A Multiresolution Study of Effective Properties of Complex Electromagnetic Systems

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***Abstract*— A systematic study of the across scale coupling phenomenology in electromagnetic (EM) scattering problems is addressed using the theory of multiresolution decomposition and orthogonal wavelets. By projecting an integral equation formulation of the scattering problem onto a set of subspaces that constitutes a multiresolution decomposition of $L_2(\mathbf{R})$, one can derive two coupled formulations. The first governs the macroscale response, and the second governs the microscale response. By substituting the formal solution of the latter in the former, a new *self-consistent* formulation that governs the macroscale response component is obtained. This formulation is written on a macroscale grid, where the effects of the microscale heterogeneity are expressed via an across scale coupling operator. This operator can also be interpreted as representing the *effective properties* of the microstructure. We study the properties of this operator versus the characteristics of the Green function and the microstructure for various electromagnetic problems, using general asymptotic considerations. A specific numerical example of TM scattering from a laminated complex structure is provided.**

***Index Terms*—Electromagnetic scattering, wavelet transforms.**

I. Introduction

A fundamentally difficult problem in wave theory is the prediction of the response field when an excitation is applied to a structure that is described over a wide range of length scales. Coupling across scales is the genesis of these difficulties. From the mathematical point of view it stems from the essentially nonlinear dependence of a system response on the coefficients of the governing equations. This nonlinearity exists even if the governing formulation is linear, as is the case in many problems of interest.

When a time-harmonic excitation is applied to such a system, it is convenient to use the wavelength λ as a discriminator of the various length scales pertaining to the problem. Although not precisely defined yet, this wavelength would usually be that associated with the response field of a corresponding "background" or homogeneous problem. We shall refer to the scales in the order of λ and above as the *macroscale* and to the range of scales much smaller than λ (say, $\lambda/50$) as the *microscale*. A complexity frequently encountered in electromagnetic or acoustic scattering is the one characterized by a microscale heterogeneity that extends over physical domains measured on the macroscale. This heterogeneity—nonstationary when observed over the macroscale dimensions—is referred to as a "complex scatterer." The traditional scattering formulations govern the complete response of the body and require a complete description of the system heterogeneity (by "complete" here is meant the incorporation of the entire range of length scales.) The inherent completeness is a source of great difficulties for the following reasons.

1) A computational approach for the solution of such scattering problems requires a numerical grid sufficiently large to cover the entire heterogeneity, yet sufficiently fine to capture the microscale component of the heterogeneity—the numerical dimension of which may become too large to handle.
2) Any change made in either the micro or the macrostructure formally introduces a completely new problem—one defined again on a microscale grid of macroscale outer dimensions; that is, there is no reduction in the required computations occasioned by a lack of change of either the micro or the macrostructure.

A step in the direction of resolving some of these difficulties is the recognition that "fine details" of the body *response* associated with heterogeneity possessing the complexity articulated above, reside essentially in the evanescent spectrum domain, thus generating a vanishingly small contribution to the radiation far field. A formulation of the scattering problem associated with complex scatterer, that is *a priori* tuned to adequately describe the response component measured on the λ scale (but not smaller) is, therefore, suggestive. This formulation is termed here as the *formulation smooth*. The governed response—a smoothed version of the complete response—is termed here as the *macroscale response*. Now, it is intuitively acceptable that a numerical scheme discretization based on a macroscale grid would not yield predictions of the macroscale response of satisfactory accuracy if no further precautions designed to capture the effects of the microscale heterogeneity on the macroscale response are exercised. It is further recognized that the coupling between the microscale heterogeneity and the macroscale response is essentially a nonlinear process. Thus, in general, the latter cannot be characterized as a response governed by the same scattering formulation with the heterogeneity simply made smooth. (Hence, the terminology; a smooth formulation is that obtained by the heterogeneity made smooth while the *formulation smooth* is the formulation governing the smoothed response. They are not the same.)

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The above observations have motivated our study. The purpose of the present work is twofold. As a first objective, we shall be concerned with a self-consistent derivation of the *formulation smooth*. That is, a new formulation of the scattering problem that is written on a *macroscale grid* and governs the macroscale component of the response field for a complex scatterer. The term "self-consistent" here is meant to imply that the effect of the microscale heterogeneity on the macroscale response is expressed via a precisely defined across scale coupling operator. The new formulation is to be applied to nonstationary microscale heterogeneity as manifested by the complex scatterer definition. The motivation for this objective is relaxing difficulty 1), above.

Since the articulated operator describes the effects of the microstructure on the macroscale response, it is interpreted as an effective properties description of the complex scatterer or effective material.

The difficulties are not resolved, however, if the precise definition or derivation of the across scales coupling operator itself requires detailed description of the structure heterogeneity or massive computations. Unfortunately, this is usually the case. Therefore, the second objective is "effective properties study." In this context, two fundamental questions can be posed as regards the *footprint* of the microscale heterogeneity in the macroscale response. The first question is: "What types of heterogeneity have a footprint?" For the heterogeneity for which a footprint exists, one might expect that only limited information of the heterogeneity is required to estimate its footprint in the macroscale response. The second question is: "What is this information?" General asymptotic study of the new formulation is provided and we show how the articulated fundamental questions can be addressed. This message, demonstrating the potential of the new approach and the details of this demonstration, are the main conclusions of the study. Regarding the first question, we show the degree to which a microscale heterogeneity effects the macroscale response depends on the singularity of the kernel associated with the integral equation formulation of the problem. Regarding the second question, an investigation as to the degree to which the effective properties operator or some of its components are invariant under classes of variations of the macro or microstructures of a complex scatterer and the characterization of these classes is provided. The motivation of this objective is relaxing (partially, at least) difficulty 2) above.

Traditional effective properties formulations are commonly used in composite materials mechanics for estimating the macroscale response of a material specimens, which have a random heterogeneity [1]–[4]. It is instructive to contrast these applications with the one suggested in this paper. The reader is referred to [5] for a discussion of this issue.

The structure of the paper is as follows. In Section II, we briefly review the tools of multiresolution theory needed for subsequent derivations and develop the smooth formulation. In Section III, we study the properties of the new formulation, essentially via a study of the effective properties operator. A numerical example is provided in Section IV and concluding remarks in Section V.

II. The Smooth Formulation

A. The Mathematical Framework

The theory of multiresolution decomposition (MRD) and orthogonal wavelets [7]–[9] provides a natural framework for the study of interscale coupling processes. There is a rapidly growing literature on the application of wavelet transformations to operators. Examples are found in [10]–[13]. However, the main concern in [10]–[13] is purely computational in its nature: to reduce the number of operations required for the inversion of certain classes of operators by applying wavelet transformations and reducing a densely populated matrix to a sparse one. While this approach may resolve difficulty 1) in Section I for scatterers possessing *smooth* heterogeneity, it will not reduce the number of unknowns for the complex scatterer case. Furthermore, the more crucial difficulty 2) is not relaxed at all. We note that multiresolution analysis and wavelets have been used recently in [14] for numerical homogenization, i.e., for generating an equation with slowly varying coefficients whose solution has the same large-scale behavior as that of the original equation. Relation of this type has been pointed out and detailed also in [5] and [6]. The work in [14] is devoted mainly to a sophisticated "decimation" process in which efficient numerical algorithm for estimating the large-scale response component is developed. It does not directly address any of the fundamental questions articulated in the introduction to this work. The problem presented in this paper requires a different viewpoint and, subsequently, a different analysis. Below, we briefly summarize the MRD theory in a manner tailored to the class of problems discussed here.

Let $\{V_j\}$ be a nested sequence of linear spaces that constitutes MRD of $L_2(R)$. Let $\phi(x)$ and $\psi(x)$ be the associated scaling function and wavelet, respectively, and define $\phi_{mn}(x)$ as the translated and dilated version of $\phi(x)$ via $\phi_{mn}(x) = 2^{m/2}\phi(2^m x - n)$, with m, n integers. A similar definition holds for ψ_{mn}. The sets $\{\phi_{jn}\}_n$ and $\{\psi_{jn}\}_n$ are orthonormal bases of V_j and of the orthogonal complement of V_j in V_{j+1}, respectively. Thus, $V_j = \mathrm{span}_n\{\phi_{jn}\}$ and $V_{j+1} = V_j \oplus \mathrm{span}_n\{\psi_{jn}\}$. An approximation of a function $u(x)$ at a resolution k can be written as the sum of two mutually orthogonal functions, namely a smooth (u^s) and a detail (u^d) components. We have

$$u(x) \simeq u^s(x) + u^d(x) \tag{1}$$

where

$$u^s(x) = \mathbf{P}_j u(x) = \sum_n s_n \phi_{jn}(x), \quad s_n = \langle u, \phi_{jn}\rangle \tag{1a}$$

$$u^d(x) = \mathbf{D}_j^k u(x) = \sum_{m=j}^{k-1} \sum_n d_{mn}\psi_{mn}(x), \quad d_{mn} = \langle u, \psi_{mn}\rangle. \tag{1b}$$

The asymptotic equality in (1) becomes exact in the limit $k \to \infty$. Here, $\langle\cdot,\cdot\rangle$ denotes the inner product of $L_2(R)$ and $j < k$ is some reference resolution—a judicious choice of which depends on the physics of the problem. The scaling functions and wavelets satisfy the orthonormality relations $\langle \phi_{jn}, \phi_{jn'}\rangle =$

$\delta_{nn'}$, $\langle \psi_{mn}, \psi_{m'n'} \rangle = \delta_{mm'}\delta_{nn'}$, and $\langle \phi_{jn'}, \psi_{mn} \rangle = 0 \quad \forall j \leq m$. Thus, $\mathbf{P}_j$ and $\mathbf{D}_j^k$ in (1a)–(1b) are projection operators satisfying $\mathbf{D}_j^k \mathbf{P}_{j'} = \mathbf{P}_{j'} \mathbf{D}_j^k = 0 \forall j \geq j'$. ϕ and ψ are either of compact support or fast decreasing and centered more or less about the origin. They can be interpreted, respectively, as defining a local low-pass and a local band-pass filters. Examples are shown in Fig. 1. From the dilation translation relations articulated above, it follows that the terms ψ_{mn} in (1b) are situated, respectively, around the points

$$x_{mn} = n2^{-m}. \qquad (2)$$

The functions $u^s(x)$ and $u^d(x)$ in (1) can be interpreted as a locally smoothed or averaged, description of $u(x)$ on the length scale 2^{-j}, and a signal describing the finer details covering length scales ranging from $2^{-(j+1)}$ to 2^{-k}, respectively. Here and henceforth we refer to the number 2^{-m} and the index m as a length scale and the resolution associated with it, respectively. A wavelet has M vanishing moments

$$\int x^m \psi(x)\, dx = 0, \quad m = 0, 1, \cdots, M-1. \qquad (3)$$

For the Haar wavelet $M = 1$; for the cubic-spline Battle–Lemarie wavelet $M = 4$ [7], [8]. This parameter can be related to the regularity of the multiresolution system and to the support of the associated wavelets and scaling functions (see [9] for details and examples).

1) Asymptotic Expressions for the Inner Products: When the function $f(x)$ in the neighborhood of $x = x_{mn}$ varies slowly compared to the lengthscale 2^{-m} and possesses M first derivatives (see (3) for M), it is possible to derive approximate expressions for the inner products $\langle f, \phi_{mn} \rangle$ and $\langle f, \psi_{mn} \rangle$. More specifically, it can be shown that for the cubic-spline Battle–Lemarie system shown in Fig. 1 (see [5])

$$\langle f, \phi_{mn} \rangle = 2^{-m/2} f(x_{mn}) - \alpha 2^{-m(2M+1/2)} O(f_{2M}) \qquad (4a)$$

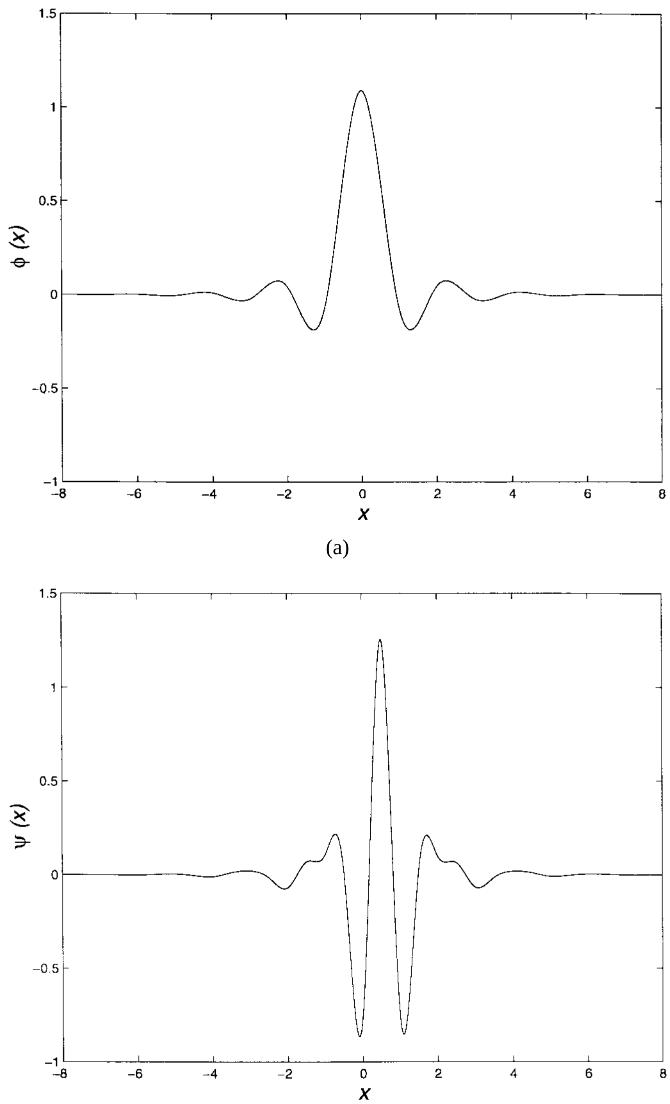
$$\langle f, \psi_{mn} \rangle = \beta 2^{-m(M+1/2)} f_M(x_{mn} + 2^{-m-1}) + \beta 2^{-m(M+5/2)} O(f_{M+2}) \qquad (4b)$$

where $f_M(x)$ is defined as

$$f_M(x) = \sigma \left(i \frac{d}{dx} \right)^M f(x) \qquad (4c)$$

where $\sigma = 1, -i$ for M even, odd. α and β are constants given by $\alpha = \frac{B_M}{2(2M)!}$ and $\beta = 2^{-M} [\frac{(2^{2M}-1)B_M}{(2M)!}]^{1/2}$. B_M is the Mth Bernoulli number and x_{mn} is a point in the wavelet grid defined by (2). Note that for the cubic-spline Battle–Lemarie system one has $M = 4$, $B_M = 1/30$, so $\alpha, \beta \ll 1$. Thus, the remaining terms in the right side of (4a) and (4b) are small. Furthermore, if $f_M = O(1)$ we get $|\langle f, \psi_{mn} \rangle| \ll 1$ for $m \gg 1$; in the limit $m \to \infty$ the inner product with the wavelet is vanishingly small if the function f is smooth. Similar results can be derived for other multiresolution systems.

2) Local Irregularity and Wavelets: The manner in which the irregularity of a function reflects on its wavelet coefficients plays a role in the multiresolution study of scattering effects [15]. It is also a central issue in the study of effective properties of complex scatterers, as the effective material operator strongly depends on the nature of the Green function



(a)

(b)

Fig. 1. The cubic-spline Battle–Lemarie multiresolution system. (a) The scaling function $\phi(x)$. (b) The wavelet $\psi(x)$.

singularity [6]. Wavelet coefficients in the presence of an irregularity have been studied in the general framework of Hölder regularity condition and Hölder spaces [9], [12]. Here we shall sacrifice generality and address the issue via a simple example. Let $g(x)$ be a function with rth derivative discontinuity at $x = x'$ ($r = 0$ corresponds to a discontinuity in $g(x)$ itself). Assume further that $g(x)$ is bounded and can be expressed as

$$g(x) = f(x) + h(x) \qquad (5)$$

where $f(x)$ is a smooth function $f(x) \in C^M$ and $h(x)$ contains the irregularity of $g(x)$ in the form

$$h(x) = \begin{cases} a^-(x-x')^r, & \text{for } x < x' \\ a^+(x-x')^r, & \text{for } x > x' \end{cases} \qquad (6)$$

in a domain defined by $|x - x'| \leq O(1)$. The rth order discontinuity magnitude across $x = x'$ is given by $(a^+ - a^-)r!$. The wavelet coefficients of $g(x)$ can now be expressed as $\langle g, \psi_{mn} \rangle = \langle f, \psi_{mn} \rangle + \langle h, \psi_{mn} \rangle$. Thus, if 2^{-m} is small compare to the typical length scale of $f(x)$ the first term can be approximated by (4b). It can be shown that if the wavelet irregularity order is higher than that of $h(x)$, the second term is bounded by [9], [12]

$$|\langle h, \psi_{mn}\rangle| \leq C2^{-m(r+1/2)}, \quad C = O(1) \tag{7}$$

where the constant C depends on the specific multiresolution system. This bound is valid for all m.

B. A Multiresolution Study of the Scattering Problem

The procedure that will be described here can be applied to a general linear operator equation. However, to render general ideas more specific and to allow for a straightforward application to scattering problems we shall refer throughout to the Fredholm integral equation of the second kind

$$u(x) = u_0(x) + \mathbf{L}u \quad a \leq x \leq b \tag{8}$$

where $\mathbf{L}$ is the integral operator defined as

$$\mathbf{L}u = \int_a^b K(x, x')\, u(x')\, dx' \tag{8a}$$

and where $u_0(x)$ is a forcing term, $u(x)$ is the response to be determined, and $K(x, x')$ is a known kernel. This equation occurs widely in diverse areas of mathematical physics and engineering. The formulation is said to be *complete* in the sense that it contains the entire range of length scales associated with the problem. In cases where the scatterer is flat and the system heterogeneity is characterized by the material properties (e.g., local constitutive relations) the scattering formulation can be written as (8) and (8a) with the following simplifications [5]:

$$K(x, x') = G(x - x')h(x'). \tag{9a}$$

Here, $G(x)$ is the Green function of an appropriately defined background problem, usually that modeled by a linear differential equation with constant coefficients, and $h(x)$ represents a material heterogeneity with variations occurring on both the microscale and macroscale—the complex scatterer. In these cases, the operation $\mathbf{L}u$ can be represented as the background operator $\mathbf{L}_b$ associated with $G(x)$, operating on the function hu

$$\mathbf{L}u = \mathbf{L}_b\,(hu) \tag{9b}$$

where

$$\mathbf{L}_b f = \int G(x - x') f(x')\, dx' \tag{9c}$$

and, by definition of the complex scatterer, the integration limits in (8a) are affected by h.

With the MRD theory, the integral equation formulation (8) can be decomposed into a pair of coupled formulations governing a length-scale resolved response. We express the response field $u(x)$ as a sum of two components; the smooth component $u^s(x)$—macroscale response and a component that describes the fine details $u^d(x)$—microscale response. If 2^{-k} is a lower bound on the lengthscales pertaining to the problem, then $u(x) \simeq \mathbf{P}_k u(x)$, and one can invoke (1) and (1b) where u^s and u^d are obtained, respectively, by the response field smoothed on a reference lengthscale 2^{-j} and the remaining fine details. s_n and d_{mn} are yet to be determined coefficients, representing the smooth and detail parts of the response. With a standard Galerkin procedure one can cast (8) in a matrix form, written for the sets of unknown coefficients $\{s_n\}$ and $\{d_{mn}\}$. the result is [5]

$$\begin{pmatrix} \mathbf{I} - \mathbf{\Phi} & -\mathbf{C} \\ -\bar{\mathbf{C}} & \mathbf{I} - \mathbf{\Psi} \end{pmatrix} \begin{pmatrix} \vec{s} \\ \vec{d} \end{pmatrix} = \begin{pmatrix} \vec{e} \\ \vec{E} \end{pmatrix} \tag{10}$$

where $\vec{s}$ and $\vec{d}$ are the (unknown) smooth and detail column vectors, representing the smooth and detail components of the response. $\mathbf{I}$ is an identity matrix, $\vec{e}$ and $\vec{E}$ are the excitation smooth and detail column vectors whose elements are the respective (known) coefficients of $u_0^s(x)$ and $u_0^d(x)$. $\mathbf{\Phi}$, $\mathbf{\Psi}$, $\mathbf{C}$, and $\bar{\mathbf{C}}$ are matrix operators whose elements are given, respectively, by

$$\begin{aligned} \Phi_{n',n} &= \langle \mathbf{P}_j \mathbf{L} \mathbf{P}_j \phi_{jn}, \phi_{jn'} \rangle = \langle \mathbf{L}\phi_{jn}, \phi_{jn'} \rangle \\ \Psi_{m'n',mn} &= \left\langle \mathbf{D}_j^k \mathbf{L} \mathbf{D}_j^k \psi_{mn}, \psi_{m'n'} \right\rangle = \langle \mathbf{L}\psi_{mn}, \psi_{m'n'} \rangle \\ C_{n',mn} &= \left\langle \mathbf{P}_j \mathbf{L} \mathbf{D}_j^k \psi_{mn}, \phi_{jn'} \right\rangle = \langle \mathbf{L}\psi_{mn}, \phi_{jn'} \rangle \\ \bar{C}_{m'n',n} &= \left\langle \mathbf{D}_j^k \mathbf{L} \mathbf{P}_j \phi_{jn}, \psi_{m'n'} \right\rangle = \langle \mathbf{L}\phi_{jn}, \psi_{m'n'} \rangle. \end{aligned} \tag{11}$$

Equation (11) show the four matrices to actually be representations of the same operator $\mathbf{L}$ each representation distinguished by the subspaces of L_2 on which the operator is to act and on which the results of the action are to be projected. Thus, for $\mathbf{\Phi}$ and $\bar{\mathbf{C}}$ the action is applied to elements in V_j whereas for $\mathbf{\Psi}$ and $\mathbf{C}$ the action is applied to elements in the complement of V_j. Further, for $\mathbf{\Phi}$ and $\mathbf{C}$ the result of the action is to be projected on V_j whereas for $\mathbf{\Psi}$ and $\bar{\mathbf{C}}$ the projection is on the complement of V_j. One further noteworthy feature for understanding $\mathbf{\Phi}$, $\mathbf{\Psi}$, $\mathbf{C}$, and $\bar{\mathbf{C}}$ is that base functions for spanning the V_j space form a one-parameter set and one can interpret the representations associated with it as physical-space representations; the base functions spanning the complement of V_j form a two-parameter set and one can interpret the representations associated with it as phase-space representations. Thus, $\mathbf{\Phi}$, $\mathbf{\Psi}$, $\mathbf{C}$, and $\bar{\mathbf{C}}$ can be interpreted as a physical space, phase-space, and mixed-domain operators, respectively.

Equation (10) provides the starting point for a multiresolution study of the scattering problem [5], [15] and for a self-consistent development of the formulation smooth. From the lower half of (10), $\vec{d}$ can be expressed in terms of $\vec{s}$ and $\vec{E}$. When this is substituted into the upper half of (10) we get a formulation governing $\vec{s}$—the formulation smooth

$$[\mathbf{I} - \mathbf{\Phi} - \mathbf{C}(\mathbf{I} - \mathbf{\Psi})^{-1}\bar{\mathbf{C}}]\vec{s} = \vec{e} + \mathbf{C}(\mathbf{I} - \mathbf{\Psi})^{-1}\vec{E}. \tag{12}$$

It has been shown that $\mathbf{\Phi}$ can be interpreted as a "smoothed version" of the operator $\mathbf{L}$ (see [5], [6], and [16]); a representation of the latter by a straightforward discretization on a reference grid separation 2^{-j}. Thus, $\mathbf{C}(\mathbf{I} - \mathbf{\Psi})^{-1}\bar{\mathbf{C}}$ has been interpreted as an effective properties operator; an operator describing the coupling across scales due to the presence of a microstructure. The dependence of this operator on the heterogeneity $h(x)$ is nonlinear. This fact is the genesis of difficulty 2) listed in Section I.

III. Properties of the Formulation Smooth

The manner in which the microstructure effects the effective material operator $\mathbf{C}(\mathbf{I}-\mathbf{\Psi})^{-1}\bar{\mathbf{C}}$ in (12) have been investigated in [5], [6], and [16] for the case of a heterogeneity with a widely separated micro and macroscales, pertaining to acoustic scattering from a thin, linearly elastic, fluid-loaded plate. Since these results play a pivotal role in the present theory, we shall briefly summarize them in the following subsections and develop new results more relevant to electromagnetic scattering. We assume that the integral operator kernel can be expressed as in (9a). Then, with $\mathbf{L}_b^*$, the adjoint operator of $\mathbf{L}_b$ in (9b), the elements can be rewritten as

$$\Phi_{n',n} = \langle h\phi_{jn}, \mathbf{L}_b^*\phi_{jn'}\rangle = \langle h\phi_{jn}, \phi_{jn'}^f\rangle \quad (13a)$$

$$\Psi_{m'n',mn} = \langle h\psi_{mn}, \mathbf{L}_b^*\psi_{m'n'}\rangle = \langle h\psi_{mn}, \psi_{m'n'}^f\rangle \quad (13b)$$

$$C_{n',mn} = \langle h\psi_{mn}, \mathbf{L}_b^*\phi_{jn'}\rangle = \langle h\psi_{mn}, \phi_{jn'}^f\rangle \quad (13c)$$

$$\bar{C}_{m'n',n} = \langle h\phi_{jn}, \mathbf{L}_b^*\psi_{m'n'}\rangle = \langle h\phi_{jn}, \psi_{m'n'}^f\rangle \quad (13d)$$

where the "fields" $\phi_{jn'}^f$ and $\psi_{m'n'}^f$ can be expressed as

$$\phi_{jn}^f(y) = \int \phi_{jn}(x)\overline{G(x-y)}\,dx = \langle \phi_{jn}, G(\cdot - y)\rangle \quad (14)$$

and a similar expression holds for $\psi_{mn}^f(y)$. The overline denotes a complex conjugate. These expressions will be used in Section III-B to study the effective material operator.

A. System Characterization and Scales Hierarchy

Two type lengthscales are defined by the physical system. The first applies to the variation of $G(x)$, induced by the dynamics of the system. We refer to this as the wavelength (λ) scale and to the associated resolution as m_λ. The possible irregularity of $G(x)$ at the origin may introduce additional variability over a relatively wide range of lengthscales. This irregularity has an essential role in the theory and it will be discussed further. The second type lengthscales is due to the system heterogeneity and applies to the variations of $h(x)$. These variations can range from the *macroscale* $(\geq\lambda)$ to the *microscale* $(\ll\lambda)$. The former and the latter are associated with the resolutions m_a and m_i, respectively, such that $2^{-m_a} \geq \lambda$ and $2^{-m_i} \ll \lambda$. Finally, the reference, or smoothing, scale should be carefully selected. Details of scale λ must be adequately described, whereas microscale details can be averaged. The following hierarchy of resolutions and scales applies:

$$m_a \leq m_\lambda \ll j \ll m_i \Longleftrightarrow 2^{-m_a} \geq 2^{-m_\lambda} \approx \lambda \gg 2^{-j} \gg 2^{-m_i}. \quad (15)$$

Essential to the present study is the concept of a "scatterer" as a localized region of *microscale* heterogeneity that is confined to a domain in space measured on the *macroscale*. Also essential is the existence of a "gap" in the scales for observing heterogeneity. Specifically, the system heterogeneity is not to posses any component in the range of scales $(2^{-m_a-1}, 2^{-m_i+1})$. We term this case a two-scale (macro/micro) variation. When referred to the j scale; i.e., the chosen smoothing scale, the macroscale variation is the smooth component and the microscale variation is the detail component. We write

$$h(x) = h^s(x) + h^d(x) \quad (16)$$

where,

$$h^s(x) = \sum_n a_n\phi_{m_a n}(x) \quad (16a)$$

is expressible as a synthesis of scaling functions and

$$h^d(x) = \sum_n b_n\psi_{m_i-1\,n}(x). \quad (16b)$$

as a synthesis of wavelets. The index n, in both these equations, span a region with a physical dimensions described on the macroscale. Note that $h^s(x)$ and $h^d(x)$ are mutually orthogonal and the associated scales are widely separated. The synthesis in (16a) and (16b) identifies the spatial averages of $h(x)$ over neighborhoods in the order of the macroscale 2^{-m_a} with h^s and further implies that spatial averages of the detail component $h^d = h - h^s$ vanish if performed over neighborhoods larger than the microscale 2^{-m_i}. Denoting $\|\cdot\|_2$ the L_2 norm, we shall assume throughout the rest of this work that

$$\|h^d(x)\|_2 = O(\|h^s(x)\|_2) = O(1). \quad (17)$$

In other words, the energy associated with the microstructure is of the order of that associated with the macrostructure and both are beyond the perturbative regime.

B. The Response Components u^s, u^d, and the Microstructure Signature

An interesting general relation between the smooth and detail components of the response field can be derived from the set (10). The integral equation excitation term u_0 is nothing but the background system response. In most applications, the latter varies on the scale of λ—the macroscale. Thus, with (4b), (4c), and (15) we obtain that the detail component of the excitation term in (10), namely $\vec{E}$, becomes vanishingly small [see discussion after (4c)]. In this case, the lower half of (10) yields

$$(\mathbf{I}-\mathbf{\Psi})\vec{d} = \bar{\mathbf{C}}\vec{s} \quad (18)$$

which states that when the excitation is described only on the macroscale, the relation between the smooth (macroscale) and detail (microscale) components of the response field is independent of the excitation. This relation will be useful in subsequent derivations.

Below we derive some results pertaining to the effect of the microstructure on the macroscale response—the microstructure signature—and relate it to the nature of G.

1) Smooth $G(x)$: We assume the integral equation kernel function $G(x)$ is regular at the origin (that is, it varies on the scale of λ for all x). Then, an estimate of the matrix elements in (11) can be obtained by invoking (4a)–(4c), (13a)–(14), and the scale hierarchy characterization (15). These estimates can be used to obtain bounds on the matrix norms. Such a procedure was first undertaken in [5]. A more systematic derivation is given in the appendix of [16]. The results are

$$||\mathbf{\Psi}|| \leq B \tag{19a}$$
$$||\bar{\mathbf{C}}|| \leq B \tag{19b}$$
$$||\mathbf{C}|| \leq O(1) \tag{19c}$$

where $||\cdot||$ denotes the matrix norm induced by the Euclidean vector norm, and the bound B is given by

$$B^2 = \beta^2 \lambda (1 - 2^{-2M})^{-1} (\lambda 2^{m_a})^{-1} (\lambda 2^j)^{-2M} \ll 1. \tag{19d}$$

These estimates are derived in [16]. With (19a)–(d), one obtains for the effective material operator $||\mathbf{C}(\mathbf{I} - \mathbf{\Psi})^{-1}\bar{\mathbf{C}}|| \ll 1$. The conclusion that under the smoothness condition on G the macroscale response practically bares no footprint of the microscale heterogeneity follows directly from this last result. Furthermore, this result establishes that a new formulation governing the macroscale response component $u^s(x)$ is readily obtained from (8)–(9c):

$$u^s(x) = u_0(x) + \mathbf{L}_b h^s u^s \tag{20}$$

where $\mathbf{L}_b$ is the background operator defined in (9c), $u_0(x)$ is the excitation term (assumed here to vary on the macroscale only) and $h^s(x)$ is the locally averaged (smooth component) of the system heterogeneity [see (16)–(16b) and discussion thereafter]. Equation (20) identifies $h^s(x)$ as a local effective property of the system heterogeneity $h(x)$. In these cases, the formulation smooth and the smoothed formulation are identical (a general comment on this issue is made in the introduction).

The inequalities in (19a)–(c) can be used to derive an estimate on the relation between u^s and u^d. From the multiscale resolved formulation (10)

$$(\mathbf{I} - \mathbf{\Psi})\vec{d} = \bar{\mathbf{C}}\vec{s} + \vec{E}. \tag{21}$$

Using (19a), and subsequently applying the triangle and operator norm inequalities, we get

$$||\vec{d}|| \leq ||\bar{\mathbf{C}}|| ||\vec{s}|| + ||\vec{E}|| \tag{22}$$

where $||u||$ denotes the Euclidean vector norm if u is a vector or the matrix norm induced by the Euclidean vector norm if u is a matrix. In cases where the excitation term u_0 is described on the macroscale only (i.e., $||\vec{E}|| = 0$, so (18) holds) we get [with (19b)]

$$||\vec{d}|| \ll ||\vec{s}|| \tag{23a}$$

or, denoting $||\cdot||_2$ the L_2 norm (taken over the support of the heterogeneity)

$$||u^d||_2 \ll ||u^s||_2. \tag{23b}$$

2) Singular $G(x)$: In cases where the integral equation background kernel $G(x)$ is singular at the origin, the regularity assumptions leading to (19a)–(c) cannot be used. Thus, the inequality $||\mathbf{C}(\mathbf{I} - \mathbf{\Psi})^{-1}\bar{\mathbf{C}}|| \ll 1$ does not hold in general. The immediate consequence is that a microstructure can have a significant effect on the macroscale response. This has been investigated and demonstrated in [6] for the problem of a fluid loaded thin elastic plate with a one-dimensional (1-D) stiffness heterogeneity subjected to a time-harmonic forcing. In this case, the kernel function $G(x)$ possesses a delta function singularity at the origin. The work in [16] uses the formulation smooth to develop a 1-D effective constitutive relation pertaining to the plate stiffness heterogeneity.

More relevant to the electromagnetic scattering problems is the case in which the kernel $G(x)$ possesses a first derivative discontinuity at the origin, as suggested by the exponent $e^{i|x-x'|}$. Indeed, an important special case that is of concern is the kernel

$$G(x) = i\frac{\pi}{\cos\theta} e^{i2\pi\cos\theta|x|}. \tag{24}$$

The lengthscale on which this kernel varies away from the origin—the wavelength—is $\lambda = 1/\cos\theta$. With this kernel and the scales hierarchy (15), the fields ϕ^f_{jn}, ψ^f_{mn} of (13a) and (14) can still be approximated by (4a) and (b) for points sufficiently far from the irregularity. These approximate expressions do not apply for all y. However, since G possesses a first derivative discontinuity at the origin, we can apply (7) with $r = 1$ to obtain the general bound

$$\left|\psi^f_{mn}(y)\right| \leq C 2^{-3m/2}. \tag{25}$$

Since $\psi^f_{mn}(y), \phi^f_{jn}(y)$ themselves possess a second derivative discontinuity at the origin, we can repeatedly use (7) to obtain

$$\left|\left\langle \psi_{mn}, z\phi^f_{jn'}\right\rangle\right| \leq C 2^{-5m/2} \tag{26a}$$
$$\left|\left\langle \psi_{mn}, z\psi^f_{m'n'}\right\rangle\right| \leq C^2 2^{-5m/2-3m'/2} \tag{26b}$$

where z is any bounded function that possesses at least two derivatives. With these bounds, one may attempt to study the operator elements (13a)–(d) and what heterogeneity component dominates each.

We start with the elements of $\mathbf{\Phi}$. With the heterogeneity elements h^s and h^d of (16) we can write

$$\Phi_{n',n} = \Phi^s_{n',n} + \Phi^d_{n',n} \tag{27}$$

where

$$\Phi^{s,d}_{n',n} = \left\langle h^{s,d}\phi_{jn}, \phi^f_{jn'}\right\rangle = \left\langle h^{s,d}, \phi_{jn}\phi^f_{jn'}\right\rangle. \tag{28}$$

The second equality holds since ϕ is real. Using now (16b), (17), and (26a), we get

$$\left|\Phi^d_{n',n}\right|^2 \leq C 2^{-5(m_i - 1)}, \quad C = O(1). \tag{29}$$

A similar derivation gives $\Phi^s_{n',n} = O(1)$. Thus

$$||\mathbf{\Phi}^d|| \ll ||\mathbf{\Phi}^s||. \tag{30}$$

In other words, the operator $\mathbf{\Phi}$ is dominated by the macroscale component of the heterogeneity. Reversing the steps, one achieves with no further approximation

$$\Phi_{n',n} \simeq \langle \mathbf{L}_b h^s \phi_{jn}, \phi_{jn'} \rangle. \tag{31}$$

The operator $\mathbf{\Psi}$ can be written as in (13b) or as (h real) $\Psi_{m'n',mn} = \langle \psi_{mn}, h\psi^f_{m'n'} \rangle$. Using (26b) yields

$$|\Psi_{m'n',mn}| \leq C^2 2^{-3m'/2-5m/2}. \tag{32}$$

This bound on the matrix element can be used to derive a bound on the norm of $\mathbf{\Psi}$. The result is (see Appendix)

$$||\mathbf{\Psi}||^2 \leq O(1)2^{-6j} \ll 1. \tag{33}$$

The second inequality follows from the hierarchy in (15), with $\lambda = O(1)$, and with the condition that the scatterer outer dimension is on the order of the wavelength. This inequality suggests a Neumann series expansion to $(\mathbf{I} - \mathbf{\Psi})^{-1}$. Then, the effective material operator can be expressed as the series

$$\mathbf{C}(\mathbf{I} - \mathbf{\Psi})^{-1}\bar{\mathbf{C}} = \sum_n \mathbf{R}_n = \sum_n \mathbf{C}\mathbf{\Psi}^n\bar{\mathbf{C}} \tag{34a}$$

where the norms of the expansion terms $\mathbf{R}_n$ are bounded by

$$||\mathbf{R}_n|| = ||\mathbf{C}\mathbf{\Psi}^n\bar{\mathbf{C}}|| \leq ||\mathbf{C}||||\mathbf{\Psi}||^n||\bar{\mathbf{C}}|| \leq 2^{-3jn}||\mathbf{C}||||\bar{\mathbf{C}}||. \tag{34b}$$

One should be careful in interpreting this inequality. The relations between the norm bounds do not necessarily translate to the norms themselves. The norm bounds of $\mathbf{R}_n$ decrease monotonically (and rapidly) with n. Nevertheless, in general one may encounter a situation where the norm itself does not decrease monotonically. An example is the case where $\mathbf{C}\bar{\mathbf{C}} = 0$ for which we have $0 = ||\mathbf{R}_0|| \leq ||\mathbf{R}_n||$ for all $n > 0$. However, if $\mathbf{C}\bar{\mathbf{C}} \neq 0$ we can always find k such that $||\mathbf{R}_0|| \gg ||\mathbf{R}_m||$ for all $m > k$. Furthermore, if j is sufficiently large [see (33)], this can hold right at $k = 0$ (that is, for all $m > 0$).

Let us assume that this indeed is the case. Then we can approximate the effective material operator by $\mathbf{C}\bar{\mathbf{C}}$ plus a small reminder

$$\mathbf{C}(\mathbf{I} - \mathbf{\Psi})^{-1}\bar{\mathbf{C}} = \mathbf{C}\bar{\mathbf{C}} + \mathbf{R}, \quad ||\mathbf{R}|| \ll ||\mathbf{C}\bar{\mathbf{C}}||. \tag{35}$$

The accumulation of near-field interactions taking place on a given small scale, say 2^{-m} ($\ll 2^{-j}$), can couple to the large-scale response either directly, or by successively cascading upward from 2^{-m} to 2^{-m+1}, then to 2^{-m+2}, etc. The last result states that when the conditions leading to (35) are satisfied, the former route is dominant (see discussion after (11) for an interpretation of the operators). This is schematized in Fig. 2. Conversely, if $\mathbf{C}\bar{\mathbf{C}} = 0$, the across-scale coupling mechanism is manifested *only* by an energy that cascades upward in a series of successive small steps.

The expression in (35) suggests the approximation of the effective material operator by $\mathbf{C}\bar{\mathbf{C}}$. This approximation is very appealing as it relaxes difficulty 2) listed in the introduction. This is because the effective material operator can now be factorized into two operators, each is linear with respect to

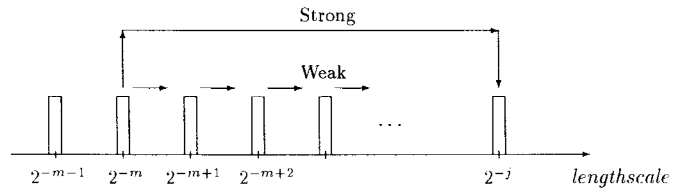


Fig. 2. A schematization of the cascading process.

the heterogeneity. To see this, let us rewrite the operator $\mathbf{C}$ as follows:

$$\mathbf{C} = \mathbf{C}^s + \mathbf{C}^d; \quad \mathbf{C}^{s,d} = \langle \mathbf{L}_b h^{s,d}\psi, \phi \rangle \tag{36}$$

and a similar decomposition written for $\bar{\mathbf{C}}$. Thus, if only the macroscale (or only the microscale) structure has been changed, one has to recompute only the operators $\mathbf{C}^s$, $\bar{\mathbf{C}}^s$ (or only the operators $\mathbf{C}^d$, $\bar{\mathbf{C}}^d$). In any case, no inversion of small scale or mixed data operator is needed to get the effective material operator. Even though the expression $\mathbf{C}\bar{\mathbf{C}}$ for the effective material operator is approximate, we shall see that it yields an accurate prediction of the macroscale response.

Another interesting and useful implication of the inequality in (33) concerns the relation between the smooth and the detail components of the response field. From (18) we now have

$$\vec{d} \simeq \bar{\mathbf{C}}\vec{s} \tag{37}$$

which means that once we predict the macroscale response $\vec{s}$ the prediction of the microscale response does not require any inversion of operator that incorporates small scale data and, further, this relation is independent of the forcing.

3) Remark: The results obtained in this section, especially in Section III-B, hold also for scatterers that possess macro and microstructures $h^{s,d}$ more general then those specified in (16a) and (b). For example, one can have more scales in the wavelet synthesis of h^d (16b), but the large gap between the micro and macroscales, as well as the “finite energy” conditions (17), must be maintained.

IV. An Application

We shall be concerned with the response field u of a planar wave velocity inhomogeneity in a two-dimensional (2-D) (x, z) space, illuminated by a plane wave u_0 of a unit amplitude and a unit wavelength $\lambda = 1$. The layers are parallel to the z axis and the incident plane wave propagates with an angle θ relative to the negative x direction. With no loss of generality, we assume that the wave velocity inhomogeneity is a manifestation of variations in the material dielectric properties. Its outer dimensions in the order of λ (say, 2–3λ), whereas the inner variations are measured on lengthscales much smaller than λ. The system configuration is shown in Fig. 3. This problem can be used, for example, as a basic model for the study of the effective electromagnetic properties of composite material laminates.

The system response is governed by the generic wave equation (assuming a unit vacuum wavelength; $\lambda = 1$)

$$\left[\frac{d^2}{dx^2} + 4\pi^2\cos^2\theta + 4\pi^2 h(x)\right]u(x) = S(x). \tag{38}$$

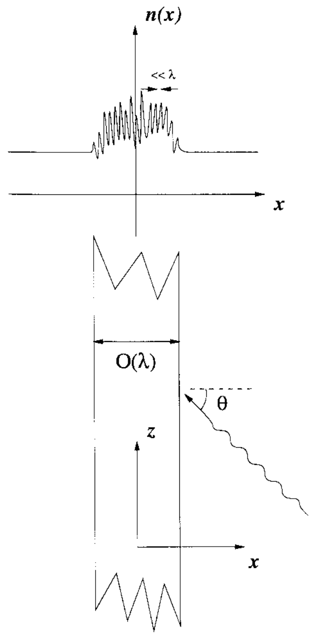


Fig. 3. A schematization of the system configuration.

The electromagnetic field solutions are obtained from $u(x)$ via $E_y(x,z) = u(x)e^{i2\pi z \sin\theta}$ for the TE case and via $H_y(x,z) = n(x)u(x)\, e^{i2\pi z \sin\theta}$ for the TM case. Here, $n(x)$ is the refraction index, $h(x)$ is the heterogeneity function

$$h(x) = \begin{cases} n^2(x) - 1 & \text{for TE} \\ n^2(x) + \frac{1}{4\pi^2}\left[\frac{n''(x)}{n(x)} - 2\frac{n'^2(x)}{n^2(x)}\right] - 1 & \text{for TM} \end{cases} \tag{39}$$

and $S(x)$ is any source term that generates the incident field. The generic wave (38) can be rewritten with the term $4\pi^2 h(x)u(x)$ in the right-hand side. Then, one can represent the solution as a convolution of the 1-D homogeneous medium Green function with the extended source term $S(x) - 4\pi^2 h(x)u(x)$. The result is the integral equation formulation (8)–(9c), where $G(x)$ and $h(x)$ are given by (24) and by (39), respectively, and $u_0(x)$ is determined by the unit amplitude incident electromagnetic plan wave in the absence of the heterogeneity $(n(x) = 1)$

$$u_0(x) = e^{i2\pi x \cos\theta}. \tag{40}$$

Thus, for complex heterogeneity $h(x)$, the results of Section III-B.2 directly apply.

We turn now to demonstrate the results of Section III-B.2 via numerical simulations. Of particular interest is the TM case, since terms of the form $n'^2(x)/[n(x)k_0]^2$ with $n(x)$ that varies on length scales much smaller than λ contribute a very significant microscale component to $h(x)$, even if the microscale component of the refraction index is small. To illustrate this, consider the refraction index

$$n(x) = 1 + e^{-x^2/2}[a + b\cos(2\pi x/L)] \tag{41}$$

with $a = -0.5$, $b = 0.05$, and $L = 1/12$. The function $n^2(x)$ is plotted in Fig. 4. Clearly it possesses a macrostructure of outer dimension in the order of several wavelengths and a small magnitude microstructure that varies on a lengthscale of a fraction of a wavelength. The corresponding heterogeneity function $h(x)$ have been computed for the TM case via (39). The result is shown in Fig. 5. Fig. 5(a) shows the complete heterogeneity function $h(x)$. It is certainly consistent with our definition of a complex scatterer. Fig. 5(b) and (c) show its macroscale $(h^s(x))$ and microscale $(h^d(x))$ components, respectively, separated according to the discussion in Section III-A (see also a remark at the end of Section III).

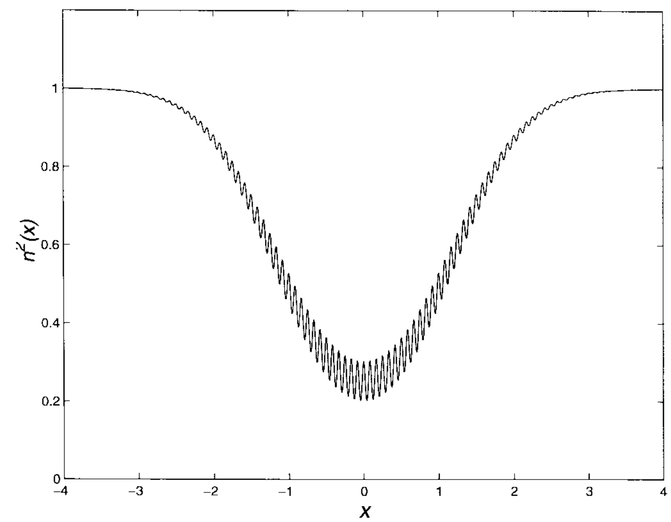


Fig. 4. Square of the refraction index of (41).

A TM scattering problem with $n(x)$ given in (41) (that is, the heterogeneity function $h(x)$ is the one shown in Fig. 5) and plane wave incidence angle $\theta = 30°$ is chosen for a test case. To generate a reference solution, the corresponding integral equation was solved by a straightforward moment method approach based on a grid sufficiently fine to capture the details of $h^d(x)$ (the point separation was chosen to be $1/64$, which corresponds to the resolution $m_i = 6$). Then, using a reference resolution $j = 3$ [see (15)], the resulting complete solution was separated to its macroscale and microscale components $u^s(x)$ and $u^d(x)$. The results are shown in Fig. 6. To examine the significance of the microstructure h^d, we have solved the same problem with only h^s present [Fig. 5(b)]. The result is shown in Fig. 7. Clearly, the resulting u^s is completely different from the one that corresponds to the heterogeneity $h(x) = h^s(x) + h^d(x)$, shown in Fig. 6. Hence, as expected, the microscale component of the heterogeneity has a significant effect on the macroscale response. We have repeated this experiment with h that corresponds to $n(x)$ the same as that in (41), but with $b = 0$ (no microstructure on the level of $n(x)$). Again, the resulting u^s deviated significantly from the macroscale response in the presence of the microstructure.

Finally, we turn to examine the smooth formulation (12) ($\vec{E} = 0$), where $\mathbf{\Phi}$ is computed with h^s only [see (27)–(31)], and the effective material operator construction suggested after (35)

$$\mathbf{C}(\mathbf{I} - \mathbf{\Psi})^{-1}\bar{\mathbf{C}} \simeq \mathbf{C}\bar{\mathbf{C}}. \tag{42}$$

The operators $\mathbf{C}$ and $\bar{\mathbf{C}}$ have been evaluated via (36). An approximate prediction of the macroscale response was then

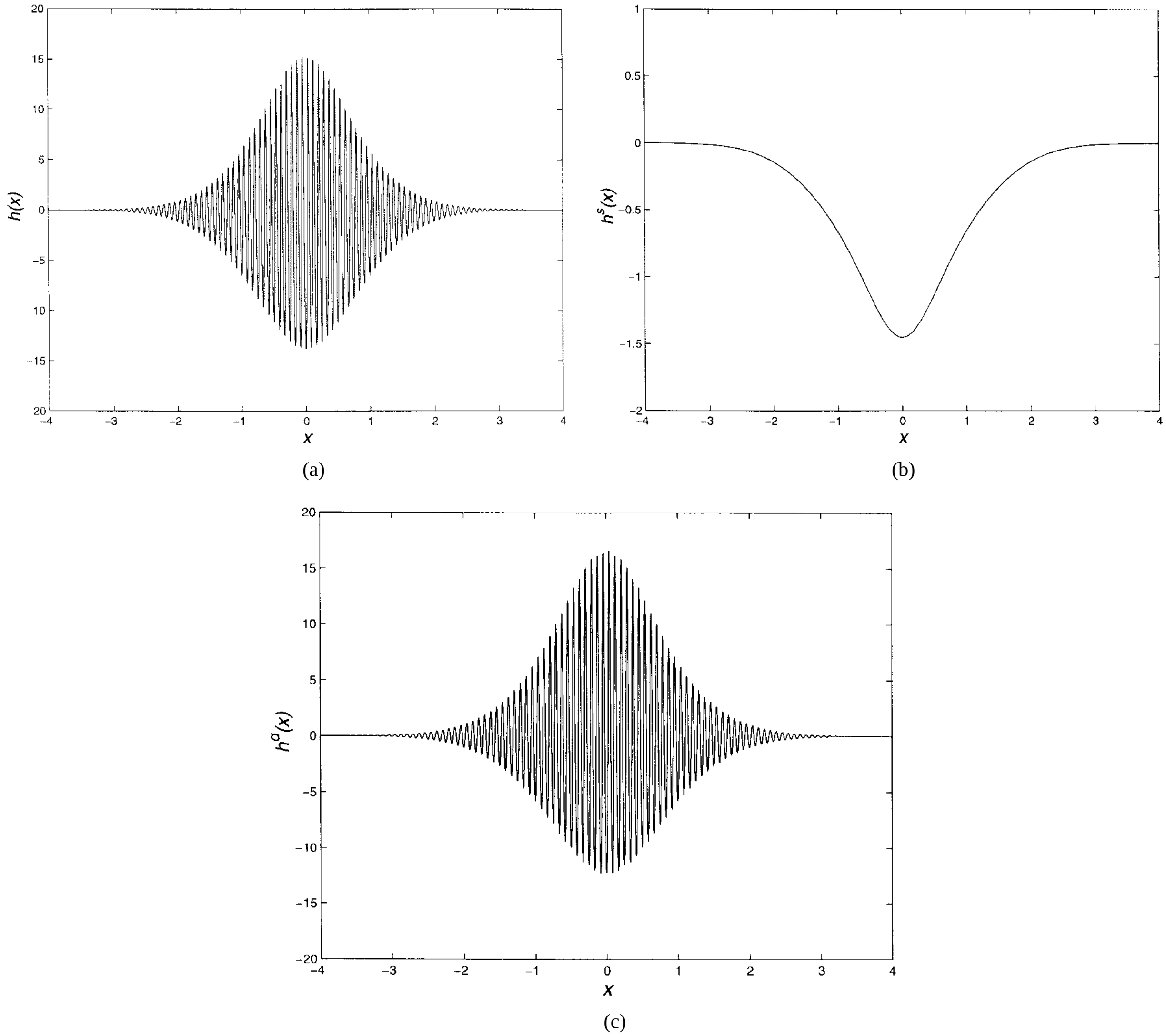


Fig. 5. The heterogeneity function that corresponds to $n(x)$ of (41) for a TM problem. (a) $h(x)$. (b) The macroscale component $h^s(x)$. (c) The microscale component $h^d(x)$.

obtained by solving the approximate smooth formulation

$$[\mathbf{I} - \mathbf{\Phi}^s - (\mathbf{C}^s + \mathbf{C}^d)(\bar{\mathbf{C}}^s + \bar{\mathbf{C}}^d)]\vec{s}_{\text{app}} = \vec{e}. \tag{43}$$

Note that this formulation does not require any inversion of small scale or mixed-scale data. Its solution requires an inversion of an operator written completely on a macroscale grid. The approximated macroscale response solution $u^s_{\text{app}}(x)$ obtained from the approximated coefficients vector $\vec{s}_{\text{app}}$ is compared in Fig. 8 to the exact macroscale solution component of Fig. 6. The L_2 norm of the normalized deviation between the lines within the domain shown is about 3%.

V. Conclusion

Multiresolution technique has been used to develop a theory for effective properties of complex scatterers. A new formulation that governs the macroscale component of the scattered field is obtained—the *smooth formulation* (12). This formulation is written on a *macroscale* grid, whereas the effects of the microscale heterogeneity on the macroscale response are expressed via a precisely defined across scales coupling operator. The latter can also be interpreted as an *effective material operator* (EMO) description of the heterogeneity. The EMO norm is predominantly determined by the singularity at the origin of the associated integral equation kernel. The more singular the kernel the higher the norm and the stronger is the effect of the microscale heterogeneity on the macroscale response. Since the articulated kernel is usually nothing but the Green function associated with the corresponding "background" problem, it follows that the effects of the microscale heterogeneity h^d on the macroscale response u^s are predominantly determined by the *near-field* behavior of the corresponding background system. This conclusion is intuitively appealing; the microscale structure can be viewed as a collection of point scatterers separated by distances much smaller then the wavelength. Then, their effect on the macroscale response is interpreted as the result of accumulated multiple near-field interactions.

The EMO properties have been studied for cases in which the associated integral equation kernel possesses a second-order singularity. It has been shown that it can be approx-

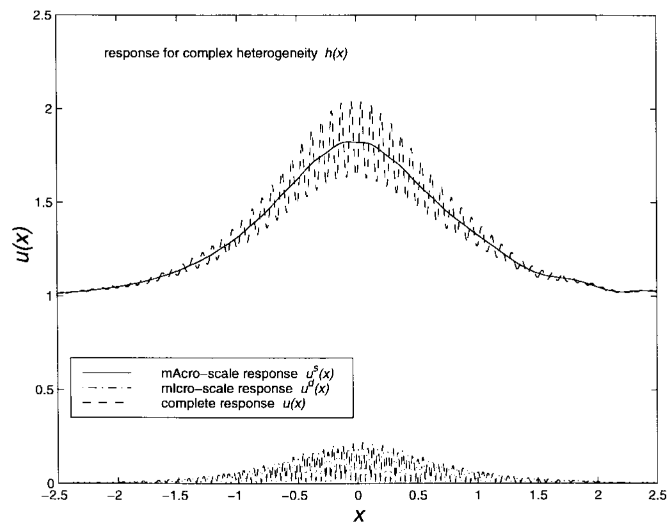


Fig. 6. Magnitude of TM solution and its components for $h(x)$ shown in Fig. 5(a). Solid and dotted lines (- - - ——— - - -): macroscale (u^s) and microscale (u^d) components. Dashed line (- - -): the complete solution.

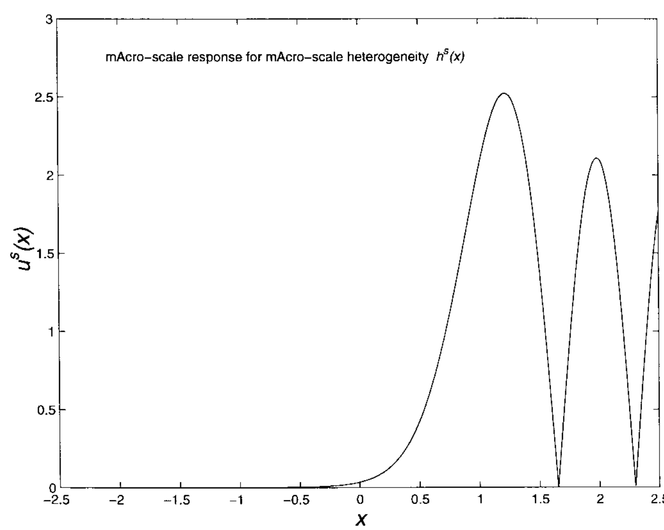


Fig. 7. The same as Fig. 6, but for h^s only [shown in Fig. 5(b)].

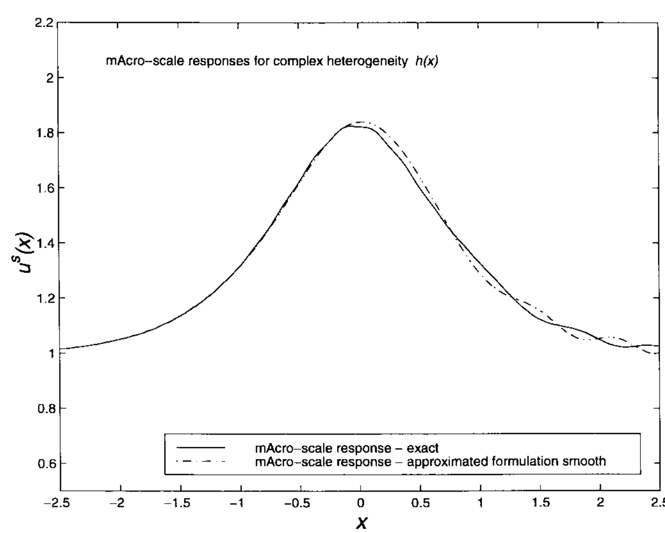


Fig. 8. Magnitudes of u^s for h shown in Fig. 5(a). Solid line (———): exact u^s. Dotted line (- - -): u^s obtained via the approximated smooth formulation (43).

imately factorized into two parts, each depending *linearly* on the heterogeneity h. Thus, in its approximated form, no inversion of small-scale information is needed. This fact has far-reaching ramifications. It means, for example, that when changes are introduced into either h^s or h^d, one need not solve a completely new problem or invert a new operator written entirely on a superfine grid. Use can be made of previous knowledge of the heterogeneity and its representing operators $\mathbf{C}$ and $\tilde{\mathbf{C}}$—the only component that depends on the altered part of h has to be re-evaluated and, in any case, this evaluation does not require operator inversion. These results, obtained via general asymptotic considerations, are demonstrated numerically for the case of 2-D scattering of TM wave from a laminated complex structure.

APPENDIX
THE BOUND ESTIMATE IN (33)

With the Euclidean matrix norm inequality $||\mathbf{\Psi}||^2 \leq \sum_{m',n',m,n} |\Psi_{m'n',mn}|^2$ and (32), one has

$$||\mathbf{\Psi}||^2 \leq C^4 \sum_{m',n',m,n} 2^{-3m'-5m}. \tag{A.1}$$

The summation over n and n' depends on m and m', respectively, as the number of grid points associated with resolution m scales like $N_0 2^m$, where $N_0 = 2^{-m_a} O(1)$ is the number of points at resolution 0. Thus

$$||\mathbf{\Psi}||^2 \leq C^4 2^{-2m_a} \sum_{m',m=j} 2^{-2m'-4m} = \frac{64}{45} C^4 2^{-2m_a} 2^{-6j}. \tag{A.2}$$

The result in (33) now follows from the condition that the scatterer outer dimension 2^{-m_a} is on the order of the wavelength λ, $\lambda = O(1)$, and $C = O(1)$.

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John J. McCoy, photograph and biography not available at the time of publication.