On the Universal Behavior of Scattering from a Rough Surface for Small Grazing Angles

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***Abstract*— It is shown that for scattering from a plane in average rough surface, the scattering cross section of the range of small grazing angles of the scattered wave demonstrates a universal behavior. If the angle of incidence is fixed (in general, it should not be small), the diffuse component of the scattering cross section for the Dirichlet problem is proportional to θ^2 where θ is the (small) angle of elevation and for the Neumann problem it does not depend on θ. For the backscattering case, these dependencies correspondingly become θ^4 and θ^0. The result is obtained from the structure of the equations that determine the scattering problem rather than the use of an approximation.**

***Index Terms*—Electromagnetic scattering, rough surfaces.**

I. Introduction

In this paper, we investigate the low-grazing-angle (LGA) behavior of the scattering amplitudes for the scalar rough surface scattering problem under the Dirichlet and Neumann boundary condition. Usually, the rough surface scattering problems are attacked using some kind of analytical approximations [5]–[7] or numerical methods [1], [4]. Unlike the majority of publications in the rough surface scattering theory, we only use the most general properties of the scattered fields for our study. Our technique is based on the exact integral equations for source functions and scattering amplitudes. We show that even at this very general level of analysis, it is possible to obtain certain properties of the scattering amplitudes at LGA's of incidence and/or scattering.

II. Notations and Basic Equations

We consider the scattering of the plane wave

$$E_{inc}(\mathbf{r}, z) = \exp[i\mathbf{q}_0\mathbf{r} - i\nu_0 z] \tag{1}$$

by the rough surface $z = \zeta(\mathbf{r})$. Here, the arbitrary three-dimensional wave vector $\mathbf{k}$ is of the form

$$\mathbf{k} = \mathbf{q} \pm \mathbf{e}\nu(\mathbf{q}), \quad \mathbf{e} = (0, 0, 1), \quad \mathbf{qe} = 0. \tag{2}$$

The vertical wave number ν is the function of $\mathbf{q}$

$$\nu(\mathbf{q}) = \begin{cases} \sqrt{k^2 - \mathbf{q}^2}, & \text{for } q < k \\ i\sqrt{\mathbf{q}^2 - k^2}, & \text{for } q > k \end{cases}, \quad \mathrm{Re}\,\nu \geq 0, \quad \mathrm{Im}\,\nu \geq 0 \tag{3}$$

and because of this relation, the plane wave with the wave vector (2) satisfies the Helmholtz equation.

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The scattering amplitude $S(\mathbf{q}, \mathbf{q}_0)$ in the region $z > \max \zeta(\mathbf{r})$ is defined by the formula

$$E_{sc}(\mathbf{r}, z) = \iint \exp[i\mathbf{qr} + i\nu(\mathbf{q})z] S(\mathbf{q}, \mathbf{q}_0) \frac{d^2q}{k\nu(\mathbf{q})}. \tag{4}$$

For $q < k$ the value $d^2q/k\nu(\mathbf{q})$ is equal to the element of the solid angle $d\Omega(\mathbf{k})$ on the sphere $\mathbf{q}^2 + \nu^2 = k^2$. Equation (4) remains the same for both the Dirichlet and the Neumann problems. Most of the following relations, however, will be different for these two problems and we consider them separately. We also use the Weyl–Sommerfeld formula

$$G_0(\mathbf{r} + \mathbf{e}z; \mathbf{r}' + \mathbf{e}z') = \frac{1}{8i\pi^2} \iint \frac{d^2q}{\nu} \exp[i\mathbf{q}(\mathbf{r} - \mathbf{r}') + i\nu|z - z'|]. \tag{5}$$

The reciprocity theorem is true both for the Dirichlet and the Neumann problems:

$$S(\mathbf{q}, \mathbf{q}_0) = S(-\mathbf{q}_0, -\mathbf{q}). \tag{6}$$

This derivation follows the general idea of [3].

Consider domain V bounded by the scattering surface $z = \zeta(\mathbf{r})$ from the bottom and some plane $z = z_* > \zeta_{\max}$ from the top. For two arbitrary solutions $E_1(\mathbf{r}, z)$ and $E_2(\mathbf{r}, z)$ of the homogeneous Helmholtz equation, the Green's theorem states that

$$\iint \left\{ E_1[\mathbf{r}, \zeta(\mathbf{r})] \frac{\partial E_2[\mathbf{r}, \zeta(\mathbf{r})]}{\partial n(\mathbf{r})} - E_2[\mathbf{r}, \zeta(\mathbf{r})] \frac{\partial E_1[\mathbf{r}, \zeta(\mathbf{r})]}{\partial n(\mathbf{r})} \right\} d\Sigma(\mathbf{r}) = \iint \left[E_1(\mathbf{r}, z_*) \frac{\partial E_2(\mathbf{r}, z_*)}{\partial z} - E_2(\mathbf{r}, z_*) \frac{\partial E_1(\mathbf{r}, z_*)}{\partial z} \right] d^2r. \tag{7}$$

Suppose now that $E_1(\mathbf{r}, z) = E(\mathbf{r}, z; \mathbf{q}_0)$ is the total field of the scattering problem with the incident field given by (1), and the scattered field is presented in the form (4). If the second field is an upward-directed plane wave

$$E_2(\mathbf{r}, z) = \exp[-i\mathbf{q}_1\mathbf{r} + i\nu_1 z] \tag{8}$$

then the right-hand part of (7) can be readily calculated and results in

$$\iint \left\{ E[\mathbf{r},\, \zeta(\mathbf{r});\; \mathbf{q}_0) \frac{\partial \exp[i\nu_1\zeta(\mathbf{r}) - i\mathbf{q}_1\mathbf{r}]}{\partial n(\mathbf{r})} - \exp[i\nu_1\zeta(\mathbf{r}) - i\mathbf{q}_1\mathbf{r}] \frac{\partial E[\mathbf{r},\, \zeta(\mathbf{r});\; \mathbf{q}_0]}{\partial n(\mathbf{r})} \right\} d\Sigma(\mathbf{r}) = 8i\pi^2\nu_0\delta(\mathbf{q}_0 - \mathbf{q}_1) \tag{9}$$

where for any function F we use the notation

$$\frac{\partial F[\mathbf{r},\, \zeta(\mathbf{r});\; \mathbf{q}_0]}{\partial n(\mathbf{r})} \equiv \left. \frac{\partial F(\mathbf{r},\, z;\; \mathbf{q}_0);}{\partial n(\mathbf{r})} \right|_{z=\zeta(\mathbf{r})}.$$

If the second field is a downward-directed plane wave, then

$$E_2(\mathbf{r},\, z) = \exp[-i\mathbf{q}_1\mathbf{r} - i\nu_1 z] \tag{10}$$

and instead of (9) we have

$$\iint \left\{ E[\mathbf{r},\, \zeta(\mathbf{r});\; \mathbf{q}_0] \frac{\partial \exp[-i\mathbf{q}_1\mathbf{r} - i\nu_1\zeta(\mathbf{r})]}{\partial n(\mathbf{r})} - \exp[-\mathbf{q}_1\mathbf{r} - i\nu_1\zeta(\mathbf{r})] \frac{\partial E[\mathbf{r},\, \zeta(\mathbf{r});\; \mathbf{q}_0];}{\partial n(\mathbf{r})} \right\} d\Sigma(\mathbf{r}) = -\frac{8i\pi^2}{k} S(\mathbf{q}_1,\, \mathbf{q}_0). \tag{11}$$

For the Dirichlet problem we have $E[\mathbf{r},\, \zeta(\mathbf{r})] = 0$ and (9) and (11) reduce to

$$\iint \exp\,[-i\mathbf{q}\mathbf{r} + i\nu\zeta(\mathbf{r})] F(\mathbf{r},\, \mathbf{q}_0)\, d\Sigma(\mathbf{r}) = -8i\pi^2\nu\delta(\mathbf{q} - \mathbf{q}_0) \tag{12}$$

and

$$\iint \exp\,[-i\mathbf{q}\mathbf{r} - i\nu\zeta(\mathbf{r})] F(\mathbf{r},\, \mathbf{q}_0)\, d\Sigma(\mathbf{r}) = \frac{8i\pi^2}{k} S(\mathbf{q},\, \mathbf{q}_0). \tag{13}$$

We denote here

$$\frac{\partial E[\mathbf{r}, \zeta(\mathbf{r});\; \mathbf{q}_0]}{\partial n(\mathbf{r})} \equiv F(\mathbf{r},\, \mathbf{q}_0).$$

For the Neumann problem $\partial E[\mathbf{r},\, \zeta(\mathbf{r})]/\partial n(\mathbf{r}) = 0$, (9) and (11) reduce to

$$\iint [\nu + \mathbf{q}\nabla\zeta(\mathbf{r})]\, \exp[-i\mathbf{q}\mathbf{r} + i\nu\zeta(\mathbf{r})] E(\mathbf{r},\, \mathbf{q}_0)\, d^2r = 8\pi^2\nu\delta(\mathbf{q} - \mathbf{q}_0) \tag{14}$$

and

$$\iint [-\nu + \mathbf{q}\nabla\zeta(\mathbf{r})]\, \exp[-i\mathbf{q}\mathbf{r} - i\nu\zeta(\mathbf{r})] E(\mathbf{r},\, \mathbf{q}_0)\, d^2r = -\frac{8\pi^2}{k} S(\mathbf{q},\, \mathbf{q}_0). \tag{15}$$

We denote here

$$E[\mathbf{r},\, \zeta(\mathbf{r});\; \mathbf{q}_0] \equiv E(\mathbf{r},\, \mathbf{q}_0). \tag{16}$$

Equations (12)–(15) are the main equations for the following analysis of the small-angle behavior of the scattering amplitudes. Equations (12) and (14) represent the extinction theorem for the Dirichlet and Neumann cases and usually are used to determine the unknown surface sources.

III. Small-Grazing-Angle Behavior of Scattering Amplitudes

A. The Dirichlet Problem

Let us compare (12) with (13). The left-hand sides of these formulas differ only by the sign of ν in the exponent. If we subtract (13) from (12), the $\sin[\nu\zeta(\mathbf{r})]$ appears under the integral and we obtain (after simple algebra)

$$S(\mathbf{q},\, \mathbf{q}_0) = -k\nu\delta(\mathbf{q} - \mathbf{q}_0) - \frac{k}{4\pi^2} \iint \exp(-i\mathbf{q}\mathbf{r})\, \sin[\nu\zeta(\mathbf{r})] \times F(\mathbf{r},\, \mathbf{q}_0)\, d\Sigma(\mathbf{r}). \tag{17}$$

The function $F(\mathbf{r},\, \mathbf{q}_0)$ in this relation is unknown and should be determined from (12), but dependence of $S(\mathbf{q},\, \mathbf{q}_0)$ on $\mathbf{q}$ and ν is given in (17) in explicit form. This dependent property of (17) is really the advantage of (17) because it allows us to investigate the behavior of $S(\mathbf{q},\, \mathbf{q}_0)$ without determining the source function $F(\mathbf{r},\, \mathbf{q}_0)$.

The main conclusion that follows from (17) is that the diffuse part of the scattering amplitude for the Dirichlet problem

$$S_D^{\text{dif}}(\mathbf{q},\, \mathbf{q}_0) \equiv S(\mathbf{q},\, \mathbf{q}_0) + k\nu\delta(\mathbf{q} - \mathbf{q}_0) = -\frac{k}{4\pi^2} \iint \exp(-i\mathbf{q}\mathbf{r})\, \sin[\nu\zeta(\mathbf{r})] \times F(\mathbf{r},\, \mathbf{q}_0)\, d\Sigma(\mathbf{r}) \tag{18}$$

contains (in its formal expansion in the Taylor series) only odd powers of ν.[1] Indeed, we have

$$q = \sqrt{k^2 - \nu^2}, \quad \mathbf{q} = q\hat{\mathbf{q}} = \hat{\mathbf{q}}\sqrt{k^2 - \nu^2} \tag{19}$$

i.e., $\mathbf{q}(\nu)$ is an even function of ν. Therefore, $\exp(-i\mathbf{q}\mathbf{r})\, \sin[\nu\zeta(\mathbf{r})]$ is an odd function of ν. Thus, if we multiply and divide the integrand in (18) by ν, we obtain

$$S_D^{\text{dif}}(\mathbf{q},\, \mathbf{q}_0) = -\frac{k\nu}{4\pi^2}\, \Phi^{(+)}\left(\hat{\mathbf{q}}\sqrt{k^2 - \nu^2},\, \mathbf{q}_0;\; \nu^2\right) \tag{20}$$

[1]The value ν is defined only in the regions $\nu \geq 0$ for $q \leq k$ or $\text{Im}\nu \geq 0$ for $q \geq k$. Because of this, it is incorrect to use the term odd function for this case. Nevertheless, we will use terms odd and even functions of ν, meaning that corresponding expansions in the Taylor series contain only odd or even powers of ν.

where

$$\Phi^{(+)}(\hat{\mathbf{q}}\sqrt{k^2-\nu^2},\, \mathbf{q}_0; \nu^2) = \iint \exp\left(-i\sqrt{k^2-\nu^2}\,\hat{\mathbf{q}}\mathbf{r}\right) \frac{\sin[\nu\zeta(\mathbf{r})]}{\nu} \times F(\mathbf{r},\, \mathbf{q}_0)\, d\Sigma(\mathbf{r}) \tag{21}$$

depends only on ν^2. If $\nu = 0$, this function is equal to some independent of ν constant

$$\Phi^{(+)}(\hat{\mathbf{q}}k,\, \mathbf{q}_0;\, 0) = \iint \exp(-ik\hat{\mathbf{q}}\mathbf{r})\zeta(\mathbf{r})F(\mathbf{r},\, \mathbf{q}_0)\, d\Sigma(\mathbf{r}) \tag{22}$$

and in the region of small ν we obtain

$$S_D^{\rm dif}(\mathbf{q},\, \mathbf{q}_0) = -\frac{k\nu}{4\pi^2}\, [\Phi^{(+)}(\hat{\mathbf{q}}k,\, \mathbf{q}_0;\, 0) + O(\nu^2)] \tag{23}$$

or

$$\lim_{\nu\to 0} \frac{S_D^{\rm dif}(\mathbf{q},\, \mathbf{q}_0)}{\nu} = -\frac{k}{4\pi^2}\, \Phi^{(+)}(\hat{\mathbf{q}}k,\, \mathbf{q}_0;\, 0). \tag{24}$$

Certainly, the applicability of (23) depends on several conditions. The most important restriction is related to expansion of $\sin[\nu\zeta(\mathbf{r})]$ in powers of ν. It is evident that we should require the restriction

$$\nu|\zeta|_{\max} \ll 1, \quad \text{or} \quad \sin\,\theta_{\rm gr} \ll \frac{1}{k|\zeta|_{\max}} \tag{25}$$

where $\theta_{\rm gr}$ is the grazing angle of the scattered wave. Some other conditions may appear in different applications (for example, the size of footprint).

The result obtained does not involve any approximation for the unknown source function F. This result is an exact consequence of Helmholtz equation, Dirichlet boundary condition, and radiation condition.

On the other hand, the scattering amplitude satisfies the reciprocity condition (6), which allows us to write the reciprocal to (24)

$$\lim_{\nu_0\to 0} \frac{S_D^{\rm dif}(\mathbf{q},\, \mathbf{q}_0)}{\nu_0} = -\frac{k}{4\pi^2}\, \Phi^{(+)}(-\hat{\mathbf{q}}_0 k,\, -\mathbf{q};\, 0). \tag{26}$$

If both ν and ν_0 are small, both (24) and (26) must be true and we obtain the result

$$\lim_{\nu\to 0}\lim_{\nu_0\to 0} \frac{S_D^{\rm dif}(\mathbf{q},\, \mathbf{q}_0)}{\nu\nu_0} = X_D(\hat{\mathbf{q}},\, \hat{\mathbf{q}}_0). \tag{27}$$

where the function $X_D(\hat{\mathbf{q}}, \hat{\mathbf{q}}_0)$ depends only on directions of the incident and scattered waves.

B. The Neumann Problem

The Neumann problem is a little more complicated than the Dirichlet problem, but we can perform a quite similar analysis. Let us add (14) for E and (15), determining S. After completing some simple algebra we obtain

$$S(\mathbf{q},\, \mathbf{q}_0) = k\nu\delta(\mathbf{q}-\mathbf{q}_0) - \frac{i\nu k}{4\pi^2}\iint \exp(-i\mathbf{q}\mathbf{r})E(\mathbf{r},\, \mathbf{q}_0) \times \sin[\nu\zeta(\mathbf{r})]\, d^2r - \frac{k\mathbf{q}}{4\pi^2}\iint \nabla\zeta(\mathbf{r}) \times \exp(-i\mathbf{q}\mathbf{r})E(\mathbf{r},\, \mathbf{q}_0)\cos[\nu\zeta(\mathbf{r})]\, d^2r. \tag{28}$$

Equation (28) for the Neumann problem is analogous to (17) obtained for the Dirichlet problem. If we consider the diffuse part of the scattering amplitude for the Neumann problem, we obtain

$$S_N^{\rm dif}(\mathbf{q},\, \mathbf{q}_0) \equiv S(\mathbf{q},\, \mathbf{q}_0) - k\nu\delta(\mathbf{q}-\mathbf{q}_0) = -\frac{k\mathbf{q}}{4\pi^2}\iint \nabla\zeta(\mathbf{r})\, \exp(-i\mathbf{q}\mathbf{r})E(\mathbf{r},\, \mathbf{q}_0) \times \cos[\nu\zeta(\mathbf{r})]\, d^2r - \frac{i\nu k}{4\pi^2}\iint \times \exp(-i\mathbf{q}\mathbf{r})E(\mathbf{r},\, \mathbf{q}_0)\sin[\nu\zeta(\mathbf{r})]\, d^2r. \tag{29}$$

Equation (29) shows that when $\nu \to 0$, $\mathbf{q} \to k\hat{\mathbf{q}}$

$$\lim_{\nu\to 0} S_N^{\rm dif}(\mathbf{q},\, \mathbf{q}_0) = T_N(\hat{\mathbf{q}},\, \mathbf{q}_0) = -\frac{k^2\hat{\mathbf{q}}}{4\pi^2}\iint \nabla\zeta(\mathbf{r})\, \exp(-ik\hat{\mathbf{q}}\mathbf{r})E(\mathbf{r},\, \mathbf{q}_0)\, d^2r. \tag{30}$$

The integral in (30) converges if the rough surface becomes plane at infinity ($\nabla\zeta \to 0$ if $r \to \infty$).

Equation (30) shows that for the Neumann problem, the scattering amplitude in the region of small scattering grazing angles does not depend on this angle.

It follows from the reciprocity condition (6) that in the region of small angles of incidence $\nu_0 \ll k$, the property that is similar to (30) is true:

$$\lim_{\nu_0\to 0} S_N^{\rm dif}(-\mathbf{q}_0,\, -\mathbf{q}) = \lim_{\nu_0\to 0} S_N^{\rm dif}(\mathbf{q},\, \mathbf{q}_0) = T_N(-\hat{\mathbf{q}}_0,\, -\mathbf{q}). \tag{31}$$

Thus, if both ν and ν_0 are small, $S_N^{\rm dif}(\mathbf{q},\, \mathbf{q}_0)$ remains some constant. In particular, the backscattering amplitude and cross section in the case of the Neumann problem are independent of the elevation angle if this angle is small.

Similar to the Dirichlet case, $E(\mathbf{r},\, \mathbf{q}_0)$ is some unknown functional of $\zeta(\mathbf{r})$ and (30) and (31) are not any approximations, but exact results that follow from the Helmholtz equation, Neumann boundary condition, and radiation condition.

IV. Comparison with Perturbation Theory

A. The Dirichlet Problem

The well-known result of perturbation theory [6] for the scattering amplitude for the Dirichlet problem is

$$S_{D,\,1}^{\rm dif}(\mathbf{q},\, \mathbf{q}_0) = 2i\nu\nu_0 k\tilde{\zeta}(\mathbf{q}-\mathbf{q}_0). \tag{32}$$

where

$$\tilde{\zeta}(\mathbf{q}) \equiv \frac{1}{4\pi^2}\iint \exp(-i\mathbf{q}\mathbf{r})\zeta(\mathbf{r})\, d^2r. \tag{33}$$

This result also can be obtained from (17) if we keep the zero-order term in $\zeta(\mathbf{r})$ for $F(\mathbf{r},\, \mathbf{q}_0)$ and expand $\sin[\nu\zeta(\mathbf{r})]$ to the first order in $\zeta(\mathbf{r})$.

Comparison of (32) with (27) shows that for the Dirichlet problem, the first perturbation term for the scattering amplitude reveals the same dependence on ν, ν_0 as the first term of the expansion in powers of ν and/or ν_0.

B. The Neumann Problem

With our definition of the scattering amplitude, the well-known first-order perturbation result for the scattering amplitude has the form

$$S_{N,\,1}^{\text{dif}}(\mathbf{q},\, \mathbf{q}_0) = -2ik[k^2 - \mathbf{q}\mathbf{q}_0]\widetilde{\zeta}(\mathbf{q} - \mathbf{q}_0). \tag{34}$$

This result can also be obtained from our basic (29) if we keep the zero-order term in powers of $\zeta(\mathbf{r})$ for $E(\mathbf{r},\, \mathbf{q}_0)$[2]

$$E_0(\mathbf{r},\, \mathbf{q}_0) = 2\, \exp(i\mathbf{q}_0\mathbf{r})$$

and keep the first-order terms in $\zeta(\mathbf{r})$ in the other integrands of (29). After calculating two integrals we obtain (34); that is the standard result for the Bragg scattering for the Neumann problem.

Let us consider now the scattering amplitude for the Neumann problem for small ν without restrictions on $\zeta(\mathbf{r})$. It follows from (29) that the second term in the right-hand side of (29) has the second order in powers of ν, but the first term has the zero-order component in powers of ν. Thus, the leading term of the expansion in powers of ν in the region of small ν is equal to

$$S_N^{\text{dif}}(\mathbf{q},\, \mathbf{q}_0) \approx -\frac{k^2\hat{\mathbf{q}}}{4\pi^2} \iint \nabla\zeta(\mathbf{r})\, \exp(-ik\hat{\mathbf{q}}\mathbf{r}) \times E(\mathbf{r},\, \mathbf{q}_0)\, d^2r \tag{35}$$

and is independent of ν. Equation (35) is valid for arbitrary ζ and $\nu \ll k$, but (34) is true for small ζ and arbitrary ν. If we consider the case when both ν and ζ are small, we should substitute in (35) $E_0(\mathbf{r},\, \mathbf{q}_0) = 2\, \exp(i\mathbf{q}_0\mathbf{r})$ instead of $E(\mathbf{r},\, \mathbf{q}_0)$. In this case, we obtain

$$\begin{aligned} S_N^{\text{dif}}(\mathbf{q},\, \mathbf{q}_0) &\approx -\frac{k^2\hat{\mathbf{q}}}{2\pi^2} \iint \nabla\zeta(\mathbf{r}) \times \exp[-i(k\hat{\mathbf{q}} - \mathbf{q}_0)\mathbf{r}]\, d^2r \\ &= -2ik(k^2 - k\hat{\mathbf{q}}\mathbf{q}_0)\widetilde{\zeta}(k\hat{\mathbf{q}} - \mathbf{q}_0) \end{aligned} \tag{36}$$

which is a particular case of (34) in the region of small ν. Thus, both formulas coincide in the region of their overlapping.

V. Comparison with the Kirchhoff Approximation

Here, we examine the small grazing angle behavior of the scattering amplitude in the Kirchhoff approximation.

[2]This is just a doubled incident field evaluated at the reference plane $z = 0$. This result can also be obtained from (14) in the zero order in $\zeta(\mathbf{r})$.

A. The Dirichlet Problem

In order to obtain the Kirchhoff approximation, we assume that the source function in (18) corresponds to the reflection from the local tangent plane (see details in [2])

$$F(\mathbf{r},\, \mathbf{q}_0) \approx -2i\, \frac{[\nu_0 + \mathbf{q}_0\nabla\zeta(\mathbf{r})]}{\sqrt{1 + [\nabla\zeta(\mathbf{r})]^2}} \times \exp[i\mathbf{q}_0\mathbf{r} - i\nu_0\zeta(\mathbf{r})]. \tag{37}$$

The Kirchhoff approximation for the scattering amplitude has the form

$$S_{D,\,K}^{\text{dif}}(\mathbf{q},\, \mathbf{q}_0) \approx \frac{ik}{2\pi^2} \iint [\nu_0 + \mathbf{q}_0\nabla\zeta(\mathbf{r})]\, \sin[\nu\zeta(\mathbf{r})] \times \exp[i(\mathbf{q}_0 - \mathbf{q})\mathbf{r} - i\nu_0\zeta(\mathbf{r})]\, d^2r. \tag{38}$$

In the limit of the small grazing angle $\nu \to 0$ we have

$$S_{D,\,K}^{\text{dif}}(\mathbf{q},\, \mathbf{q}_0) \approx \frac{ik\nu}{2\pi^2} \iint [\nu_0 + \mathbf{q}_0\nabla\zeta(\mathbf{r})]\zeta(\mathbf{r}) \times \exp[i(\mathbf{q}_0 - \mathbf{q})\mathbf{r} - i\nu_0\zeta(\mathbf{r})]\, d^2r. \tag{39}$$

Obviously, Kirchhoff approximation exhibits the same behavior of the small grazing angle as the exact solution (23). It is worth noting that formula (39) can be obtained directly from (23) by the same approximation of the source function.

B. The Neumann Problem

We make the same assumption that the source function in (29) corresponds to the reflection from the local tangent plane (see details in [8])

$$E(\mathbf{r},\, \mathbf{q}_0) = 2\, \exp[i\mathbf{q}_0\mathbf{r} - i\nu_0\zeta(\mathbf{r})]\Phi(\mathbf{r},\, \mathbf{q}_0). \tag{40}$$

Equation (29) for the diffuse scattering amplitude now takes the form

$$\begin{aligned} S_{N,\,K}^{\text{dif}}(\mathbf{q},\, \mathbf{q}_0) \approx &-\frac{k\mathbf{q}}{2\pi^2} \iint \nabla\zeta(\mathbf{r}) \\ &\times \exp[i(\mathbf{q}_0 - \mathbf{q})\mathbf{r} - i\nu_0\zeta(\mathbf{r})]\, \cos[\nu\zeta(\mathbf{r})]\, d^2r \\ &- \frac{i\nu k}{2\pi^2} \iint \exp[i(\mathbf{q}_0 - \mathbf{q})\mathbf{r} - i\nu_0\zeta(\mathbf{r})] \\ &\times \sin[\nu\zeta(\mathbf{r})]\, d^2r. \end{aligned} \tag{41}$$

In the limit of small grazing angles we have

$$S_{N,\,K}^{\text{dif}}(\mathbf{q},\, \mathbf{q}_0) \approx -\frac{k^2\hat{\mathbf{q}}}{2\pi^2} \iint \exp[i(\mathbf{q}_0 - k\hat{\mathbf{q}})\mathbf{r} - i\nu_0\zeta(\mathbf{r})] \times \nabla\zeta(\mathbf{r})\, d^2r. \tag{42}$$

Similarly to the exact solution (30), the scattering amplitude in Kirchhoff approximation approaches a finite limit at small grazing angles. Equation (42) can be obtained from the exact small grazing angle asymptote by applying Kirchhoff approximation to the dipole source function in (30).

VI. Conclusions

The scattering cross section σ to the unit of solid angle is related to the scattering amplitude S by[3]

$$\sigma(\mathbf{q},\, \mathbf{q}_0) = \frac{4\pi^2}{k^2}\left\langle |S(\mathbf{q},\, \mathbf{q}_0)|^2\right\rangle \tag{43}$$

where $\langle \cdots \rangle$ denotes the averaging.

It follows from our results that if the angle of incidence is fixed in the region of small grazing angles of scattered waves the ratio of the scattering cross sections σ_D/σ_N is proportional to the square of the small elevation angle θ

$$\frac{\sigma_D(\mathbf{q},\, \mathbf{q}_0)}{\sigma_N(\mathbf{q},\, \mathbf{q}_0)} \sim \nu^2 \sim \sin^2\theta, \quad \nu \ll k. \tag{44}$$

In the case of backscattering ($\mathbf{q} = -\mathbf{q}_0$, $\nu = \nu_0$), the second power changes to the fourth

$$\frac{\sigma_D(\mathbf{q},\, -\mathbf{q})}{\sigma_N(\mathbf{q},\, -\mathbf{q})} \sim \nu^4 \sim \sin^4\theta, \quad \nu \ll k. \tag{45}$$

An interesting question arises from the analysis of the experimental data concerning the relation between the scattering cross sections of vertically polarized electromagnetic waves (σ_{VV}) and horizontally polarized waves (σ_{HH}). For a one-dimensional rough interface between a vacuum and a perfectly conducting material, a scattering of the horizontally polarized wave corresponds to the Dirichlet problem and a scattering of the vertically polarized wave corresponds to the Neumann problem. If the cross-polarization scattering is small, we can use the results obtained for the Dirichlet and for the Neumann problems as a good approximation for the electromagnetic wave scattering. Thus, our consideration is of interest for the relation between σ_{VV} and σ_{HH} cross sections.

It is interesting to compare our results to the small grazing angle results obtained in [9]. The theory presented in [9] is based on the surface model in the form of "bosses" randomly distributed on the plane. For this surface model, the method of images allows us to obtain an exact solution for a single boss on the plane if an exact solution for a finite body consists of the boss and its mirror image is available. Single-boss scattering amplitudes in Twersky's treatment [9, eq. (11)] reveal the same small-grazing-angle dependence as (23) and (31). This single-boss scattering amplitude agrees with the perturbation theory (Born scattering) in the case of a small boss. However, for the scattering cross sections for the ensemble of bosses randomly distributed on the plane V, Twersky reports $\sigma_D(\mathbf{q},\, \mathbf{q}_0) = O(\nu^4)$ and $\sigma_N(\mathbf{q},\, \mathbf{q}_0) = O(\nu^2)$. This difference appears in the process of averaging and is caused, in our opinion, by some arbitrary assumptions involved in the calculations. It is clear that these results cannot, in general, be correct because they do not agree with the perturbation theory limit.

[3] This formula is valid for the finite scattering area. In the limiting case of infinite scattering surface, the limit

$$\sigma_0(\mathbf{q},\, \mathbf{q}_0) = \lim_{A\to\infty} \frac{4\pi^2}{k^2 A}\left\langle |S(\mathbf{q},\, \mathbf{q}_0)|^2\right\rangle$$

exists, which describes the scattering cross section from the unit of scattering surface (see, e.g., [2]).

In Barrick's paper [10] the problem of grazing scattering above the impedance surface with the boundary condition

$$\frac{\partial E[\mathbf{r},\, \zeta(\mathbf{r})]}{\partial n} + ik\mathcal{Z}E[\mathbf{r},\, \zeta(\mathbf{r})] = 0 \tag{46}$$

was considered. The result obtained in this paper shows the θ^4 behavior of the backscattering cross-section for any impedance $\mathcal{Z}$ for small θ. If $\mathcal{Z} \to \infty$, we should obtain the Dirichlet problem and results obtained in Barrick's paper and in this paper agree with each other. However, in the case $\mathcal{Z} = 0$ we obtain the Neumann problem and these results contradict each other. This contradiction is caused by the singular behavior of the scattering cross section as a function of two parameters ν and $\mathcal{Z}$. If we consider this problem by the perturbation method, we obtain the following for the scattering amplitude:

$$S^{(1)}_{\mathcal{Z}}(\mathbf{q},\, \mathbf{q}_0) = -2ik\, \frac{\nu\nu_0(k^2 - \mathbf{q}\mathbf{q}_0 - k^2\mathcal{Z}^2)}{(\nu + k\mathcal{Z})(\nu_0 + k\mathcal{Z})}\, \tilde{\zeta}(\mathbf{q}-\mathbf{q}_0). \tag{47}$$

If $\mathcal{Z} = \infty$, we obtain (32) for the Dirichlet case if $\mathcal{Z} = 0$ and (34) for the Neumann case. However, if we consider finite $\mathcal{Z}$ and small $\nu \ll k\mathcal{Z}$, the result is proportional to ν, similarly to the Dirichlet case. Thus, for small $\mathcal{Z}$ and small ν, the result depends on the relation between ν and $k\mathcal{Z}$. If $\mathcal{Z} \ll \nu/k \ll 1$, the dependence of the scattering amplitude on ν agrees with the solution of the Neumann problem, but if $\nu/k \ll \mathcal{Z} \ll 1$, the Neumann problem can not serve as a good approximation for the impedance problem. Note that the critical angle $\theta = \arcsin(\mathcal{Z})$ corresponds to the Brewster angle. The real behavior of the scattering amplitude as a function of ν will depend on the relation between the scattering angle, beamwidth, and Brewster angle. In principle, both cases are possible under different conditions.

The more detailed version of the paper is published in [11].

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References

[1] D. E. Barrick, "Near-grazing illumination and shadowing of rough surfaces," *Radio Sci.*, vol. 30, pp. 536–580, 1995.
[2] M. I. Charnotskii and V. I. Tatarskii, "Tilt-invariant theory of rough-surface scattering: I," *Waves Random Media*, vol. 5, pp. 361–380, 1995.
[3] D. Maystre, "Rigorous vector theories of diffraction gratings," in *Progress in Optics*, E. Wolf, Ed. Amsterdam, The Netherlands: North-Holland, 1984, vol. 21, pp. 1–67.
[4] C. L. Rino and H. D. Ngo, "Application of beam simulation to scattering at low grazing angles—I: Methology and validation," *Radio Sci.*, vol. 29, pp. 1365–1379, 1994.
[5] W. T. Shaw and A. J. Dougan, "Half-space Green's function and applications to scattering of electromagnetic waves from ocean-like surfaces," *Waves Random Media*, vol. 5, pp. 341–359, 1955.
[6] A. G. Voronovich, *Wave Scattering From Rough Surfaces.* Berlin, Germany: Springer-Verlag, 1994.
[7] ——, "On the theory of electromagnetic waves scattering from sea surface at low grazing angles," *Radio Sci.*, vol. 31, pp. 1519–1530, 1995.
[8] M. I. Charnotskii and V. I. Tatarskii, "Tilt-invariant theory of rough-surface scattering," *Waves Random Media*, pt. II, to be published.
[9] V. Twersky, "On scattering and reflection of sound by rough surfaces," *J. Acoust. Soc. Amer.*, vol. 29, pp. 209–225, 1957.

[10] D. E. Barrick, "Grazing behavior of scatter and propagation above any rough surface," *IEEE Trans. Antennas Propagat.*, this issue, pp. 73–83.
[11] V. I. Tatarskii and M. I. Charnotskii, "Universal behavior of scattering amplitudes for scattering from a plane in average rough surface for small grazing angles," *Waves Random Media,* to be published.

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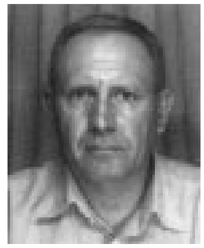

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