Effective Impedance Boundary Conditions for an Inhomogeneous Thin Layer on a Curved Metallic Surface

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***Abstract*— Effective impedance boundary conditions for an inhomogeneous thin layer coated on a perfectly conducting object are considered. The permittivity of the thin layer is inhomogeneous along both the normal and tangential directions. Explicit forms of the first- and second-order approximate impedance boundary conditions are derived first for a two-dimensional (2-D) thin layer for the TE and TM case. Numerical results are presented. The case of Maxwell's equations for a three-dimensional inhomogeneous thin layer is also considered.**

***Index Terms*—Coatings, electromagnetic scattering, impedance boundary condition.**

I. Introduction

APPROXIMATE boundary conditions, which provide an approximate relation between the electric and magnetic fields on a chosen surface, have been widely used in problems of wave propagation, diffraction, and guidance to simulate the material and geometric properties of surfaces. Such a mathematically derived boundary condition is usually referred to as an effective impedance boundary condition. The general purpose of the effective impedance boundary conditions is to simplify the analytical or numerical solution of wave-scattering problem involving complex structures by, e.g., converting a two (or more) media problem into a single medium problem. Many studies have been carried out in this area (see, e.g., [1]–[8]). However, all are limited to the case of one (or several) homogeneous layer(s).

In a previous paper [9], we have extended the idea to the case of an inhomogeneous layer coated on a planar surface. In the present paper, we consider the effective impedance boundary conditions for an inhomogeneous (along both the normal and tangential directions) thin layer coated on a curved surface. Inhomogeneous thin layers can be found in many applications such as thin gratings, corrugated surfaces, and coated edges or junctions. Due to the inhomogeneity of the coating, the effective impedance boundary conditions are in general nonuniform. The case of a two-dimensional (2-D) inhomogeneous thin layer coated on a metallic cylinder (of an arbitrary cross section) is considered first. An asymptotic expansion of the field solution in a power series of the thickness is used after a suitable scaling along the normal direction with the thickness of the thin layer (see, e.g., [9], [11], [12]). The first- and second-order effective impedance boundary conditions are then derived. In the special case of a homogeneous layer on a metallic circular cylinder, the exact solution of the scattering problem is known and this has been used to check the accuracy of our effective impedance boundary conditions. The general case of Maxwell's equations for a three-dimensional inhomogeneous thin layer on a curved metallic surface is also considered.

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II. Inhomogeneous Thin Layer in Two Dimensions

A. The TE Case and the First-Order Impedance Boundary Condition

In this section, we consider an electromagnetic scattering problem for a 2-D inhomogeneous thin layer coated on a metallic cylinder of an arbitrary smooth cross section Γ.

Outside but sufficiently close to Γ, we denote by $\mathbf{r}_\Gamma$ the orthogonal projection of a point $\mathbf{r}$ on Γ, s a curvilinear abscissa (tangential coordinate) of $\mathbf{r}_\Gamma$, and

$$n = |\mathbf{r} - \mathbf{r}_\Gamma|. \tag{1}$$

Then (s, n) is a parameterization of the neighborhood of the surface Γ. The unit normal to the surface Γ at $\mathbf{r}_\Gamma$ is denoted by $\hat{\mathbf{n}}$. Denote by $c(s, n)$ the curvature at the point (s, n) of the surface

$$\Gamma_n \equiv \{\mathbf{r} = \mathbf{r}_\Gamma + n\hat{\mathbf{n}}\} \tag{2}$$

which is "parallel" to the surface Γ. In a special case when the metallic object is a circular cylinder with radius a, one has

$$c(s, n) = \frac{1}{a + n}.$$

The length element ds_n on the curve Γ_n at the point $\mathbf{r}$ is related to the length element ds on the curve Γ at the point $\mathbf{r}_\Gamma$ by

$$ds_n = [1 + c(s, 0)n]\, ds.$$

Thus, one has

$$\partial_{s_n} = \frac{1}{1 + c(s, 0)n}\, \partial_s. \tag{3}$$

In the present paper, the convention notation for the partial derivative is used, e.g., $\partial_s \equiv \frac{\partial}{\partial s}$. The Laplacian $\triangle$ has the following form in the local coordinates system (s,n) when the field has no variation along the axis of the cylinder (see e.g. [10], [11]):

$$\triangle = \partial_n^2 + c(s,n)\partial_n + \partial_{s_n}^2. \tag{4}$$

Now consider the scattering problem for a 2-D perfectly conducting object (with surface Γ) coated with an inhomogeneous thin layer of a thickness h. Inside the layer the permittivity ϵ is an arbitrary bounded function of s and n, and the permeability has a constant value μ_1. Outside the inhomogeneous layer there is vacuum with a permittivity ϵ_0 and a permeability μ_0. In this section, we consider a TE plane wave (i.e., the electric field is parallel to the axis of the cylinder) impinging on the object from the exterior region. The time-dependence of all fields is assumed to be $e^{-j\omega t}$. Let the electric field $\mathbf{E} = E(s,n)\mathbf{e}_z$, where $\mathbf{e}_z$ is the unit vector along the axis of the cylinder. Then the amplitude $E(s,n)$ of the electric field satisfies the following Helmholtz equation

$$\triangle E + \omega^2 \mu_1 \epsilon(s,n) E = 0. \tag{5}$$

The boundary condition at $n = 0$ is

$$E = 0, \quad n = 0 \tag{6}$$

and the conditions on the surface $n = h$ are

$$E|_{n=h^-} = E|_{n=h^+}, \quad \frac{1}{\mu_1}(\partial_n E)|_{n=h^-} = \frac{1}{\mu_0}(\partial_n E)|_{n=h^+}. \tag{7}$$

Let the variable

$$\tau = n/h. \tag{8}$$

We investigate the asymptotic behavior of $E(s,n;h)$ inside the thin layer as h goes to zero, under the assumption that the "scaled" permittivity profile $\epsilon(s,\tau)$ remains unchanged. More precisely, we want to derive approximate conditions satisfied by $\partial_n E|_{n=h^+}$ and $E|_{n=h^+}$. Inside the thin layer, one can expand $E(s,n;h)$ in the following form:

$$E(s,n;h) = E^{(0)}(s,\tau) + hE^{(1)}(s,\tau) + h^2 E^{(2)}(s,\tau) + \cdots. \tag{9}$$

The curvature $c(s,n)$ can be expanded as

$$c(s,h\tau) = c(s,0) + h\tau c'(s,0) + \cdots \tag{10}$$

where $c'(s,0) = [\partial_n c(s,n)]_{n=0}$.

The impedance boundary condition that we are looking for is an approximate relation between $\partial_n E(s,n;h)$ and $E(s,n;h)$ at the surface $n = h^+$. From (9) and (7), one obtains

$$\begin{aligned} E|_{n=h^+} &= E^{(0)}|_{\tau=1} + hE^{(1)}|_{\tau=1} + h^2 E^{(2)}|_{\tau=1} \\ &\quad + h^3 E^{(3)}|_{\tau=1} + \cdots \end{aligned} \tag{11}$$

$$\begin{aligned} \partial_n E|_{n=h^+} &= \frac{\mu_0}{\mu_1 h}\big[\partial_\tau E^{(0)}|_{\tau=1} + h\partial_\tau E^{(1)}|_{\tau=1} \\ &\quad + h^2 \partial_\tau E^{(2)}|_{\tau=1} \\ &\quad + h^3 \partial_\tau E^{(3)}|_{\tau=1} + \cdots\big]. \end{aligned} \tag{12}$$

Note that the derivatives in the expression (4) have the following forms:

$$\begin{aligned} \partial_n &= h^{-1}\partial_\tau \\ \partial_{s_n}^2 &= \partial_s^2 - [2c(s,0)\tau\partial_s^2 + \partial_s c(s,0)\tau\partial_s]h + \cdots. \end{aligned}$$

Thus, substituting (9) into the Helmholtz equation (5) and matching the coefficients of the h^{-2}, h^{-1}, h^0, $\cdots$, terms, respectively, one obtains

$$\partial_\tau^2 E^{(0)} = 0 \tag{13}$$

$$\partial_\tau^2 E^{(1)} + c(s,0)\partial_\tau E^{(0)} = 0 \tag{14}$$

$$\begin{aligned} &\partial_\tau^2 E^{(2)} + \partial_s^2 E^{(0)} + c(s,0)\partial_\tau E^{(1)} + \tau c'(s,0)\partial_\tau E^{(0)} \\ &\quad + \omega^2 \mu_1 \epsilon(s,\tau) E^{(0)} = 0. \end{aligned} \tag{15}$$

$$\cdots$$

The boundary condition (6) becomes

$$E^{(1)}(s,0) + hE^{(1)}(s,0) + h^2 E^{(2)}(s,0) + \cdots = 0$$

which gives

$$E^{(i)}(s,0) = 0, \quad i = 0,1,2,\cdots. \tag{16}$$

It then follows from (13) and (14) that

$$E^{(0)}(s,\tau) = C_0(s)\,\tau \tag{17}$$

$$E^{(1)}(s,\tau) = -\frac{\tau^2}{2}c(s,0)C_0(s) + C_1(s)\,\tau \tag{18}$$

where $C_0(s)$ and $C_1(s)$ are certain functions depending only on the tangential coordinate s. One thus obtains

$$\begin{aligned} &\partial_\tau E^{(0)}|_{\tau=1} + h\partial_\tau E^{(1)}|_{\tau=1} \\ &\quad = C_0(s) + hC_1(s) - hc(s,0)C_0(s) \end{aligned} \tag{19}$$

$$\begin{aligned} &E^{(0)}|_{\tau=1} + hE^{(1)}|_{\tau=1} \\ &\quad = C_0(s) + hC_1(s) - \frac{1}{2}hc(s,0)C_0(s). \end{aligned} \tag{20}$$

Taking the first two terms in (11) and (12) and using the relation (20), one obtains the following first-order approximate impedance boundary condition on the surface $n = h^+$ (noting that $C_0(s) = E^{(0)}|_{\tau=1}$):

$$E|_{n=h^+} - \frac{\mu_1}{\mu_0}h(\partial_n E)|_{n=h^+} - \frac{1}{2}hc(s,0)E|_{n=h^+} = 0. \tag{21}$$

Thus, to calculate the scattered field outside the inhomogeneous thin layer, one can replace the original scattering problem with the following simple boundary-value problem when the thickness h is much less than the wavelength:

$$\begin{cases} \triangle E + \omega^2 \mu_0 \epsilon_0 E = 0 & n \geq h \\ E - \frac{\mu_1}{\mu_0}h\,\partial_n E - \frac{1}{2}h\,c(s,0)E = 0, & n = h \\ E - E^{\text{in}} \text{ satisfies the radiation condition at infnity.} & \end{cases} \tag{22}$$

One notes that the permittivity profile $\epsilon(s,\tau)$ does not appear in the first-order approximate impedance boundary condition (21). In the next subsection, we derive a second-order impedance boundary condition in which $\epsilon(s,\tau)$ appears.

B. Second-Order Impedance Boundary Condition

Substituting (17) and (18) into (15) yields

$$\partial_\tau^2 E^{(2)} = -\omega^2\mu_1\tau\epsilon(s,\tau)C_0 - \tau\left[\partial_s^2 C_0(s) - c^2(s,0)C_0(s) + c'(s,0)C_0(s)\right] - c(s,0)C_1(s). \quad (23)$$

Integrating the above equation with respect to τ, one obtains

$$\partial_\tau E^{(2)} = -\omega^2\mu_1 C_0(s)\int_0^\tau \tau_1\epsilon(s,\tau_1)\,d\tau_1 - \frac{1}{2}\tau^2\left[\partial_s^2 C_0(s) - c^2(s,0)C_0(s) + c'(s,0)C_0(s)\right] - c(s,0)C_1(s)\tau + C(s) \quad (24)$$

$$E^{(2)} = -\omega^2\mu_1 C_0(s)\int_0^\tau\left[\int_0^{\tau_1}\tau_2\epsilon(s,\tau_2)\,d\tau_2\right]d\tau_1 - \frac{1}{6}\tau^3\left[\partial_s^2 C_0(s) - c^2(s,0)C_0(s) + c'(s,0)C_0(s)\right] - \frac{1}{2}c(s,0)C_1(s)\tau^2 + C(s)\tau \quad (25)$$

where $C(s)$ is a certain function depending only on the abscissa s. Putting $\tau = 1$ in (25), one obtains

$$C(s) = E^{(2)}|_{\tau=1} + \omega^2\mu_1 C_0(s)\int_0^1\left[\int_0^{\tau_1}\tau_2\epsilon(s,\tau_2)\,d\tau_2\right]d\tau_1 + \frac{1}{6}\left[\partial_s^2 C_0(s) - c^2(s,0)C_0(s) + c'(s,0)C_0(s)\right] + \frac{1}{2}c(s,0)C_1(s). \quad (26)$$

Substituting (26) into (24) with $\tau = 1$ yields

$$\partial_\tau E^{(2)}|_{\tau=1} = E^{(2)}|_{\tau=1} - \frac{1}{3}\omega^2\mu_1 C_0(s)\tilde{\epsilon}(s) - \frac{1}{3}\left[\partial_s^2 C_0(s) - c^2(s,0)C_0(s) + c'(s,0)C_0(s)\right] - \frac{1}{2}c(s,0)C_1(s). \quad (27)$$

where

$$\tilde{\epsilon}(s) = 3\left\{\int_0^1\tau_1\epsilon(s,\tau_1)\,d\tau_1 - \int_0^1\left[\int_0^{\tau_1}\tau_2\epsilon(s,\tau_2)\,d\tau_2\right]d\tau_1\right\}. \quad (28)$$

In the special case when the thin layer has no variation along the normal direction, i.e., $\epsilon(s,\tau) = \epsilon(s)$, one has $\tilde{\epsilon}(s) = \epsilon(s)$.

Since

$$C_0(s) = E^{(0)}|_{\tau=1} \quad (29)$$

$$C_1(s) = E^{(1)}|_{\tau=1} + \frac{1}{2}c(s,0)E^{(0)}|_{\tau=1} \quad (30)$$

one obtains from (19) and (27) that

$$\begin{aligned}
&\partial_\tau E^{(0)}|_{\tau=1} + h\partial_\tau E^{(1)}|_{\tau=1} + h^2\partial_\tau E^{(2)}|_{\tau=1}\\
&= E^{(0)}|_{\tau=1} + h\left[E^{(1)}|_{\tau=1} + \frac{1}{2}c(s,0)E^{(0)}|_{\tau=1}\right]\\
&\quad - hc(s,0)E^{(0)}|_{\tau=1} + h^2E^{(2)}|_{\tau=1}\\
&\quad - \frac{1}{3}h^2\omega^2\mu_1E^{(0)}|_{\tau=1}\tilde{\epsilon}(s) - \frac{1}{3}h^2\left[\partial_s^2E^{(0)}\right|_{\tau=1}\\
&\quad - c^2(s,0)E^{(0)}|_{\tau=1} + c'(s,0)E^{(0)}|_{\tau=1}]\\
&\quad - \frac{1}{2}h^2c(s,0)\left[E^{(1)}|_{\tau=1} + \frac{1}{2}c(s,0)E^{(0)}|_{\tau=1}\right]\\
&= \left[E^{(0)}|_{\tau=1} + hE^{(1)}|_{\tau=1} + h^2E^{(2)}|_{\tau=1}\right]\\
&\quad - \frac{1}{2}hc(s,0)\left[E^{(0)}|_{\tau=1} + hE^{(1)}|_{\tau=1}\right]\\
&\quad - \frac{1}{3}h^2\omega^2\mu_1E^{(0)}|_{\tau=1}\tilde{\epsilon}(s) - \frac{1}{3}h^2\left[\partial_s^2E^{(0)}|_{\tau=1} - \frac{1}{4}c^2(s,0)E^{(0)}|_{\tau=1} + c'(s,0)E^{(0)}|_{\tau=1}\right]. \quad (31)
\end{aligned}$$

One thus obtains the following second-order approximate impedance boundary condition [cf., (11), (12)]:

$$\frac{\mu_1}{\mu_0}h\,\partial_n E = E - \frac{1}{2}hc(s,0)\,E - \frac{1}{3}h^2\omega^2\,\mu_1\,\tilde{\epsilon}(s)\,E - \frac{1}{3}h^2\left[\partial_s^2E - \frac{1}{4}c^2(s,0)\,E + c'(s,0)E\right] \qquad n = h^+. \quad (32)$$

Therefore, to calculate the scattered field outside the inhomogeneous thin layer, one can replace the original scattering problem with the following boundary-value problem when the thickness h is much less than the wavelength:

$$\begin{cases}\triangle E + \omega^2\mu_0\epsilon_0 E = 0, & n \geq h\\ E - \frac{\mu_1}{\mu_0}h\,\partial_n E - \frac{1}{2}hc(s,0)E - \frac{1}{3}h^2\omega^2\mu_1\,\tilde{\epsilon}\,E & \\ \quad -\frac{1}{3}h^2\left[\partial_s^2E - \frac{1}{4}c^2(s,0)\,E + c'(s,0)\,E\right] = 0, & n = h\\ E - E^{\text{in}} \text{ satisfies the radiation condition at infnity} & \end{cases} \quad (33)$$

where $\tilde{\epsilon}(s)$ is given by (28).

1) Numerical Example: As a numerical example, we consider a TE plane wave incident on a scatterer consisting of a perfectly conducting circular cylinder (with radius a) and a dielectric coating layer (with thickness h). Thus, one has $c(s,n) = \frac{1}{a+n}$. The permeability has a constant value μ_0 in the whole space. The total electric fields on the surface of the cylindrical scatterer are calculated by solving the system (33) (see the Appendix) and are shown in Fig. 2 for a homogeneous dielectric coating with $\epsilon = 2\epsilon_0$ or an inhomogeneous dielectric coating with

$$\epsilon = \epsilon_0\left(3 - \frac{r-a}{h}\cos\theta\right), \qquad a < r < a+h.$$

In this example, we choose $ka = 5$ and $kh = 0.2\pi$. In the case of a homogeneous dielectric coating, the exact solution of the scattering problem for the homogeneous dielectric coating is known (see the Appendix) and this is used to check the accuracy of our effective impedance boundary conditions in this numerical example. Fig. 2 shows that the numerical

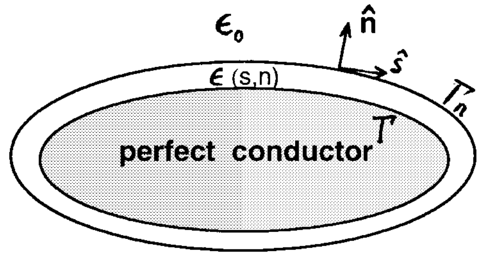

Fig. 1. The scattering configuration.

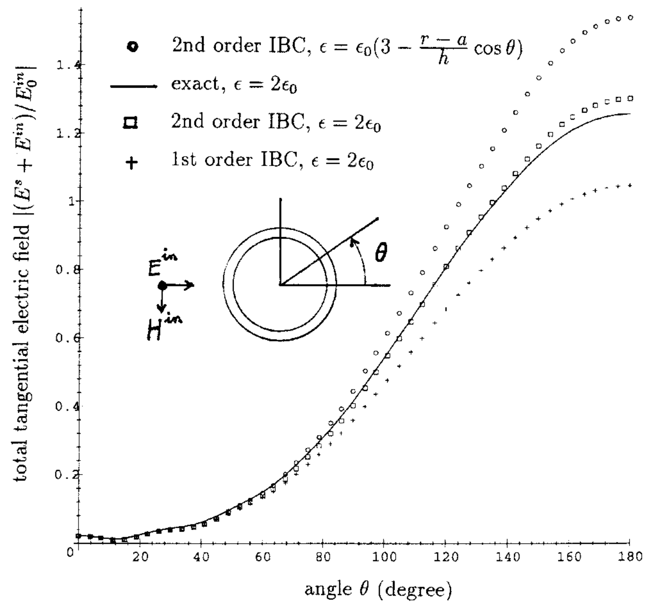

Fig. 2. The total electric field on the surface of a scatterer consisting of a perfectly conducting circular cylinder (with radius a) and a thin dielectric coating layer (with thickness h). $ka = 5$, $kh = 0.2\pi$.

result calculated with the second-order impedance boundary condition (the boxes in Fig. 2) is in good agreement with the exact solution (the solid curve in Fig. 2) for the homogeneous dielectric thin coating.

C. The TM Case

For the TM case, the magnetic field $\mathbf{H} = H(s,n)\mathbf{e}_z$. The boundary condition is

$$\partial_n H(s,n) = 0, \qquad n = 0. \tag{34}$$

A similar asymptotic analysis can be carried out without any extra difficulty to derive the approximate boundary conditions for an inhomogeneous thin layer in the TM case. For example, when the layer has a permeability $\mu(s,n)$ and a constant permittivity ϵ_0, one can obtain the following approximate impedance boundary conditions:

- first-order approximate impedance boundary condition

$$\partial_n H + h\omega^2\epsilon_0\tilde{\mu}(s)H + h\partial_s^2 H = 0 \tag{35}$$

- the second-order approximate impedance boundary condition

$$\begin{aligned}\partial_n H + h\omega^2\epsilon_0\tilde{\mu}(s)H + h\partial_s^2 H + \frac{h^2}{2}\partial_s[c(s,0)\partial_s H]\\ - c(s,0)h^2\omega^2\epsilon_0 \tilde{\tilde{\mu}}(s)H = 0\end{aligned} \tag{36}$$

where

$$\tilde{\mu}(s) = \int_0^1 \mu(s,\tau)\,d\tau \tag{37}$$

$$\tilde{\tilde{\mu}}(s) = \int_0^1\int_0^\tau \mu(s,\tau_1)\,d\tau_1\,d\tau. \tag{38}$$

III. Inhomogeneous Thin Layer in Three Dimensions

In this section, we derive an effective impedance boundary condition for an inhomogeneous thin dielectric layer for which both permittivity and permeability depend on all three-space coordinates. The thin dielectric layer is coated on a metallic object (with a smooth surface Γ). Inside the thin layer, one has the following Maxwell's equations:

$$\nabla \times \mathbf{E} = j\omega\mu\Big(s, \frac{n}{h}\Big)\mathbf{H} \tag{39}$$

$$\nabla \times \mathbf{H} = -j\omega\epsilon\Big(s, \frac{n}{h}\Big)\mathbf{E} \tag{40}$$

where $s = (s_1, s_2)$ is the curvilinear abscissas, and (s,n) is the parameterization of the neighborhood of the surface Γ as described in the previous section. In this section, we assume that the parameters ϵ and μ are bounded and piecewise differentiable with respect to s. The boundary condition is

$$\mathbf{E} \times \hat{\mathbf{n}} = 0, \quad n = 0. \tag{41}$$

On the surface $n = h$ the tangential components of the electric and magnetic fields (i.e., $\mathbf{E} \times \hat{\mathbf{n}}$ and $\mathbf{H} \times \hat{\mathbf{n}}$) are continuous.

Let Γ_n $(n > 0)$ be the family of the surfaces parallel to Γ (the surface of the perfect conductor), i.e., $\Gamma_n = \{(s,n), s \in \Gamma\}$. For any smooth function u or any vector field $\mathbf{v}$ defined on a surface Γ_n, one can get an extension of each one in a neighborhood of the surface Γ_n by setting

$$\tilde{u}(\mathbf{r}) = u(\mathbf{r}_{\Gamma_n}), \quad \tilde{\mathbf{v}}(\mathbf{r}) = \mathbf{v}(\mathbf{r}_{\Gamma_n})$$

where $\mathbf{r}_{\Gamma_n}$ is the orthogonal projection of $\mathbf{r}$ on the surface Γ_n. One can then define the following surface differential operators (see e.g. [10]):

1) the surface divergence of a tangential field $\mathbf{v}$ on Γ_n: $\mathrm{div}_{\Gamma_n}\mathbf{v} \equiv (\mathrm{div}\,\tilde{\mathbf{v}})|_{\Gamma_n}$;
2) the vector rotational of a function on Γ_n: $\mathbf{curl}_{\Gamma_n} u \equiv [\mathbf{curl}(u\hat{\mathbf{n}})]|_{\Gamma_n}$;
3) the scalar rotational of a tangential field on Γ_n: $\mathrm{curl}_{\Gamma_n}\mathbf{v} \equiv \hat{\mathbf{n}} \cdot (\mathbf{curl}\,\tilde{\mathbf{v}})|_{\Gamma_n}$.

Then, one has the following formulas:

$$\begin{aligned}\nabla \times \mathbf{v} = (\mathrm{curl}_{\Gamma_n}\mathbf{v}_{\Gamma_n})\hat{\mathbf{n}} + \mathbf{curl}_{\Gamma_n}(\mathbf{v}\cdot\hat{\mathbf{n}})\\ + [R_n - c(s,n)]\mathbf{v}\times\hat{\mathbf{n}} - \partial_n(\mathbf{v}\times\hat{\mathbf{n}})\end{aligned} \tag{42}$$

$$\nabla \cdot \mathbf{v} = (\mathrm{div}_{\Gamma_n}\mathbf{v}_{\Gamma_n}) + c(s,n)\mathbf{v}\cdot\hat{\mathbf{n}} + \partial_n(\mathbf{v}\cdot\hat{\mathbf{n}}) \tag{43}$$

where R_n is the tensor of curvature at (s,n), $c(s,n)$ is the mean curvature at (s,n), and $\mathbf{v}_\Gamma$ is the orthogonal projection of the vector $\mathbf{v}$ on Γ, i.e.,

$$\mathbf{v}_\Gamma = -\hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \mathbf{v}. \tag{44}$$

We introduce the following asymptotic expansions of the electric and magnetic fields:

$$\begin{cases} \mathbf{E}(s,n;h) = \mathbf{E}^{(0)}(s,\tau) + h\mathbf{E}^{(1)}(s,\tau) + h^2\mathbf{E}^{(2)}(s,\tau) + \cdots, \\ \mathbf{H}(s,n;h) = \mathbf{H}^{(0)}(s,\tau) + h\mathbf{H}^{(1)}(s,\tau) + h^2\mathbf{H}^{(2)}(s,\tau) + \cdots \end{cases} \tag{45}$$

where the variable τ is defined by (8).

To obtain an effective boundary condition, we need to find a relation between the tangential component of the electric field and the tangential component of the magnetic field. Substituting (45) into Maxwell's equations (39) and (40) and matching the coefficients of the h^{-1}, h^0, h, h^2, $\cdots$, terms, respectively, one obtains

$$\partial_\tau(\mathbf{E}^{(0)} \times \hat{\mathbf{n}}) = 0 \tag{46}$$

$$\partial_\tau(\mathbf{H}^{(0)} \times \hat{\mathbf{n}}) = 0 \tag{47}$$

$$\begin{aligned} &\partial_\tau(\mathbf{E}^{(1)} \times \hat{\mathbf{n}}) + \left(\mathrm{curl}_\Gamma \mathbf{E}_\Gamma^{(0)}\right)\hat{\mathbf{n}} + \mathbf{curl}_\Gamma(\mathbf{E}^{(0)} \cdot \hat{\mathbf{n}}) \\ &\quad + [R_n|_{n=0} - c(s,0)]\mathbf{E}^{(0)} \times \hat{\mathbf{n}} - \partial_n(\mathbf{E}^{(0)} \times \hat{\mathbf{n}}) \\ &\quad = j\omega\mu(s,\tau)\mathbf{H}^{(0)} \end{aligned} \tag{48}$$

$$\begin{aligned} &\partial_\tau(\mathbf{H}^{(1)} \times \hat{\mathbf{n}}) + \left(\mathrm{curl}_\Gamma \mathbf{H}_\Gamma^{(0)}\right)\hat{\mathbf{n}} + \mathbf{curl}_\Gamma(\mathbf{H}^{(0)} \cdot \hat{\mathbf{n}}) \\ &\quad + [R_n|_{n=0} - c(s,0)]\mathbf{H}^{(0)} \times \hat{\mathbf{n}} - \partial_n(\mathbf{H}^{(0)} \times \hat{\mathbf{n}}) \\ &\quad = -j\omega\epsilon(s,\tau)\mathbf{E}^{(0)} \end{aligned} \tag{49}$$

$$\cdots.$$

The boundary condition (41) becomes

$$(\mathbf{E}^{(i)} \times \hat{\mathbf{n}})|_{\tau=0} = 0, \quad i = 0,1,2,3,\cdots. \tag{50}$$

From (46) and (50), one obtains

$$(\mathbf{E}^{(0)} \times \hat{\mathbf{n}})(s,\tau) = 0. \tag{51}$$

Substituting (45) into the condition $\nabla \cdot (\epsilon\mathbf{E}) = \nabla \cdot (\mu\mathbf{H}) = 0$ [obtained by taking the divergence of Eqs. (39) and (40)] and matching the coefficients of the h^{-1}, h^0, h, h^2, $\cdots$, terms, respectively, one obtains

$$\partial_\tau[\epsilon(s,\tau)\mathbf{E}^{(0)} \cdot \hat{\mathbf{n}}] = 0 \tag{52}$$

$$\partial_\tau[\mu(s,\tau)\mathbf{H}^{(0)} \cdot \hat{\mathbf{n}}] = 0 \tag{53}$$

$$\cdots$$

which gives

$$(\mathbf{E}^{(0)} \cdot \hat{\mathbf{n}})(s,\tau) = \frac{A_1(s)}{\epsilon(s,\tau)} \tag{54}$$

$$(\mathbf{H}^{(0)} \cdot \hat{\mathbf{n}})(s,\tau) = \frac{A_2(s)}{\mu(s,\tau)} \tag{55}$$

where $A_1(s)$ and $A_2(s)$ are certain functions depending only on s. The normal components of (48) and (49) give

$$\mathrm{curl}_\Gamma \mathbf{E}_\Gamma^{(0)} = j\omega\mu(s,\tau)\mathbf{H}^{(0)} \cdot \hat{\mathbf{n}} \tag{56}$$

$$\mathrm{curl}_\Gamma \mathbf{H}_\Gamma^{(0)} = -j\omega\epsilon(s,\tau)\mathbf{E}^{(0)} \cdot \hat{\mathbf{n}}. \tag{57}$$

From the above four equations, one obtains

$$A_1(s) = -\frac{j}{\omega}\mathrm{curl}_\Gamma \mathbf{E}_\Gamma^{(0)} \tag{58}$$

$$A_2(s) = \frac{j}{\omega}\mathrm{curl}_\Gamma \mathbf{H}_\Gamma^{(0)}. \tag{59}$$

Therefore, it follows from the tangential component of (48) that [cf., (51)]

$$\begin{aligned} \partial_\tau(\mathbf{E}^{(1)} \times \hat{\mathbf{n}}) = &-\mathbf{curl}_\Gamma\left[\frac{j}{\omega\epsilon(s,\tau)}\mathrm{curl}_\Gamma \mathbf{H}_\Gamma^{(0)}\right] \\ &+ j\omega\mu(s,\tau)\mathbf{H}_\Gamma^{(0)}. \end{aligned} \tag{60}$$

Integrating the above equation from $\tau = 0$ to $\tau = 1$, one obtains (noting that $(\mathbf{E}^{(1)} \times \hat{\mathbf{n}})|_{\tau=0} = 0$)

$$\begin{aligned} (\mathbf{E}^{(1)} \times \hat{\mathbf{n}})|_{\tau=1} = &-\frac{j}{\omega}\left[\int_0^1 \frac{d\tau}{\epsilon(s,\tau)}\right]\mathbf{curl}_\Gamma \mathrm{curl}_\Gamma \mathbf{H}_\Gamma^{(0)} \\ &- \frac{j}{\omega}\left(\int_0^1 \mathbf{curl}_\Gamma\left[\frac{1}{\epsilon(s,\tau)}\right]d\tau\right)\mathrm{curl}_\Gamma \mathbf{H}_\Gamma^{(0)} \\ &+ j\omega\left[\int_0^1 \mu(s,\tau)\,d\tau\right]\mathbf{H}_\Gamma^{(0)}. \end{aligned} \tag{61}$$

Therefore, one obtains the following approximate impedance boundary condition on the surface $n = h^+$ [cf., (45) and (51)]:

$$\begin{aligned} &(\mathbf{E} \times \hat{\mathbf{n}}) = \\ &jh\left\{\frac{1}{\omega}\left[\int_0^1 \frac{d\tau}{\epsilon(s,\tau)}\right]\mathbf{curl}_\Gamma \mathrm{curl}_\Gamma(\hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \mathbf{H})\right. \\ &\quad + \frac{1}{\omega}\left(\int_0^1 \mathbf{curl}_\Gamma\left[\frac{1}{\epsilon(s,\tau)}\right]d\tau\right)\mathrm{curl}_\Gamma(\hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \mathbf{H}) \\ &\quad \left. - \omega\left[\int_0^1 \mu(s,\tau)\,d\tau\right](\hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \mathbf{H})\right\}. \end{aligned} \tag{62}$$

In a similar way, one can derive higher order impedance boundary conditions.

IV. Conclusion

In the present paper, we have derived the effective impedance boundary conditions for an inhomogeneous (along both the normal and tangential directions) thin layer coated on a curved metallic surface. Explicit forms of the first-order and second-order effective impedance boundary conditions have been derived through an asymptotic analysis after a suitable scaling with the thickness. The present effective impedance boundary conditions are useful in simplifying the analytical or numerical solution of wave scattering problem involving complex structures.

Appendix
Numerical Solution for (33) in the Case of Circular Cylinder

In this appendix, we describe a numerical method for solving the boundary-value problem (33) when the metallic object is a circular cylinder with radius a. For this special case, one has $c(s,n) = \frac{1}{a+n}$. Consider a plane wave incident on the cylinder (coated with a dielectric layer of thickness h; the permeability in the whole space is μ_0), as depicted by the small scattering configuration in Fig. 2. Thus, one has

$$E^{\mathrm{in}} = E_0^{\mathrm{in}} e^{jkr\cos\theta} = E_0^{\mathrm{in}} \sum_{m=-\infty}^{\infty} (j)^m J_m(kr) e^{jm\theta} \tag{A.1}$$

where $k = \omega\sqrt{\epsilon_0\mu_0}$ is the incident wave number and $J_m(x)$ is the Bessel function of order m. The scattered field outside the scatterer can be expanded according to

$$E^s = E_0^{\text{in}} \sum_{m=-\infty}^{\infty} (j)^m S_m H_m^{(1)}(kr) e^{jm\theta} \tag{A.2}$$

where $H_m^{(1)}$ is the Hankel function of the first kind. Substituting (A.2) into the first-order impedance boundary condition (21), one obtains the following approximation for the scattered field on the surface $r = a + h$:

$$\begin{aligned} E^s|_{r=a+h} \approx E^{s(1)}|_{r=a+h} &= -E_0^{\text{in}} \sum_{m=-\infty}^{\infty} (j)^m \\ &\times \frac{\left(1-\frac{h}{2a}\right)J_m(k(a+h)) - khJ'_m(k(a+h))}{\left(1-\frac{h}{2a}\right)H_m^{(1)}(k(a+h)) - khH_m'^{(1)}(k(a+h))} \\ &\times H_m^{(1)}(k(a+h))e^{jm\theta} \end{aligned} \tag{A.3}$$

where $J'_m(x)$ and $H_m'^{(1)}(x)$ denote the derivatives of $J_m(x)$ and $H_m^{(1)}(x)$, respectively. For the second-order impedance boundary condition (32), we first neglect the small inhomogeneous term $\frac{\tilde{\epsilon}}{3\epsilon_0}h^2k^2$, then solve the problem with the remaining homogeneous boundary condition, and finally add the inhomogeneous term $\frac{\tilde{\epsilon}}{3\epsilon_0}h^2k^2$ back to the solution. The final approximate result is

$$\begin{aligned} E^s|_{r=a+h} &\approx E^{s(2)}|_{r=a+h} \\ &= E_0^{s(2)}|_{r=a+h} + \frac{k^2h^2}{3\epsilon_0}\tilde{\epsilon}(\theta)\left(E^{\text{in}} + E_0^{s(2)}\right) \end{aligned} \tag{A.4}$$

where $\tilde{\epsilon}(\theta)$ is calculated by (28) and

$$\begin{aligned} E_0^{s(2)} &= -E_0^{\text{in}} \sum_{m=-\infty}^{\infty} (j)^m \\ &\times \frac{b_m J_m(k(a+h)) - khJ'_m(k(a+h))}{b_m H_m^{(1)}(k(a+h)) - khH_m'^{(1)}(k(a+h))} \\ &\times H_m^{(1)}(k(a+h))e^{jm\theta}. \end{aligned} \tag{A.5}$$

and where $b_m = 1 - \frac{h}{2a} + \frac{h^2}{4a^2} + \frac{m^2h^2}{3(a+h)^2}$ [here $\frac{m^2h^2}{3(a+h)^2}$ is due to the term $\frac{1}{3}h^2\partial_s^2 E|_{n=h^+}$ in (32)].

When the coating dielectric layer is homogeneous, one can obtain an exact solution given by (A.2) with [13]

$$S_m = \frac{k_2 J_m(k(a+h))P_m - kJ'_m(k(a+h))Q_m}{-k_2 H_m^{(1)}(k(a+h))P_m + kH_m'^{(1)}(k(a+h))Q_m} \tag{A.6}$$

where $k_2 = k\sqrt{\epsilon/\epsilon_0}$, and

$$\begin{aligned} P_m &= H_m'^{(1)}(k_2(a+h))H_m^{(2)}(k_2 a) \\ &\quad - H_m'^{(2)}(k_2(a+h))H_m^{(1)}(k_2 a), \\ Q_m &= H_m^{(1)}(k_2(a+h))H_m^{(2)}(k_2 a) \\ &\quad - H_m^{(2)}(k_2(a+h))H_m^{(1)}(k_2 a). \end{aligned}$$

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