Virtual Array Synthesis Method for Planar Array Antennas

Leo I. Vaskelainen

Abstract— **A new array antenna synthesis method, which we call the virtual array synthesis method, is presented. In this method, the excitation values of a virtual array are synthesized using some known synthesis method. The geometry of the virtual array can be chosen so that there will be a suitable synthesis method for that geometry and the synthesis of the virtual array can be done accurately enough. In the synthesis method presented, the excitation values of the virtual array are transformed into the excitation values of the actual array geometry. Matrix operations are simple and large arrays can be easily synthesized.**

I. Introduction

THERE are several synthesis methods for planar arrays having elements on a rectangular lattice and rectangular boundary [1], [6]; for example, the Fourier transform method or polynomial expansions. Polynomial representation with roots radially displaced from the unit circle can be used for equispaced linear arrays and, with a restricted number of destination array factors, for equispaced planar arrays [4]. Contour transformation can be used to synthesize contoured footprints that are symmetric about both the x and y axes [2]. For more general array geometry and destination beam shape, the least squares synthesis or quadratic programming type synthesis method [3] must be used, often combined with iterative methods [5], which, with large arrays, leads to difficult calculation and often to iterative optimizing methods.

In Section II, it is shown that by one matrix inversion and two matrix multiplications with real-valued easily calculated matrices, the excitation values of an array can be transformed into the excitation values of another array so that the integral of the squared absolute value of the difference between these two array factors is minimized. If we thus have a suitable synthesis method for some array geometry, it is easy to convert this synthesis result for another geometry.

For planar arrays with a rectangular lattice, the Woodward synthesis is a very fast and simple synthesis method. In Section II, the one-dimensional (1-D) Woodward synthesis represented in [1] or [6] is rewritten for two-dimensional (2-D) arrays. A large Woodward synthesized array can be used as a virtual array during the synthesis process and the excitation values of that virtual array can be transformed into the excitation values of the actual array geometry, as shown in Section IV. In Section V some examples of the use of the virtual array synthesis method are calculated. The virtual array synthesis method also gives an easy method to synthesize the excitation values for thinned arrays of any geometry and find the less significant elements for removal. An example is given in Section V.

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The author is with VTT Information Technology Telecommunications, Espoo, FIN-02044 VTT Finland.

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II. Virtual Array Method

If we have an unknown array antenna with K isotropic elements, each having position vector $\vec{r}_k$, and excitation a_k the radiation pattern $E(\hat{u})$ of the array antenna for the direction of a unit vector $\hat{u}$ can be written

$$E(\hat{u}) = \sum_{k=1}^{K} a_k \cdot e^{jk_0 \vec{r}_k \cdot \hat{u}} \tag{1}$$

where $k_0 = 2\pi/\lambda$ is the wave number in free-space. On the other hand, we can write the radiation pattern $D(\hat{u})$ of a known array antenna with P isotropic elements, each having position vector $\vec{\rho}_p$ and excitation α_p

$$D(\hat{u}) = \sum_{p=1}^{p} \alpha_p \cdot e^{jk_0 \vec{\rho}_p \cdot \hat{u}}. \tag{2}$$

The known array antenna can be "virtual" in the sense that it is used only as a destination array, which is easy to synthesize but can be impractical to realize. If we define the error function

$$\varepsilon(\hat{u}) = E(\hat{u}) - D(\hat{u}) \tag{3}$$

and calculate the integral of the square of the absolute value of error for all directions (over the solid angle $\Omega = 4\pi$)

$$\begin{aligned} s &= \oint_\Omega \varepsilon(\hat{u})\varepsilon^*(\hat{u})\, d\Omega \\ &= \oint_\Omega \left\{ \sum_{k=1}^{K} a_k e^{ijk_0 \vec{r}_k \cdot \hat{u}} - \sum_{p=1}^{p} \alpha_p e^{jk_0 \vec{\rho}_p \cdot \hat{u}} \right\} \\ &\quad \cdot \left\{ \sum_{k=1}^{K} a_k^* e^{ijk_0 \vec{r}_k \cdot \hat{u}} - \sum_{p=1}^{p} \alpha_p^* e^{jk_0 \vec{\rho}_p \cdot \hat{u}} \right\} d\Omega \end{aligned} \tag{4}$$

it turns out that $\partial s/\partial \mathrm{Re}(a_i) = 0$ and $\partial s \partial \mathrm{Im}(a_i) = 0$ when

$$\sum_{k=1}^{K} \left\{ a_k \oint_\Omega e^{-jk_0(\vec{r}_i - \vec{r}_k)\cdot \hat{u}}\, d\Omega \right\} - \sum_{p=1}^{P} \left\{ \alpha_p \oint_\Omega e^{-jk_0(\vec{r}_i - \vec{\rho}_p)\cdot \hat{u}}\, d\Omega \right\} = 0, \qquad i = 1 \cdot K. \tag{5}$$

The selection of the error function of (3) leads to a very simple solution for excitation values a_k as we can see. The cost function s does not optimize the phase of the gain function as the type of cost function used, for example, in [7] can. The analytical solution for the integrals in (5) does not allow the use of different weight values in different directions as is usually used in the least-squares type optimization methods [8].

Type $\oint_\Omega e^{-jk_o \vec{r}_0 \cdot \hat{u}}\, d\Omega$ integrals are easily calculated in a local polar coordinate system, where the direction of $\vec{r}$ is $\theta = 0$

$$\oint_\Omega e^{-jk_o \vec{r}\cdot\hat{u}}\, d\Omega = \int_0^\pi \sin(\theta)\cdot e^{-jk_0|\vec{r}|\cos(\theta)}\, d\theta \cdot \int_0^{2\pi} d\varphi = 4\pi\, \frac{\sin(k_0|\vec{r}|)}{k_0|\vec{r}|}. \tag{6}$$

By defining a $K \times K$ matrix $\boldsymbol{S}_{rr}$ with elements

$$S_{rr}(i,k) = 4\pi\, \frac{\sin(k_0|\vec{r}_i - \vec{r}_k|)}{k_0|\vec{r}_i - \vec{r}_k|} \tag{7}$$

and a $K \times P$ matrix $\boldsymbol{S}_{rv}$ with elements

$$S_{rv}(i,p) = 4\pi\, \frac{\sin(k_0|\vec{r}_i - \vec{\rho}_p|)}{k_0|\vec{r}_i - \vec{\rho}_p|} \tag{8}$$

we can write a matrix equation for the excitations a_i that minimizes the integral s of the square of the absolute value of error for all the directions defined in (4)

$$\begin{aligned} \boldsymbol{S}_{rr}\boldsymbol{A} &= \boldsymbol{S}_{rv}\boldsymbol{A}_v \\ \boldsymbol{A} &= [a_1 \quad a_2 \quad \cdots \quad a_K]^T \\ \boldsymbol{A}_v &= [\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_K]^T \end{aligned} \tag{9}$$

where T is the transpose of a matrix.

No normalizing of the array factors (or element excitation amplitudes) is assumed in (9). In the cost function s in (4) the array factors of the virtual array and the actual array can have any common scalar multiplier.

For the virtual array synthesis method we need an easy synthesis method for some virtual array geometry. In Section III, we extend the 1-D Woodward synthesis method for 2-D equally spaced arrays and in Section IV, we combine the Woodward synthesis and the virtual array synthesis method.

Equation (9) can be used for any array geometry if one can find a synthesis method for the virtual array. If the virtual array is a planar array, the actual antenna must also be planar or at least symmetrical to the plane of the virtual array so that it also has a symmetrical array factor. The virtual array synthesis method is, thus, most convenient to use to synthesize planar arrays, but (9) can also be used to modify some array geometries, for example, to synthesize excitation values for thinned arrays of any geometry.

III. Two-Dimensional Woodward Synthesis

We start by considering a planar array in the y-z plane having M elements in z direction and N elements in y direction. Element spaces are d_z in z direction and d_y in y direction. For this array we write the normalized array factor where the excitations in (1) are

$$\alpha^w_{mn} = \frac{1}{MN}\, e^{-j(2\pi k/M)(m-(M+1)/2))} \cdot e^{-j(2\pi l/N)(n-(N+1/2))} \tag{10}$$

and element mn is in location $z = d_z\ (m - (M+1)/2)$, $y = d_y\ (n - (N+1)/2)$.

When these excitations and element locations are substituted in (1) beams $F_{kl}(\theta,\varphi)$ can be readily calculated (see, for example, [1]). The orders k and l $(k = 0, \pm 1, \pm 2 \cdots, l = 0, \pm 1, \pm 2 \cdots)$ represent different beams.

These beams $F_{kl}(\theta,\varphi)$ are orthogonal in the sense that they have zeros in the same directions

$$\begin{aligned} \theta_{st} &= \arccos\left(\frac{s}{M}\cdot\frac{\lambda}{d_z}\right) \\ \varphi_{st} &= \arcsin\left(\frac{t}{N}\cdot\frac{\lambda}{d_y}\Big/\sqrt{1-\left(\frac{s}{M}\cdot\frac{\lambda}{d_z}\right)^2}\right) \end{aligned} \tag{11}$$

while in orthogonal direction $s = k, t = l$ $F_{kl}(\theta,\varphi)$ has a maximum $(F_{kl}(\theta_{kl},\varphi_{kl}) = 1)$. There may also be other maximums (grating lobes), which are directed in orthogonal directions (s,t) and which obey the equations $s - k = pM$ and $t - l = qN$, where p and q are integers $(0, \pm 1, \pm 2, \cdots)$.

From now on we select only those orthogonal directions where there is only one maximum and the orthogonal directions receive real values, which means that we must select the (s,t) values according to (12a) and (12b)

$$\begin{aligned} -M/2 &\le s < M/2 \\ -N/2 &\le t < N/2 \end{aligned} \tag{12a}$$

$$\begin{aligned} &-Md_z/\lambda \le s < Md_z/\lambda \\ &-\frac{Nd_y}{\lambda}\sqrt{1-\left(\frac{k}{M}\cdot\frac{\lambda}{d_z}\right)^2} \le\ < \frac{Nd_y}{\lambda}\sqrt{1-\left(\frac{k}{M}\cdot\frac{\lambda}{d_z}\right)^2}. \end{aligned} \tag{12b}$$

These directions are selected symmetrically around the direction $\theta = \pi/2, \varphi = 0$, which is not necessary, but is usually convenient.

By using these orthogonal beams it is very easy to synthesize shaped array factors. If we have a destination beam $D(\theta,\varphi)$, we can start by selecting the numbers s and t according to (12) and calculating the corresponding orthogonal directions from (11). Next orders k and l can be chosen so that $k = s$ and $l = t$ when $D(\theta_{st},\varphi_{st}) \neq 0$. The coefficients for the orthogonal beams are

$$\lambda_{kl} = D(\theta_{kl},\varphi_{kl}) \tag{13}$$

and the array factor is the sum of the orthogonal beams $F_{kl}(\theta,\varphi)$ with amplitudes λ_{kl}

$$F(\theta,\varphi) = \sum_{k,l} \lambda_{kl}\cdot F_{kl}(\theta,\varphi). \tag{14}$$

We know that $F(\theta_{kl},\varphi_{kl}) = D(\theta_{kl},\varphi_{kl})$ in all directions $(\theta_{kl},\varphi_{kl})$ and that in all the other orthogonal directions that are selected according to (12) $F(\theta_{kl}, \varphi_{kl}) = 0$. If d_z or d_y is greater than $\lambda/2$, there may be grating lobes outside the range of (12) in real angles. Between the Woodward directions $F(\theta, \varphi) \neq D(\theta, \varphi)$ and $F(\theta, \varphi)$ can, in fact, receive values in a very broad range. The only way to control the fit of the Woodward synthesized beam to the destination function is to increase the numbers M and N (the size of the array) and the Woodward synthesis thus leads to unnecessarily large arrays.

The excitations α_{mn} can now be calculated by adding up the excitations of the individual beams from (10)

$$\alpha_{mn} = \frac{1}{MN} \cdot \sum_{k,l} \lambda_{kl} \cdot e^{-j(2\pi k/M)(m-(M+1/2))} \cdot e^{-j(2\pi l/N)(n-(N+1/2))}. \quad (15)$$

If we now write $N_w \times 1$-vector Λ containing the λ_{kl} values in any selected order (N_w = number of orthogonal directions, where $D(\theta_{kl}, \varphi_{kl}) \neq 0$) and $N_v \times 1$ vector $\boldsymbol{A}$ including the α_{mn} values (N_v = number of elements in the virtual array), (16) can be presented in matrix form

$$\boldsymbol{A}_v = \boldsymbol{V}\Lambda \quad (16)$$

where the elements of the $\boldsymbol{V}$ matrix are

$$V_{i1,i2} = \frac{1}{MN} e^{-j[(2\pi k/M)(m-(M+1)/2)]} \cdot e^{-j[(2\pi l/N)(n-(N+1)/2)]}. \quad (17)$$

where $i1$ corresponds to the order of mn combination and $i2$ to the order of kl combination.

In Figs. 3 and 4, we present a Woodward synthesized result when the destination function is the "threebox" function seen in Figs. 1 and 2. It is clearly seen that when the directions of the orthogonal beams are fixed, it is in many cases difficult to fit them with the destination function in a convenient way. In Fig. 5, a Woodward synthesis is presented in which four times the number of elements are used and a rather good fit with the destination function is attained. It is thus possible to use an "oversized" Woodward synthesized array as a virtual array and calculate the excitations of the actual array from (9). In (9), the destination function values are for the array factor and, when needed, the values must be corrected by the gain function of the element.

IV. Virtual Array Method Combined with Woodward Synthesis

The virtual array synthesis procedure can be started by selecting the size of the virtual $M \times N$ element evenly spaced array. The size of the virtual array must be so large that the result of the Woodward synthesis can be used as a destination function for the final synthesis. The size of the virtual array also determines the number of directions in which the Woodward synthesized beam is known to have exactly the same values as the destination function has. When the actual array is smaller than the virtual array, it cannot have as accurate details in its gain function as the virtual array has and the errors of the gain function in directions between the Woodward directions of the virtual array will partially be filtered out in the final synthesis.

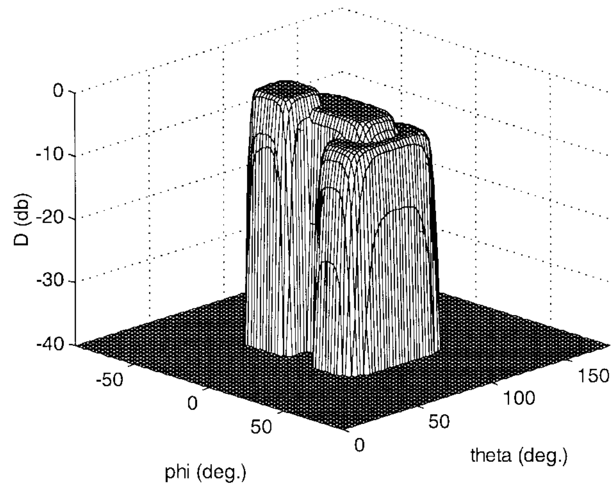

Fig. 1. Destination function "threebox."

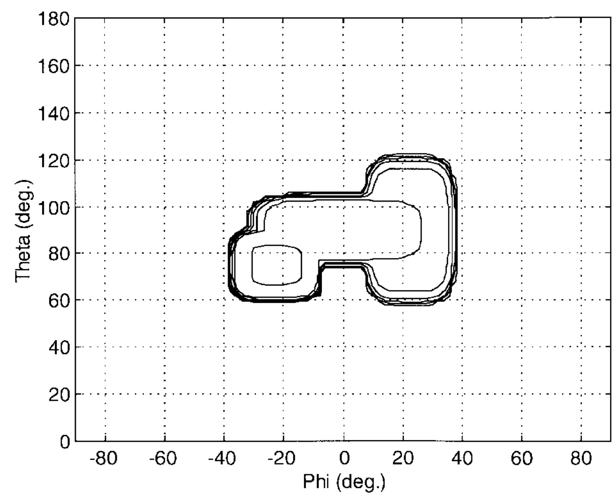

Fig. 2. Contour map of destination function "threebox."

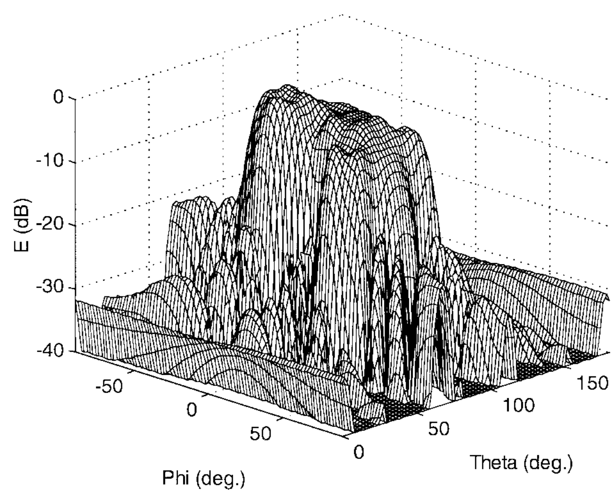

Fig. 3. Woodward synthesis of destination beam in Fig. 1, using a 15 × 15 element array with $d_z = 0.55\lambda$ and $d_y = 0.55\lambda$.

When the geometries of the actual array and the virtual array are selected, the Woodward directions can be calculated from (11), the elements of the matrices $\boldsymbol{S}_{rr}$ and $\boldsymbol{S}_{rv}$ from (7) and (8)

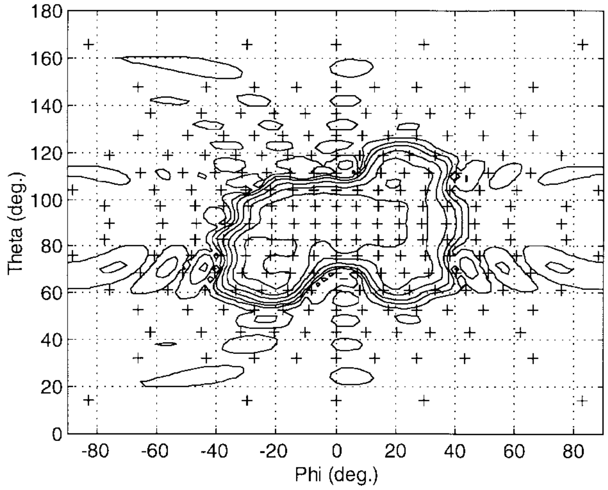

Fig. 4. Contour map of Fig. 3. Levels 0, -3, -6, -10, -15, -20, and -30 dB below the maximum of the destination function are presented. Orthogonal directions $(+)$ are also presented.

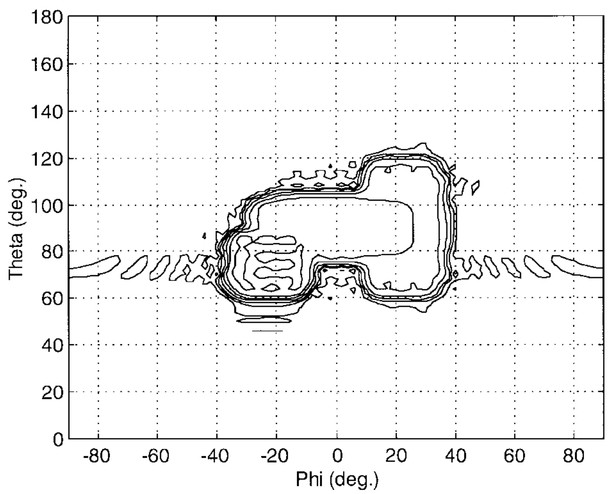

Fig. 5. Woodward synthesis of destination beam in Fig. 1 using a 30 × 30 element array with $d_z = 0.55\lambda$ and $d_y = 0.55\lambda$. In the contour map levels 0, -3, -6, -10, -15, -20, and -30 dB below the maximum of the destination function are presented.

and the elements of the matrix V from (17). The elements of the Λ matrix are the values of the destination function $D(\theta, \varphi)$ in these directions. From (9) and (16), we can determine the unknown excitations A of the actual array

$$\boldsymbol{A} = \boldsymbol{S}_{rr}^{-1} \boldsymbol{S}_{rr} \boldsymbol{V} \Lambda. \tag{18}$$

The element spaces d_z and d_y selected for the virtual array must be so small that the Woodward beams have no grating lobes in the real directions. If the virtual array has any grating lobes, (19) tries to fit the gain function of the actual array also with these grating lobes, too. Also, the actual array can't have any grating lobes. The maximum usable element spaces are readily deduced from (12).

The choice of the size of the virtual array has some influence to the accuracy of the synthesis. The array factor of the Woodward synthesized virtual array may have substantial errors between the orthogonal directions. It is thus advantageous to use virtual array so large that the actual array cannot reproduce

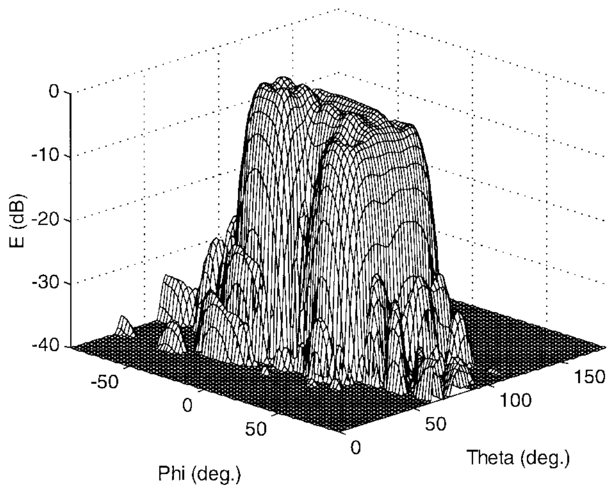

Fig. 6. Virtual array synthesis of a 15 × 15 element array with $d_z = 0.55\lambda$ and $d_y = 0,55\lambda$ using a 48 × 48 element Woodward synthesized array as a virtual destination array.

the finest details of the array factor of the virtual array and the detailed errors are smoothened out in the array factor of the actual array. The maximum dimensions of the virtual array should thus be 1.5–3 times the maximum dimensions of the actual array.

The elements of Λ, which are values of the destination function $D(\theta, \varphi)$ in orthogonal directions, include the phases of the destination function. For finding the optimum choice of the destination pattern phases, (18) must be used as a part of some iterative optimizing process. Usually for contoured beams and a planar array a zero-phased destination beam gives a minimum sidelobe pattern and is a good starting point for selection of the phases.

V. Examples of Virtual Array Synthesis

First, we synthesize the same array, which is synthesized in Figs. 3 and 4 using the virtual array method and a 48 × 48 element array as the virtual array. The synthesis result is presented in Figs. 6 and 7. When this result is compared to the Woodward synthesized result, a much better fit with the destination function and lower sidelobe level is obtained. The amplitudes and phases of the two synthesis results are compared in Appendix A. They are very similar and typical for a zero-phase destination array factor.

Equation (9) also gives a fast way to calculate new excitation values when a certain number of elements are removed from the array. If we calculate from (7) $\boldsymbol{S}_{rr}^{nr}$ for not removed elements, reorder the elements of matrix $\boldsymbol{S}_{rv}$ calculated from (8), and divide $\boldsymbol{S}_{rv}$ into submatrices $\boldsymbol{S}_{rv} = [\boldsymbol{S}_{rr}^{nr} \; \boldsymbol{S}_{rv}^{rnr}]$, we can use the array with nonremoved elements as the virtual array and synthesize the changes in excitations

$$\boldsymbol{S}_{rr}^{nr}(\boldsymbol{A} + \boldsymbol{dA}) = [\boldsymbol{S}_{rr}^{nr} \quad \boldsymbol{S}_{rv}^{rnr}][A \quad A_r]^T. \tag{19}$$

$\boldsymbol{S}_{rv}^{rnr}$ represents terms calculated from (8) between removed and nonremoved elements, $\boldsymbol{A}$ contains the excitations of the nonremoved elements, $\boldsymbol{A}_r$ contains the excitations of the removed elements, and $\boldsymbol{dA}$ represents the excitation changes of the nonremoved elements.

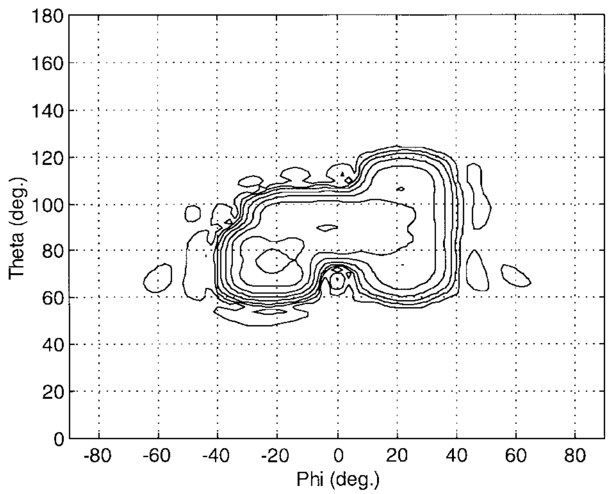

Fig. 7. Contour map of Fig. 6.

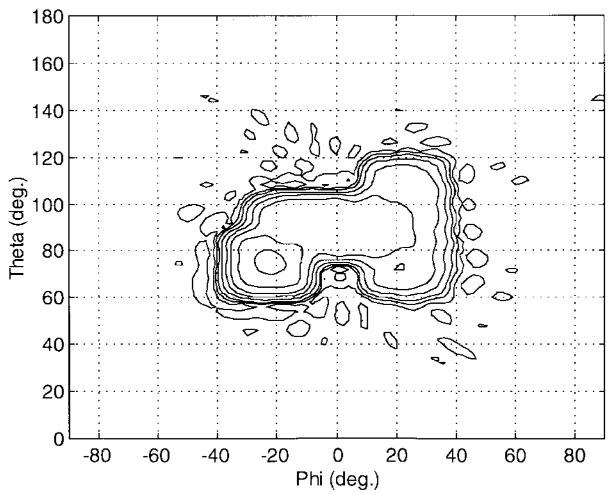

Fig. 8. A 505-element circular array (elements in a grid $d_z = 0,55\lambda$ and $d_y = 0.55\lambda$) synthesized and optimally reduced to a 225 element array using a 81 × 81 element Woodward synthesized array as a virtual destination array. In the contour map levels 0, −3, −6, −10, −15, −20, and −30 dB below the maximum of the destination function are presented.

From (20), the excitation changes of the nonremoved elements are readily determined

$$\boldsymbol{dA} = (\boldsymbol{S}_{rr}^{nr})^{-1}\boldsymbol{S}_{rv}^{rnr}. \tag{20}$$

In Fig. 8, a circular array with 505 elements is synthesized and then reduced in several steps to 225 elements (same number of elements as in the earlier arrays) by removing the elements that have the lowest excitation amplitudes and using (20) to recalculate the new excitation values. The final array geometry is presented in Fig. 8.

Some improvement has been obtained by the proper choice of the element locations compared to Figs. 6 and 7, especially on the slopes of the array factor.

VI. Conclusions

The new virtual-array synthesis method is a very fast and simple synthesis method, especially for large planar arrays when a 2-D Woodward synthesized array can be used as the virtual array. The virtual array method can also be used for any three-dimensional array for which a suitable virtual array geometry and synthesis method can be found.

The solution always minimizes the integral of the squared difference between the array factors of the actual array and the virtual array. In that sense, the solution is optimal, but, on the other hand, there is no way to compromise between the sidelobe level and the fit in the region of the shaped main beam. Excitations of arrays using stretched apertures described in [7] for footprint patterns can be optimized without any iterations or use of the derivatives of the cost function.

The accuracy of the synthesis result is not possible to predict, but by proper selection of the virtual array size and geometry, very good synthesis results can be obtained.

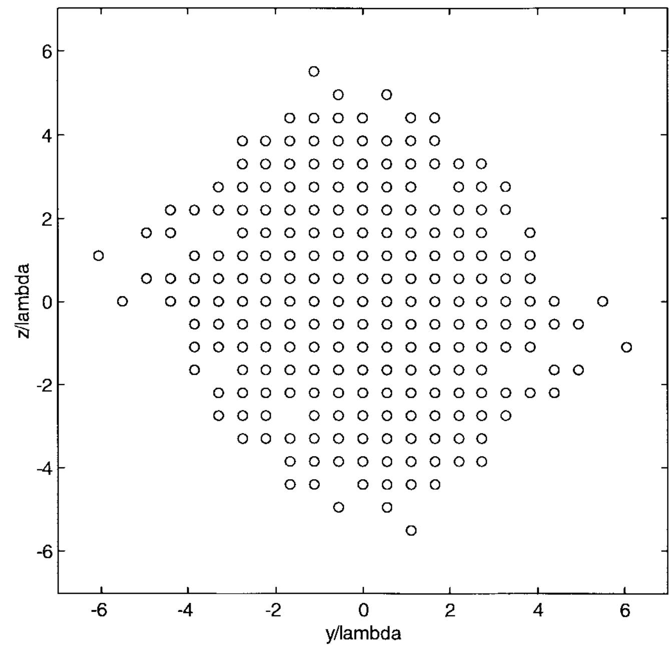

Fig. 9. Final array geometry after element reduction for calculation of Fig. 8.

TABLE I
Excitation Values in Central Line $Y = 0$

	Woodward-synthesis		Virtual-array. synthesis	
z/λ	|a|	φ [°]	|a|	φ [°]
-3.85	0.012	-173.84	0.054	175.31
-3.30	0.060	161.64	0.092	156.64
-2.75	0.032	136.00	0.013	98.03
-2.20	0.037	-92.84	0.053	-107.25
-1.65	0.057	-147.35	0.040	-141.92
-1.10	0.183	37.41	0.193	47.10
-0.55	0.702	14.17	0.786	13.69
0.00	1.000	0.00	1.000	0.00
0.55	0.702	-14.17	0.786	-13.69
1.10	0.183	-37.41	0.193	-47.10
1.65	0.057	147.35	0.040	141.92
2.20	0.037	92.84	0.053	107.25
2.75	0.032	-136.00	0.013	-98.03
3.30	0.060	-161.62	0.092	-156.64
3.85	0.012	173.84	0.054	-175.31

APPENDIX A
EXCITATION VALUES COMPARISON BETWEEN WOODWARD SYNTHESIS AND VIRTUAL ARRAY SYNTHESIS

In Table I, the normalized absolute values and phase angles of the synthesized excitations are calculated for elements in line $y = 0$. Woodward synthesis and virtual array synthesis methods are used (see Figs. 4 and 5).

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Leo I. Vaskelainen was born in Pieksämäki, Finland, in 1944. He received the degree of Dipl.Eng. from Helsinki University of Technology, Espoo, Finland, in 1971.

He is currently a Senior Research Engineer in Technical Research Centre of Finland (VTT), Information Technology, Espoo, responsible for research, design, and development of antenna analysis and synthesis, mobile communication systems, and radar systems. From 1970 to 1975 he worked as an Assistant in Helsinki University of Technology, Radio Laboratory. From 1975 to 1994 he worked in the VTT Telecommunications Laboratory as a Research Engineer, Senior Research Engineer, and as a Section Leader of the Radio Section with responsibilities including research of nuclear electromagnetic pulse (NEMP), electromagnetic compatibility of electrical equipment, antenna techniques, microwave techniques, and radar techniques.