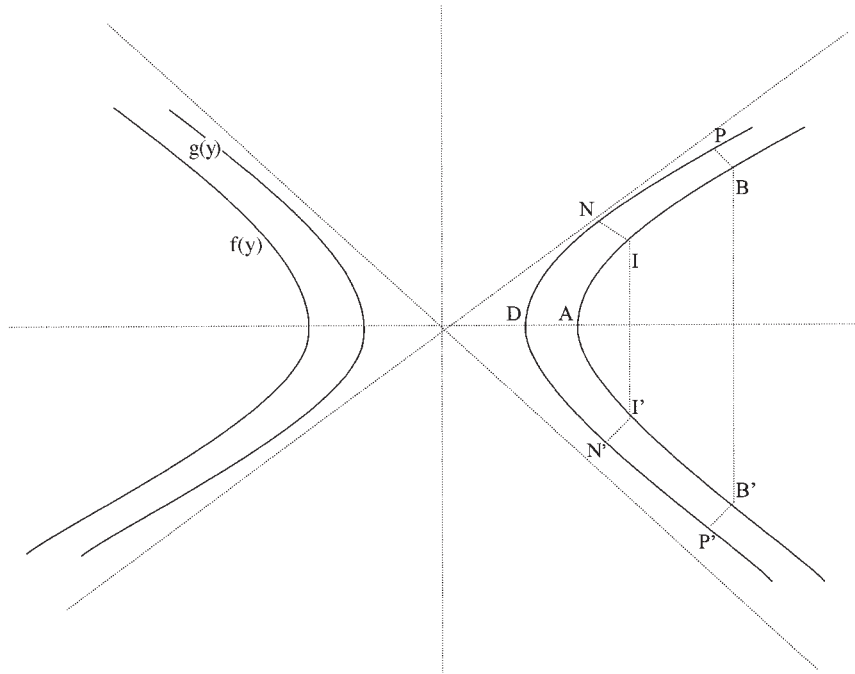


Figure 6 Hyperbolic arch form before $f(y)$ and after $g(y)$, 1 mm of proclination.

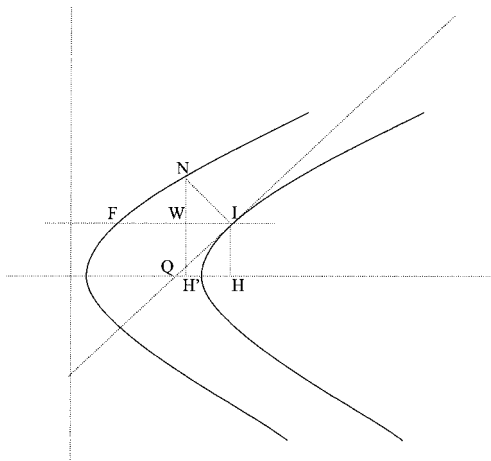


Figure 7 Geometric identification of point N on the perpendicular of the tangent of the hyperbolic curve in I.

(Figure 2). Solving the definite integrals between the extremities represented by the lateral incisors I and I' for $f(y)$:

$$II' = \int_I^{I'} \sqrt{1 + f'(y)^2} \quad \text{with } -10.43 < y < +10.43$$

and N and N' for $g(y)$:

$$NN' = \int_N^{N'} \sqrt{1 + g'(y)^2} \quad \text{with } -10.99 < y < +10.99.$$

The inter-incisive arch lengths amounted to 23.38 and 24.89 mm, respectively. The increase was 1.51 mm.

The ellipse (Figure 3) was described by the functions $(x^2/42.27^2) + (y^2/26.2^2) = 1$ before the proclination and $(x^2/43.27^2) + (y^2/27.29^2) = 1$ after 1 mm of proclination and had eccentricity values of 0.78 and 0.79. The inter-incisive arch lengths were 22.03 mm for the first equation (II') and 23.24 mm for the second (NN'), with a difference of 1.21 mm.

The catenary (Figure 4) was represented by the equation $y = 14.25 \coshyp (x/14.25)$ before proclination and $y = 13.25 \coshyp (x/13.25)$ after 1 mm of proclination. The arch II' was 22.44 mm and the arch NN' 24.51 mm. Their difference amounted to 2.07 mm.

Table 1 Summary of space variation by linear and angular proclination of lower incisors.

Curve	Frequency (%) ¹	Δl^2 each 1 mm (mm)	Δl^2 each 1° with controlled tipping (mm)	Δl^2 each 1° with non-controlled tipping (mm)
Parabola	4 ³	1.51	0.54	0.43
Ellipse	96	1.21 ⁴	0.43 ⁴	0.34 ⁴
Hyperbola	4 ³	1.61	0.57	0.46
Circumference		1.21	0.43	0.34
Catenary		2.07	0.74	0.59
Average		1.52	0.54	0.43

¹De La Cruz *et al.* (1995).

² Δl = length variation of inter-incisive arch after proclination.

³4 per cent = sum of the frequency of parabola and hyperbola.

⁴Ellipse with eccentricity value of 0.78.

The two functions of the circle (Figure 5) were $x^2 + y^2 = 18.13^2$ for $f(y)$ and $x^2 + y^2 = 19.13^2$ for $g(y)$. The inter-incisive arch length II' was 21.91 mm before 1 mm of proclination and the NN' was 23.12 mm after proclination, a difference of 1.21 mm.

To calculate the equations of the hyperbola (Figure 6), the known data were approximated. The results were $(x^2/83.13^2) - (y^2/27.51^2) = 1$ for $f(y)$ and $(x^2/82.13^2) - (y^2/28.46^2) = 1$ for $g(y)$. The two inter-incisive arch lengths amounted to 24.15 mm (II') and 25.76 mm (NN'). The increase after 1 mm of proclination was 1.61 mm.

The linear inclination was then converted into angular inclination, by applying the trigonometric theorems distinguishing between the different tipping (Figure 1a,b). In the initial presumption, the incisor was inclined 90 degrees with respect to the Ricketts occlusal functional plane. The result of this trigonometric equation

$$FF' = F'T \sin \beta \Rightarrow y = 22.5 \sin x$$

was the number of degrees corresponding to 1 mm. Within the range of 90–135 degrees of lower incisor inclination ($0 < x < 45$ degrees), 1 mm of proclination averaged 2.63 degrees in controlled tipping and 3.33 degrees in uncontrolled tipping. The increases in inter-incisive arch length after 1 mm of proclination were 0.54 mm in parabola, 0.43 in ellipse, 0.57 in hyperbola, 0.43 in circle, and 0.74 in catenary with controlled tipping, and 0.43 mm in parabola, 0.34 in ellipse, 0.46 in hyperbola, 0.34 in circle,

and 0.59 in catenary with uncontrolled tipping (Table 1).

Discussion

The numerous studies on the arch form have demonstrated the difficulty in finding a universal geometric curve that can represent the dental arch. It was therefore decided to calculate the inter-incisive arch length variation for each type of geometric arch form, described in previous studies: ellipse, parabola, hyperbola (De La Cruz *et al.*, 1995), circumference (Tweed, 1966), and catenary (MacConaill and Scher, 1949).

Steiner (1960) and Tweed (1954) probably used only one type of arch form to connect the inter-incisive arch length with the incisor proclination.

Tweed (1954) maintained that by proclining the incisor by 1 degree, the arch length would increase by 0.8 mm. In his technique, he applied the Bonwill–Hawley model to prepare the specific wire form required for the patient during orthodontic treatment. However, in that model the inter-canine arch is a circumference. Applying the mathematical calculation sequence from this investigation to the circumference form, an increase of 0.34 mm is obtained (the same increase as obtained in the ellipse, the most common arch form; De La Cruz *et al.*, 1995). This increase is approximately half the value of Tweed's result.

Steiner (1960) calculated the incisor inclination in millimetres and found an increase of 2 mm for

each millimetre of proclination. A similar value (2.07 mm) was obtained in the catenary curve.

Comparing the values obtained in this investigation with the results of Steiner and Tweed, it is not easy to understand why there are these differences. For the arch forms studied by Steiner and Tweed, the initial assumptions, the type of orthodontic movement of the incisor (controlled or uncontrolled tipping) or the calculation method are not exactly identified. In addition, they did not give either references, or a mathematical, or a clinical explanation.

In the analysis of Germane *et al.* (1991) the increase in mandibular arch length (1.04 mm) was lower than in this study (1.21 mm). The difference can be explained by the average dental arch dimensions used and by the spline function model.

At the start of this investigation some assumptions were defined. These are important factors influencing the numerical results.

In the analysis the inter-incisive area was expanded without modifying the canine position, due to the fact that the majority of researchers have demonstrated that any increase in inter-canine width is unstable (Sillman, 1964; Knott, 1972; Grander, 1976; Little *et al.*, 1981; Uhde *et al.*, 1983; Cozzani and Giannelly, 1985; Felton *et al.*, 1987; De La Cruz *et al.*, 1995) and the dental arch tends to return to the original dimension and form. However, Walter (1962) and Herberger (1981), found that the expansion in inter-canine area is maintained. Moreover, it is known that the final orthodontic result will not vary with permanent retention and the arch can be expanded without relapsing.

Average values for the dimensions of the teeth and of the dental arch were used. For example, in the ellipse form changing the long axis (the distance between the line connecting the second molars and the inter-incisive point) and/or the short axis (the distance between the second molars), the form of the curve looks either like a circumference (wide and round), or a parabola or hyperbola (long and narrow). The relationship between the axis is the eccentricity value. In the ellipse, it amounted to 0.78. Therefore, if the eccentricity value is changed, the inter-incisive arch length could vary.

Moreover, as the equation of the hyperbola was obtained by approximation of the known data, the information was insufficient to accurately identify the curve inside the Cartesian axis system.

The arch form was studied in the Ricketts occlusal functional plane. Any curve can be distorted by the inclination of the plane on which it is drawn. The occlusal functional plane is a plane independent from the incisor position, so if the incisor is inclined, it will not change. The curve can be expanded but it maintains the same form. It was presumed that the incisor was positioned 90 degrees with respect to the reference plane. Converting the inclination from millimetres into degrees, a direct relationship was not found between the linear and the angular values: i.e. if the first was doubled, the second would not double and the nearer the inclination was to 90 degrees, the more space was gained with the same inclination angle. Therefore, the average length corresponding to 1 degree was calculated in the range 0–45 degrees (90–135 degrees with respect to the occlusal functional plane).

This model, even though being an 'abstract' analysis with many approximations, is a good description of the average variations of the dental arch form and can, therefore, be utilized for clinical evaluations. Moreover, it can be applied to other arch forms, such as spline function, which has recently been identified as a curve fitting routinely within acceptable limits of the dental arch (BeGole and Lyew, 1998).

Further research will involve the use of this model in clinical practice. At present a system is being studied, which enables identification of the specific arch form of each patient and for accurate measurement of the widths of the dental arch and the inclination of the lower incisor. Therefore, during orthodontic planning, this should enable calculation with greater precision the space gained or lost by the maximum allowed inclination of the lower incisors.

Conclusions

The results of this study have shown that, by changing the arch form without increasing the inter-canine width and maintaining the same

transversal dimensions in each form for each millimetre of proclination, a minimum value of 1.21 mm in elliptic arch form ($e = 0.78$) and a maximum value of 2.07 mm in catenary can be gained. That is to say that the increase in arch length for the different conics and for the catenary vary within a range of 0.86 mm. In addition, controlled tipping of the incisors allow achievement of an inter-incisive arch one-fifth longer than with uncontrolled tipping.

Inter-incisive arch length can vary depending on the type of arch and, as result, the space required or space available will also vary. Therefore, when planning treatment, it would be advisable to evaluate the type of arch form.

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Acknowledgements

We would like to express our thanks to the journal *Ortognatodonzia Italiana* for allowing the publication of part of the text and of the illustrations (Mutinelli *et al.*, 1999).

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Appendix

The parabola is the geometric locus of the points of the plane, which are equidistant from a fixed point, known as the focus, and a fixed straight line, known as the directrix (Zwirner, 1988). The points of the curve are plotted on the Cartesian system. The model has the following characteristics: it is symmetrical with respect to the axis of abscissa, its vertex is on the abscissa, it has negative concavity and its inter-molar width is on the ordinates. The general equation is

$$x = ay^2 + by + c. \quad (\text{eqn 2.0})$$

The points of the curve are: A, the parabola vertex on the abscissa, obtained by Pythagoras' theorem from the triangle ABO, with co-ordinates 31.67 and 0 mm (Figure 2); B, the upper extremity of inter-molar width (symmetrical to B', the lower extremity) positioned on the axis of the ordinates, with co-ordinates 0 and 26.2 mm (Figure 2).

By placing together the known data and the formulae of the vertex co-ordinates, into a system

$$\begin{cases} x = ay^2 + by + c \\ x_A = (4ac + b^2)/4a = 31.67 \\ y_A = -(b/2a) = 0 \\ B \equiv (0; 26.2) \end{cases} \quad (\text{eqn 2.1})$$

the specific equation of the dental arch is calculated:

$$f(y) = -0.046y^2 + 31.67 \quad (\text{eqn 2.2})$$

The co-ordinates of the lateral incisors I and I' (Figure 2) are then calculated on the curve. The

inter-incisive width is 20.6 mm with respect to the axis of the abscissa and, consequently, the ordinates are 10.3 mm for I and –10.3 for I'. The co-ordinates of the abscissa are obtained from the equations of the dental arch, substituting y by 10.3 (+20.7 for I and I').

All teeth are then proclined 1 mm from the initial position (Figure 2), which results in a new vertex D, with the co-ordinates 32.67 and 0 mm, and point N, lying 1 mm away from I, on the perpendicular of the tangent to the curve in I (Figure 2). The co-ordinates of N are obtained from the equation of the tangent line $y = mx + q$; m is the first derivative of function, calculated in I:

$$m = f'(y) = 2 \cdot (-0.046 \cdot 10.3). \quad (\text{eqn 2.3})$$

The known term of the equation (q) of the straight line is obtained by substituting x and y with the co-ordinates of I. The intersection of this line with the abscissas is Q (37.61; 0). The angle $O\hat{Q}I$, calculated by using trigonometrical theorems, is 43.53 degrees [$\text{tg}^{-1}(IH/HQ)$]. Angle $Q\hat{I}L$ is equal to $O\hat{Q}L$, because they are alternate internal angles of two parallel lines intersected by a secant line (Cateni and Fortini, 1982). Angle $N\hat{I}L$ is the difference between $Q\hat{I}L$ and a right angle (46.47 degrees), thus the co-ordinates of N are:

$$N_x = I_x + \cos N\hat{I}L = 27.50 \quad (\text{eqn 2.4})$$

$$N_y = I_y + \sin N\hat{I}L = 10.99. \quad (\text{eqn 2.5})$$

The new equation with the points D and N is:

$$g(y) = -0.43y^2 + 32.67. \quad (\text{eqn 2.6})$$

The length of the arches II' and NN' can then be calculated. It is the result of the definite integral of the square root of the first derivative of the square function equation plus 1, between the two extremities (I and I', N and N'; Zwirner, 1988).

For the first arch, the formula is:

$$II' = \int_I^{I'} \sqrt{1 + f'(y)^2} \quad (\text{eqn 2.7})$$

with $-10.43 < y < 10.43$.

The first derivative of the function $f(y)$ is $2'(-0.046)y$. So:

$$\begin{aligned} II' &= \int_{-10.30}^{+10.30} \sqrt{1 + 0.008y^2} = [(y/2)\sqrt{1 + 0.008y^2} - \\ &(1/0.184) \ln | -0.092y + \sqrt{1 + 0.008y^2} | + c]_{-10.30}^{+10.30} \\ &= 23.38. \end{aligned} \quad \text{eqn (2.8)}$$

Using the same formula for the second function in the range $-10.99 < y < +10.99$, it gives a value for arch length NN' of 24.89 mm. Therefore, after the proclination of the incisors by 1 mm, the arch length increases by 1.51 mm.

The ellipse is the geometric locus of the points of the plane, where the distances of the points from two fixed points, known as foci, are constant (Zwirner, 1988).

In this Cartesian system (Mutinelli *et al.*, 1999) the centre of the ellipse is positioned in the centre of the axis (Figure 3). The vertices are the extremities of the inter-molar width (B and B') and the inter-incisive point (A), which is the extremity of the altitude (OA) of the triangle BB'A. Its length is the sum of the mesio-distal diameter of the second lower molar (10.06 mm; Maggiore and Ripari, 1985) plus the distance between the inter-molar line and the inter-incisive point (31.67 mm).

The generic equation of the ellipse with its centre at the centre of the axis is

$$(x^2/a^2) + (y^2/b^2) = 1. \quad \text{(eqn 3.0)}$$

Using this definition, a is (OA) (42.27) and b is (OB) (BB'/2 = 26.2). If a and b are substituted by these values, specific equation of the elliptic dental arch is:

$$(x^2/42.27^2) + (y^2/27.29^2) = 1. \quad \text{(eqn 3.1)}$$

The co-ordinates of point I are 38.87 and 10.30 (inter-incisive width = 20.60 mm). After proclination of I by 1 mm, the co-ordinates of N can be calculated, using the same method as for the parabola (Figure 3). The co-ordinates of N are 39.69 and 10.87.

Therefore, the equation of the new ellipse with points N, N', and D (vertex) is:

$$(x^2/43.27^2) + (y^2/27.29^2) = 1. \quad \text{(eqn 3.2)}$$

The eccentricities ($e = c/a$ with $c^2 = a^2 - b^2$) of the two curves are 0.78 for the first and 0.79 for the second (Zwirner, 1988).

The length of the arches II' and NN' can then be calculated, using the definitive integral:

$$II' = \int_I^{I'} \sqrt{1 + f'(y)^2} \quad \text{(eqn 1.0)}$$

$$NN' = \int_N^{N'} \sqrt{1 + g'(y)^2}. \quad \text{(eqn 1.1)}$$

A computerized program (Mathcad 7 Professional, MathSoft Inc., Cambridge, UK) is used to obtain this solution. The arch II' is 22.03 mm and arch NN' is 23.24, with a difference of 1.21 mm.

The catenary is a plane curve, that is, the curve formed when a length of flexible calibrated thread, which has its extremities fixed at a determined height, is allowed to hang freely (Paolucci, 1971).

The catenary, in the Cartesian axes (Figure 4), lies symmetrical to the y axis at the distance a from the origin and is represented by the equation

$$y = a \cosh y (x/a). \quad \text{(eqn 4.0)}$$

Vertex F (O; a) and point B ($a + 31.67$; 26.2), the extremity of inter-molar width, are plotted on the curve. By placing together all the co-ordinates and the generic equation of the catenary into a system, a numerical value for a of 14.25 can be obtained, and consequently the specific equation $f(y)$ of the curve:

$$y = 14.25 \cosh y (x/14.25). \quad \text{(eqn 4.1)}$$

From this equation the co-ordinates of I (10.30; 18.14) are available. After proclination by 1 mm, the distance from axis origin is

$$a' = a - 1 = 13.25 \quad \text{(eqn 4.2)}$$

and the new equation $g(y)$ is

$$y = 13.25 \cosh y (x/13.25). \quad (\text{eqn 4.3})$$

The co-ordinates of N are calculated by the same method applied to the parabola, using the tangent line in I and its perpendicular. The results are:

$$N_x = I_x + \cos N\hat{I}L = 10.92 \quad (\text{eqn 4.4})$$

$$N_y = I_y + \sin N\hat{I}L = 17.35. \quad (\text{eqn 4.5})$$

Solving the definitive integral between the extremities I and I' and N and N':

$$II' = \int_I^{I'} \sqrt{1 + f'(y)^2} \quad (\text{eqn 1.0})$$

$$NN' = \int_N^{N'} \sqrt{1 + g'(y)^2}. \quad (\text{eqn 1.1})$$

the length of the arches II' (22.44 mm) and NN' (24.51 mm) is known and their difference after the 1 mm proclination (2.07 mm).

The circumference or circle is the locus of the points of the plane, which are equidistant from a point named the centre (Zwirner, 1988).

In a Cartesian system, with the centre of the circumference in the origin of the axis and a radius r , the generic equation of the circle is

$$x^2 + y^2 = r^2. \quad (\text{eqn 5.0})$$

It is assumed that the inter-canine arch is like a circle arch, as in the Bonwill–Hawley model (Tweed, 1966; Figure 5). The inter-canine width is 31.4 mm (Knott, 1972). It is the base of an equilateral triangle, which subtends a central angle of 120 degrees and a circumference angle of 60 degrees (Cateni and Fortini, 1982). Applying the trigonometrical theorems, the length of the radius is:

$$r = (HC/\cos 30^\circ) = 18.13. \quad (\text{eqn 5.1})$$

The specific equation $f(y)$ is:

$$x^2 + y^2 = 18.13^2. \quad (\text{eqn 5.2})$$

After proclination of the incisors by 1 mm (Figure 5), the radius becomes 19.13 mm and the new equation $g(y)$ is:

$$x^2 + y^2 = 19.13^2. \quad (\text{eqn 5.3})$$

Knowing the inter-incisive width, the ordinate of I (14.91) is obtained from the first equation and the co-ordinates of N (–15.74; 10.87) and of N' (15.74; 10.87) from the second, using the same method as for the other curves. Solving the definite integral of the two functions between the extremities, the length of II' of 21.91 mm and NN' of 23.12 mm, and their difference of 1.21 mm is obtained.

The hyperbola is the geometric locus of the points of the plane, where the difference of the distances of its points from two fixed points, known as foci, is constant (Zwirner, 1988). The generic equation of hyperbola is

$$(x^2/a^2) - (y^2/b^2) = 1. \quad (\text{eqn 6.0})$$

It is symmetrical to the Cartesian axes and their origins coincide (Figure 6). The curve intersects only the axis of abscissas at the points $+a$ and $-a$, which are the vertices of the hyperbola. The two straight lines $y = \pm(b/a)x$ are the asymptote; that is, when a generic point of the curve extends to infinity towards the line, the distance of that point from the line approximates to 0 (Zwirner, 1988). Consequently, the hyperbola lies between the asymptotes, like the axis of abscissa.

To find the equation of the hyperbolic dental arch form, an arbitrary value for the coefficient of the asymptotes was used because of insufficient data. The coefficient is calculated, using the definition of asymptote. The angle between the two lines is less than the angle between the lines, connecting points A and B and A' and B' (Figure 5). The asymptotes and the curve have to coincide at infinity. The angular coefficient of the asymptotes is the result of BT/AT minus its percentage (BT/AT = 26.2/31.67 = 0.827). To choose the value, two conditions are laid down:

1. The depth of the inter-incisive arch has to be similar to the other curves (for the ellipse 3.4 mm, for the parabola 4.9 mm, for the circumference 3.2 mm and for catenary 3.89 mm).

2. The eccentricity of the curve has to be higher than 1 (De La Cruz *et al.*, 1995).

0.827 is reduced by 95, 90, 85, and 80 per cent down to 20 per cent, and the value of the angular coefficient (m) for each percentage is calculated.

The equations of asymptotes are:

$$y = \pm mx. \quad (\text{eqn 6.1})$$

Point A, the vertex of hyperbola, has the co-ordinates a and 0, point B $a + 31.67$ and 26.2 and $m = (b/a)$. By placing together the generic equation of hyperbola and the co-ordinates of the points, and the relationship between b and a into a system,

$$\begin{cases} (x^2/a^2) - (y^2/b^2) = 1 \\ A \equiv (a; 0) \\ B \equiv (a + 31.67; 26.2) \\ m = (b/a) \end{cases} \quad (\text{eqn 6.2})$$

the equation of the curve becomes:

$$[(a + 31.67)^2/a^2] - [26.2^2/(ma)^2] = 1. \quad (\text{eqn 6.3})$$

Solving this quadratic equation, the value of a is obtained:

$$a = (26.2^2 - 31.67^2 m^2)/63.34 m^2. \quad (\text{eqn 6.4})$$

After calculating the different values of m , 0.827 is decreased by 60 per cent, because the depth of the inter-incisive arch is 5.63 mm and the eccentricity 1.05 (De La Cruz *et al.*, 1995).

The equation $f(y)$ of hyperbola becomes:

$$(x^2/83.13^2) - (y^2/27.51^2) = 1. \quad (\text{eqn 6.5})$$

The co-ordinates of point I are 88.77 and 10.30.

After proclination by 1 mm, the points of the curve $g(y)$ are calculated. They are D, the vertex 1 mm away from A (a' ; O) with $a' = a - 1 = 82.08$ and N, the point on the perpendicular line at the tangent of the curve $f(y)$ in I and 1 mm away from I (Figure 5). To identify N, the equation of the tangent in I is calculated (Figure 6), and the trigonometric theorems are applied, following the same method as for the parabola and

the other curves. N has as its co-ordinates 88.08 and 11.03.

By placing together the generic equation of the hyperbola and the co-ordinates of the different points into a system,

$$\begin{cases} (x'^2/a'^2) - (y'^2/b'^2) = 1 \\ N \equiv (88.08; 11.03) \\ a' = 82.13 \end{cases} \quad (\text{eqn 6.6})$$

the specific equation of the hyperbolic dental arch $g(y)$, after 1 mm of proclination, is obtained:

$$(x^2/82.13^2) - (y^2/28.46^2) = 1. \quad (\text{eqn 6.7})$$

The asymptotes of $g(y)$ are $y = \pm 0.343x$ and their angle (18.97 degrees) is higher than the angle of the asymptotes of the first curve (18.31 degrees).

The length of the inter-incisive arch before (II') and after (NN') 1 mm proclination are the result of the definitive integrals:

$$II' = \int_I^{I'} \sqrt{1 + f'(y)^2} \quad \text{with } -10.30 < y < +10.30 \quad (\text{eqn 6.8})$$

$$NN' = \int_N^{N'} \sqrt{1 + g'(y)^2} \quad \text{with } -11.03 < y < +11.03. \quad (\text{eqn 6.9})$$

The length of II' is 24.15 mm and the length of NN' is 25.76, an increase of 1.61 mm.

The angular inclination of the incisor, is then calculated with respect to the Ricketts occlusal functional plane, correlated with the type of orthodontic movement (Figure 1a,b) and the space gained (Mutinelli *et al.*, 1999). On the sagittal plane with controlled tipping (Figure 1a), the incisor proclination is the distance between points P and Q, or their projections F and F'. The angle ($P\hat{T}F$) (β) can be calculated, applying the trigonometric theorems (Zwirner, 1988) to the right-angled triangle FF'T. The average length of the lower incisors is 22.5 mm (Maggiore and Ripari, 1985):

$$FF' = F'T \sin \beta \quad (\text{eqn 7.0})$$

$$\Rightarrow y = 22.5 \sin x. \quad (\text{eqn 7.1})$$

The equation is not a straight line, but a curve. Consequently, there is no direct proportion between the angular and linear value; that is, if the linear value is doubled, the angular value will not double.

The equation is represented by a sinusoid with period of 2π , which increases from 0 to $\pi/2$ (0–90 degrees). Within the range 0 to $\pi/4$ (0–45 degrees) the curve is steep. If the linear length is converted into angular width, a length of 0.393 mm will be obtained for an angle β of 1 degree and 15.91 mm for an angle of 45 degrees, i.e. 0.353 mm for each degree with an average for the range 0–45 degrees of 0.38 mm. Within the range 90–135 degrees of incisor inclination calculated with

respect to the occlusal functional plane, a 1 mm proclination with controlled tipping is equal, on average, to 2.63 degrees. Therefore, when the incisor is proclined, the nearer the initial inclination is to 90 degrees the more space is gained.

Using the same method in uncontrolled tipping of a tooth, 1 mm of proclination can be expressed as 1 degree, changing the length of $F'T$ and TP (Figure 1b.). The rotation centre lies between the apical fourth and fifth. $F'T$ decreases to $4/5$ of the tooth length, that is, 18 mm. The average β angle becomes 3.33 degrees in the range 0–45 degrees.

The linear and angular values for each arch form can now be calculated (Table 1).

