The EM Field of Constant Current Density Distributions in Parallelepiped Regions

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***Abstract*— The electromagnetic field vectors $\overline{A}$, $\overline{H}$, $\overline{E}$ arising from a constant current density $\overline{J}$ in an electrically small orthogonal parallelepiped region v are obtained analytically and exactly, up to order $(kr)^4$, at any point (x, y, z) a distance r from the center of v. They are then applied to the solution of an electric field integrodifferential equation (EFIDE) for which the region V has been divided into small parallelepiped cells. These new results are directly applicable to the evaluation of electromagnetic field interaction with natural media.**

***Index Terms*—Electromagnetic fields, hybrid methods.**

I. Introduction

With assumed time dependence $\exp(i\omega t)$, we are concerned with the basic integrals [1]–[6]

$$\overline{A}(\overline{r}) = \frac{\mu}{4\pi}\iiint_V \overline{J}(\overline{r'})\frac{e^{-ikR}}{R}\,dV'$$

$$\overline{H}(\overline{r}) = \frac{1}{4\pi}\nabla\times\iiint_V \overline{J}(\overline{r'})\frac{e^{-ikR}}{R}\,dV' \tag{1}$$

$$\overline{E}(\overline{r}) = -\frac{i\omega\mu}{4\pi}\left(1+\frac{1}{k^2}\nabla\nabla\cdot\right)\iiint_V \overline{J}(\overline{r'})\frac{e^{-ikR}}{R}\,dV' \tag{2}$$

in which $R = |\overline{r}-\overline{r'}|$, $k=\omega\sqrt{\mu\varepsilon}=2\pi/\lambda$, and $\overline{J}$ the (electric) current (or source) density. V is the volume where $\overline{J}\neq 0$ and, in particular, we are concerned here with $\overline{J}(\overline{r})$ continuous in V and with field points $\overline{r}$ in V. Closely related to the above integrals is the following equation holding both in the exterior and interior V of an inhomogeneous and isotropic medium $\varepsilon_v[\varepsilon_v = \varepsilon_v(\overline{r}) \neq \varepsilon \text{ in } V]$

$$\overline{E}(\overline{r}) = \overline{E}^i(\overline{r}) + \frac{1}{4\pi\varepsilon}(k^2+\nabla\nabla\cdot)\iiint_V(\varepsilon_v-\varepsilon)\overline{E}(\overline{r'}) \cdot\frac{e^{-ikR}}{R}\,dV'. \tag{3}$$

Here $\overline{E}^i$ is a given incident field and $i\omega[\varepsilon_v(\overline{r'})-\varepsilon]\overline{E}(\overline{r'})$ the equivalent polarization current density $\overline{J}(\overline{r'})$ in V. For $\overline{r}$ in V (3) is often mentioned in the literature as the "electric field integral equation" (EFIE) and provides the starting point for electromagnetic (EM) field evaluations inside dielectric media, human tissue, heating and hyperthermia applications, dielectric waveguides, etc. [6]–[13].

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The attention in the present paper is focused on orthogonal parallelepiped regions or cells $v = 2a\times 2b\times 2c$ into which any large region V can be divided, as a necessary step for solving the electric field integrodifferential equation (EFIDE) in it. A further restriction is to consider $\overline{J}$ constant in v. As long as v is electrically small this assumption is common in numerical evaluations of volume integrals. Thus, for small ka, kb, kc, and $\overline{J}$ constant in v the field quantities $\overline{H}$, $\overline{E}$ are evaluated in this paper analytically and exactly up to order $(kr)^4$ at any point (x, y, z) at a distance r from the center of v, both in the interior and the near exterior of v.

These near-field $\overline{E}$ values are directly usable for the evaluation of the integrodifferential term of the EFIDE (3) at the centers of the "self-cell" and its adjacent ones. Moreover, by comparing these values of $\overline{E}$ at adjacent cells with those obtained by the far field approximation $R\cong r$ one is led to the optimal size and shape of v and to great economy in matrix size and computer time.

II. The Vector Potential $\overline{A}$ and Its First Derivatives

Let $\overline{J} = \hat{u}J_u$ in $v = 2a\times 2b\times 2c$, with J_u and the unit vector $\hat{u}$ constant, and the origin of coordinates at the center of v. Assuming ka, kb, $kc \ll 1$ and keeping the first four terms in the expression of $\exp(-ikR)$, where $R^2 = (x'-x)^2+(y'-y)^2+(z'-z)^2$, we obtain from (1)

$$\begin{aligned}\frac{4\pi}{\mu J_u}&A_u(x, y, z)\\ &= \int_{-a}^{a}\int_{-b}^{b}\int_{-c}^{c}\frac{e^{-ikR}}{R}\,dx'dy'dz'\\ &\cong \int_{-a}^{a}\int_{-b}^{b}\int_{-c}^{c}\left(\frac{1}{R}-ik-\frac{k^2R}{2}+\frac{ik^3R^2}{6}\right)dx'dy'dz'\end{aligned} \tag{4}$$

at any point $\overline{r}(x, y, z)$ interior or in the near exterior of v such that $kr \ll 1$. The four basic indefinite integrals that come up in the course of integrating the first two terms of (4) are

$$\int\frac{dx}{\sqrt{x^2+A^2}} = \sinh^{-1}\left(\frac{x}{A}\right) \tag{5}$$

$$\int\sqrt{x^2+A^2}\,dx = \frac{x}{2}\sqrt{x^2+A^2}\,\frac{A^2}{2}\sinh^{-1}\left(\frac{x}{A}\right) \tag{6}$$

$$\begin{aligned}\int\sinh^{-1}&\left(\frac{A}{\sqrt{x^2+B^2}}\right)dx\\ &= x\sinh^{-1}\left(\frac{A}{\sqrt{x^2+B^2}}\right)+A\sinh^{-1}\left(\frac{x}{\sqrt{A^2+B^2}}\right)\\ &\quad -B\tan^{-1}\left(\frac{Ax}{B\sqrt{x^2+A^2+B^2}}\right)\end{aligned} \tag{7}$$

$$\begin{aligned}\int \Bigg[B \sinh^{-1}\left(\frac{A}{\sqrt{x^2+B^2}}\right) + A \sinh^{-1}\left(\frac{B}{\sqrt{x^2+A^2}}\right)\\ -x \tan^{-1}\left(\frac{AB}{x\sqrt{x^2+A^2+B^2}}\right)\Bigg] dx\\ = Bx \sinh^{-1}\left(\frac{A}{\sqrt{x^2+B^2}}\right) + Ax \sinh^{-1}\left(\frac{B}{\sqrt{x^2+A^2}}\right)\\ + AB \sinh^{-1}\left(\frac{x}{\sqrt{A^2+B^2}}\right)\\ - \frac{B^2}{2} \tan^{-1}\left(\frac{Ax}{B\sqrt{x^2+A^2+B^2}}\right)\\ - \frac{A^2}{2} \tan^{-1}\left(\frac{Bx}{A\sqrt{x^2+A^2+B^2}}\right)\\ - \frac{x^2}{2} \tan^{-1}\left(\frac{AB}{x\sqrt{x^2+A^2+B^2}}\right).\end{aligned} \tag{8}$$

The first two integrals (5) and (6) are quite common [14]. The other two, (8) in particular, are not easy to find in tables; however, the reader can verify them immediately by differentiating their right-hand-side. The final result is

$$\begin{aligned}\frac{4\pi}{\mu J_u} A_u(x, y, z)\\ = F(x, y, z) + F(x, y, -z) + F(x, -y, z)\\ + F(x - y, -z) + F(-x, y, z) + F(-x, y, -z)\\ + F(-x, -y, z) + F(-x, -y, -z) - ik8abc\\ = \Sigma 8[F(x, y, z)] - ik8abc.\end{aligned} \tag{9}$$

The notation $\Sigma 8[F(x, y, z)]$ is defined in (9). Also,

$$\begin{aligned}F(x, y, z) &= S_x(x, y, z) + S_y(x, y, z) + S_z(x, y, z)\\ &\quad - \frac{(a+x)^2}{2} \tan^{-1}(a_x) - \frac{(b+y)^2}{2} \tan^{-1}(a_y)\\ &\quad - \frac{(c+z)^2}{2} \tan^{-1}(a_z)\\ &= \{1 + CP + CP[CP]\}\\ &\quad \cdot \left[S_x(x, y, z) - \frac{(a+x)^2}{2} \tan^{-1}(a_x)\right]\end{aligned} \tag{10}$$

$$\begin{aligned}S_x(x, y, z) &= (b+y)(c+z) \sinh^{-1}\left[\frac{a+x}{\sqrt{(b+y)^2+(c+z)^2}}\right]\\ S_y(x, y, z) &= CP[S_x(x, y, z)]\\ S_z(x, y, z) &= CP[S_y(x, y, z)]\\ &= CP[CP[S_x(x, y, z)]]\end{aligned} \tag{11}$$

$$\begin{aligned}a_x(x, y, z) &= \frac{(b+y)(c+z)}{(a+x)\sqrt{(a+x)^2+(b+y)^2+(c+z)^2}}\\ a_y(x, y, z) &= CP(a_x), \quad a_z(x, y, z) = CP(a_y)\end{aligned} \tag{12}$$

where $CP[f(x, y, z)]$ denotes the function that follows from $f(x, y, z)$ by cyclical permutation of x, y, z and a, b, c, namely, by substituting $a \pm x, b \pm y, , c \pm z$ by $b \pm y, c \pm z, x \pm a$, respectively. The last two nonsingular terms of (4) are integrated only for the derivation of A_u.

Coming now to $\partial A_u/\partial x$ we can evaluate it in two ways: By differentiating (9) and by going back to (4) and passing $\partial/\partial x$ after the triple integral, keeping all four terms of (4), both procedures yield the same result (from the first two terms) in agreement with the general theory of singular integrals [1]–[4]. The final result [from all four terms of (4)] is

$$\frac{4\pi}{\mu J_u} \frac{\partial A_u}{\partial x} = D(x, y, z) - D(-x, y, z) + i\tfrac{8}{3}k^3 abcx \tag{13}$$

where

$$\begin{aligned}D(x, y, z)\\ = \sum_{yz} 4\Bigg\{ -(a+x) \tan^{-1}(a_x) + (c+z)\\ \cdot \sinh^{-1}\left(\frac{b+y}{\sqrt{(a+x)^2+(c+z)^2}}\right) + (b+y)\\ \cdot \sinh^{-1}\left(\frac{c+z}{\sqrt{(a+x)^2+(b+y)^2}}\right) - \frac{k^2}{12}\\ \cdot \Bigg[2(b+y)(c+z)\sqrt{(a+x)^2+(b+y)^2+(c+z)^2}\\ + (c+z)(3(a+x)^2+(c+z)^2)\\ \cdot \sinh^{-1}\left(\frac{b+y}{\sqrt{(a+x)^2+(c+z)^2}}\right)\\ + (b+y)(3(a+x)^2+(b+y)^2)\\ \cdot \sinh^{-1}\left(\frac{c+z}{\sqrt{(a+x)^2+(b+y)^2}}\right)\\ -2(a+x)^3 \tan^{-1}(a_x)\Bigg]\Bigg\}.\end{aligned} \tag{14}$$

The symbol $\Sigma_{yz} 4[f(x, y, z)]$ denotes the sum of four terms arising from the function $f(x, y, z)$ evaluated [besides the point (x, y, z)] at the points $(x, y, -z)$, $(x, -y, z)$, $(x, -y, -z)$ as well. The derivative $\partial A_u/\partial y$ follows from (13) and (14) by replacing x, y, a, b by y, x, b, a, respectively. These results serve, also, to express the magnetic field $\overline{H} = (1/\mu)\nabla \times \overline{A}$ everywhere.

A first conclusion, concerns the convergent nature of the integrals. Indeed, no matter how υ shrinks to zero, both A_u and $\partial A_u/\partial x$ tend to zero in and out of υ, as required by theory [1]–[4]. Finally, we examine the analyticity and continuity properties of A_u, $\partial A_u/\partial x$. Based on the limiting values $\sinh^{-1} x \cong x$, $\tan^{-1} x \cong x$ as $x \to 0$ and $\sinh^{-1} x \cong \ln(2x)$, $\tan^{-1}(\pm x) = \pm\pi/2$ as $x \to +\infty$ it is easy to conclude that all quantities in (10)–(12) and (14) remain finite and continuous, as the point (x, y, z) crosses some side of υ, even through an edge or tip of the parallelepiped. Thus, A_u, $\partial A_u/\partial x$ remain finite and continuous over all space, as required by the theory of convergent singular integrals; this was observed to hold for spherical υ, as well [2], [3].

III. Second Derivatives of A_u and the Electric Field

Apart from the singular term $1/R$ second derivatives can be obtained either by differentiating (13), (14) or by interchanging $\partial^2/\partial x^2$, $\partial^2/\partial x \partial y$ with the triple integrals of the these terms in (4). Differentiation of (13) and (14) is valid for all terms and the final results are

$$\begin{aligned}\frac{4\pi}{\mu J_u}\frac{\partial^2 A_u}{\partial x^2} =&\Sigma 8\Big\{-\tan^{-1}(a_x)-\frac{k^2}{2}[S_y(x,y,z)\\ &+S_z(x,y,z)-(x+a)^2\tan^{-1}(a_x)]\Big\}\\ &+i\tfrac{8}{3}k^3abc\\ \frac{\partial^2 A_u}{\partial y^2}=&CP\left[\frac{\partial^2 A_u}{\partial x^2}\right];\quad \frac{\partial^2 A_u}{\partial z^2}=CP\left[\frac{\partial^2 A_u}{\partial y^2}\right]\end{aligned}\tag{15}$$

$$\begin{aligned}\frac{4\pi}{\mu J_u}\frac{\partial^2 A_u}{\partial x\partial y}=&d(x,y,z)+d(x,y,-z)+d(-x,-y,z)\\ &+d(-x,-y,-z)-d(x,-y,z)\\ &-d(x,-y,-z)-d(-x,y,z)\\ &-d(-x,y,-z)+0[(kr)^4]\end{aligned}\tag{16}$$

$$\begin{aligned}d(x,y,z)=&\sinh^{-1}\left[\frac{c+z}{\sqrt{(a+x)^2+(b+y)^2}}\right]-\frac{k^2}{4}\\ &\cdot\Big\{(c+z)\sqrt{(a+x)^2+(b+y)^2+(c+z)^2}\\ &+[(a+x)^2+(b+y)^2]\\ &\cdot\sinh^{-1}\left[\frac{c+z}{\sqrt{(a+x)^2+(b+y)^2}}\right]\Big\}.\end{aligned}\tag{17}$$

The derivative $(\partial^2 A_u)/(\partial x\partial z)$ follows from (16) and (17) by replacing b, y, c, z by c, z, b, y, respectively.

The first test that the above expressions should satisfy is the Helmholtz equation

$$\nabla^2 A_u + k^2 A_u = \begin{cases}-\mu J_u, & \text{inside } v\\ 0, & \text{outside } v.\end{cases}\tag{18}$$

To this end we first observe that based on the identities [14]

$$\begin{aligned}\tan^{-1} z_1+\tan^{-1} z_2 &= \tan^{-1}\frac{z_1+z_2}{1-z_1z_2}\\ \tan^{-1}x+\tan^{-1}\frac{1}{x}&=\begin{cases}\pi/2, & \text{for } x>0\\ -\pi/2, & \text{for } x<0\end{cases}\end{aligned}\tag{19}$$

it is easy to establish the relation

$$\begin{aligned}&\tan^{-1}(a_x)+\tan^{-1}(a_y)+\tan^{-1}(a_z)\\ &\quad=\begin{cases}\pi/2, & \text{if } a_x>0\\ -\pi/2, & \text{if } a_x<0.\end{cases}\end{aligned}\tag{20}$$

Observing from (12) that a_x, a_y, a_z are all either positive or negative and that for points (x, y, z) interior to v all six quantities $a\pm x, b\pm y, c\pm z$ are positive we conclude that

$$\begin{aligned}&\Sigma 8[\tan^{-1}(a_x)+\tan^{-1}(a_y)+\tan^{-1}(a_z)]\\ &\quad=4\pi\quad\text{for}\quad |x|<a,\ |y|<b,\ |z|<c.\end{aligned}\tag{21}$$

while further consideration leads to the conclusion that $\Sigma 8[\tan^{-1}(a_x)+\tan^{-1}(a_y)+\tan^{-1}(a_z)]=0$ at any point exterior to v. Keeping this and (21) in mind, we may immediately verify that the Helmholtz equation (18) is indeed satisfied both in and out of v. Based on this first result and Maxwell's equations $\mu\overline{H}=\nabla\times\overline{A}$, $i\omega\varepsilon\overline{E}=\nabla\times\overline{H}-\overline{J}$ one may further verify that $\overline{H}$ and $\overline{E}$ satisfy throughout space the equations $\nabla x\nabla\times\overline{H}-k^2\overline{H}=\nabla\times\overline{J}=0$ and $\nabla\times\nabla\times\overline{E}-k^2\overline{E}=-i\omega\mu\overline{J}$.

Considering next the analytic and continuity properties of the second derivatives it is better to examine and explain them in connection with the electric field $\overline{E}$. Without actual loss of generality and to avoid lengthy expressions, it is better to restrict ourselves to the case $\hat{u}=\hat{x}$ and $\overline{J}=\hat{x}J_x$; cyclical permutation leads easily to the more general case $J_y, J_z\neq 0$. Then, in and out of v: $H_x=0$, $\mu H_y=\partial A_x/\partial z$, $\mu H_z=-\partial A_x/\partial y$, while from $i\omega\varepsilon\overline{E}=\nabla\times\overline{H}-\overline{J}$

$$\begin{aligned}E_x=&-\frac{1}{i\omega\varepsilon\mu}\left(\frac{\partial^2 A_x}{\partial y^2}+\frac{\partial^2 A_x}{\partial z^2}+\mu J_x\right)\\ =&\frac{1}{i\omega\varepsilon\mu}\left[\frac{\partial^2 A_x}{\partial x^2}+k^2A_x\right]\\ =&\frac{J_x}{i4\pi\omega\varepsilon}\Big\{\Sigma 8\Big[-\tan^{-1}(a_x)+\frac{k^2}{2}(2S_x+S_y+S_z\\ &-(b+y)^2\tan^{-1}(a_y)-(c+z)^2\tan^{-1}(a_z))\Big]\\ &-i\frac{16}{3}k^3abc\Big\}\end{aligned}\tag{22}$$

$$E_y=\frac{1}{i\omega\varepsilon\mu}\frac{\partial^2 A_x}{\partial x\partial y},\quad E_z=\frac{1}{i\omega\varepsilon\mu}\frac{\partial^2 A_x}{\partial x\partial z}.\tag{23}$$

The analyticity of $\overline{E}(x, y, z)$ in and out of v is again obvious. However, as (x, y, z) now crosses either of the surfaces $x=a$ or $x=-a$ of v, four of the arguments a_x of $\Sigma 8[\tan^{-1}(a_x)]$ in (23) jump from $+\infty$ to $-\infty$ and each such $\tan^{-1}(a_x)$ from $+\pi/2$ to $-\pi/2$. Therefore, E_x exhibits a step discontinuity $(J_x/i\omega\varepsilon)$, consistent with the continuity relation $\nabla\cdot\overline{E}=\rho/\varepsilon=-\nabla\cdot\overline{J}/i\omega\varepsilon$ or here with the constant surface charge density $\sigma=J_x/i\omega$ generated on $x=a$ by the abrupt change of the value of J_x [2], [3]. E_x remains continuous at the other surfaces $y=\pm b$, $z=\pm c$ to which J_x is parallel and on which no surface charge accumulates.

Coming now to E_y and E_z we observe from (16) and (17) that they become infinite at the edges $x=\pm a$, $y=\pm b$ and $x=\pm a$, $z=\pm c$ to which they are perpendicular, respectively. The infinity is logarithmic, arising from terms of the form $\sinh^{-1}[(c+z)/\rho]+\sinh^{-1}[(c-z)/\rho]$—for E_y—where $\rho=\sqrt{(a+x)^2+(b+y)^2}$ is the distance from the edge; as $\rho\to 0$ these two terms behave as $\ln[2(c+z)/\rho]+\ln[2(c-z)/\rho]$ for $|z|<c$, as $\ln[(c+z)/(|c-z|)]$ for $z>c$ and as $\ln[(c-z)/(|c+z|)]$ for $z<-c$. Therefore, in the last two cases, E_y is analytic at the extension of the edge in the exterior of v (it simply approaches infinity near the tip), but in the interior $|z|<c$ E_y goes to ∞ logarithmically. This behavior may be explained by the abrupt termination of σ at these edges and has no counterpart for surfaces S with continuous normal [2], [3], where σ, arising again from the normal component of $\overline{J}$ on S, vanishes continuously rather than abruptly on S. It may be verified further by the exactly similar behavior near the edges $y=0$, w of the E_y component of the electrostatic field of a constant surface charge density σ spread over an infinite (in z) strip $0\leq y\leq w$ on the plane $x=0$; this field can be obtained by simple integrals of the E_y-field due

to infinite line charges $\sigma dy'$ over $0 \leq y' \leq w$. Therefore, this logarithmic infinite behavior is characteristic of (dielectric) edges of v or S. However, when v is part of a larger volume V with continuous $\overline{J}$ in it, all these discontinuities and infinities cancel one another due to the opposite field values from the adjacent to v cells.

Before closing this section, following are some new results for the value of $\overline{E}(x, y, z)$ at the center of v:

$$E_y(0, 0, 0) = E_z(0, 0, 0) = 0$$

$$\begin{aligned} E_x(0, 0, 0) = &-\frac{J_x}{i\pi\omega\varepsilon}\Bigg\{2\tan^{-1}\left(\frac{bc}{a\sqrt{a^2+b^2+c^2}}\right) \\ &- k^2\Bigg[2bc\sinh^{-1}\left(\frac{a}{\sqrt{b^2+c^2}}\right) \\ &+ ac\sinh^{-1}\left(\frac{b}{\sqrt{a^2+c^2}}\right) \\ &+ ab\sinh^{-1}\left(\frac{c}{\sqrt{a^2+b^2}}\right) \\ &- b^2\tan^{-1}\left(\frac{ac}{b\sqrt{a^2+b^2+c^2}}\right) \\ &- c^2\tan^{-1}\left(\frac{ab}{c\sqrt{a^2+b^2+c^2}}\right)\Bigg] + i\frac{4k^3abc}{3}\Bigg\}. \end{aligned} \tag{24}$$

In particular, for cubical v, $a = b = c$ and

$$\begin{aligned} &E_x(0, 0, 0) \\ &= -\frac{J_x}{\pi i\omega\varepsilon}[1.047\,20 - 1.586\,72(ka)^2 + i1.333\,33(ka)^3] \\ &= -\frac{J_x}{3i\omega\varepsilon}[1 - 1.515\,20(ka)^2 + i1.273\,24(ka)^3]. \end{aligned} \tag{25}$$

In the last expression, the constant value $-J_x/(3i\omega\varepsilon)$ from the singular term $1/R$ in the preceding integral (2) is the same as at the center of a spherical v [2], [3]. For cubes, it has been known for a long time; for parallelepipeds, the first term of (24) can also be found in [6], [9], and [10]. Here, we include higher order terms and are able to evaluate $\overline{E}$ at any interior or near-exterior point.

IV. Application to the Solution of the EFIDE

The E-field expressions (3), (4), (16), and (17) can be used immediately for the solution of the EFIDE (3), in which region V has been divided into a number of parallelepiped cells $v = 2a \times 2b \times 2c$. The last (integrodifferential) term on the right of (3) is the E-field in V arising from $\overline{J} = i\omega(\varepsilon_v - \varepsilon)\overline{E}$ and values of it at any fixed point of each cell (in particular its center) due to $\overline{J}$ in v itself and in its neighboring cells are directly obtainable, as long as kr is small. The influence of more remote cells is easily evaluated using the "far-field" expressions (26) and (27) given below. An important question, is how small kr should be to insure the accuracy of these near-field values. This question, directly related to the important question of how small ka, kb, kc should be taken, can be answered reliably if we compare these near-field values with those obtained very simply from (2), when, at points exterior to v, the ratio r/α (α being the greater among a, b, c) is large enough to justify the approximation $R \cong r$. The resulting far-field (dipole) expressions are [6]

$$\begin{aligned} E_x = &-\frac{J_x}{i4\pi\omega\varepsilon}(8abc)\frac{e^{-ikr}}{r^3} \\ &\cdot\left[1 + ikr - k^2r^2 - \frac{x^2}{r^2}(3 + 3ikr - k^2r^2)\right] \end{aligned} \tag{26a}$$

$$\begin{aligned} \cong &-\frac{J_x}{i4\pi\omega\varepsilon}\frac{8abc}{r^3}\Bigg[1 - \frac{3x^2}{r^2} - \frac{k^2r^2}{2}\left(1 + \frac{x^2}{r^2}\right) \\ &+ i\frac{2}{3}k^3r^3 + \frac{k^4r^4}{8}\left(3 - \frac{x^2}{r^2}\right) - i\frac{k^5r^5}{15}\left(2 - \frac{x^2}{r^2}\right) \\ &+ O(k^6r^6)\Bigg] \end{aligned} \tag{26b}$$

$$\begin{pmatrix} E_y \\ E_z \end{pmatrix} = \frac{J_x}{i4\pi\omega\varepsilon}(8abc)\frac{xe^{-ikr}}{r^5}\begin{pmatrix} y \\ z \end{pmatrix}(3 + 3ikr - k^2r^2) \tag{27a}$$

$$\begin{aligned} \cong &\frac{J_x}{i4\pi\omega\varepsilon}(8abc)\frac{x}{r^5}\begin{pmatrix} y \\ z \end{pmatrix} \\ &\cdot\left[3 + \frac{k^2r^2}{2} + \frac{k^4r^4}{8} - i\frac{k^5r^5}{15} + O(k^6r^6)\right]. \end{aligned} \tag{27b}$$

It is worth noticing that: 1) Both near- and far-field expressions for E_x, E_y, E_z yield—for real J_x—imaginary values for zero and second order terms and the same real and constant third-order terms, namely, $[-4J_xk^3abc/(3\pi\omega\varepsilon)]$ for E_x and zero for E_y, E_z; 2) apart from the multiplying factor $8abc$ the far-field expressions are insensitive to the shape of v; they stand for the dipole field that prevails away enough from v and contrasts with the near-field expressions, which depend strongly on the shape of v; and 3) comparing with terms of order k^4r^4 and k^5r^5 and taking $kr = 1/3$, for instance, one easily concludes that retaining terms of order up to and including k^3r^3 in (26b), (27b) provides an accuracy better than 0.1% for E_y, E_z and, apart from directions for which $x^2/r^2 \cong 1/3$, better than 0.5% for E_x. Even in the latter case, the accuracy continues to be better than 0.1%, when it is realized that E_y and/or E_z are now the prevailing components. Numerical and plotted results that follow verify these remarks.

Proceeding now with comparisons of same-order-terms from near- and far-field expressions, we consider two shapes for v: $a = b = c$ (cube) and $b = c = a/2$. Numerical results are shown in Tables I and II for the normalized and dimensionless field quantity $\overline{e} = (i4\pi\omega\varepsilon/J_x)\overline{E}$. Only positive values of x, y, z in half of the first octant need be considered, owing to configurational symmetry. The following are worth noticing.

1) The cube is the more advantageous shape, as far as overlapping and least differing values between near- and far-field expressions are concerned. The percent difference of the dominant constant term for adjacent cells in the x and y directions does not exceed 18.1% for cubes. For longer parallelepipeds $b = c = a/2$, it reaches 178% (and for much elongated ones $b = c = a/5$ it grows to 1897%!) One has to move quite a bit away from v in the last two cases for the far-field expressions to take

TABLE I
CUBE: $a = b = c$. VALUES OF THE DOMINANT TERM AND THE COEFFICIENT OF THE k^2a^2-TERM FROM NEAR- AND FAR-FIELD EXPRESSIONS AT THE CENTERS OF THE SELF CELL AND NEARBY EXTERIOR ONES. SHOWN ARE e_x AND e_y VALUES FOR THE NORMALIZED DIMENSIONLESS FIELD $\overline{e} = i4\pi\omega\varepsilon\overline{E}/J_x$ AND CORRESPONDING PERCENT DIFFERENCES ($a = \alpha$)

1	2	3	4	5	6	7	8	9	10	11	12	13
point (x,y,z)	Constant term of e_x		% diff. from col.2	Coeff. of $(ka)^2$-term of e_x		% diff. from col.5	Constant term of e_y		% diff. from col.8	Coeff. of $(ka)^2$-term of e_y		% diff. from col.11
	From (22)	From (26b)		From (22)	From (26b)		From (16),(17) (23)	From (27b)		From (16),(17) (23)	From (27b)	
(0,0,0)	-4.18879			6.34687			0			0		
(2a,0,0)	1.69373	2.00000	-18.08	3.60593	4.00000	-10.93	0	0		0	0	
(0,2a,0)	-0.846863	-1.00000	-18.08	2.14741	2.00000	6.86	0	0		0	0	
(2a,2a,0)	0.172739	0.176777	-2.34	2.09094	2.12132	-1.45	0.539378	0.530330	1.68	0.622330	0.707107	-13.62
(0,2a,2a)	-0.345479	-0.353553	-2.34	1.47865	1.41421	4.36	0	0		0	0	
(2a,2a,2a)	0	0		1.54142	1.53960	0.12	0.196166	0.192450	1.89	0.354229	0.384900	-8.66
(4a,0,0)	0.246786	0.250000	-1.30	1.95534	2.00000	2.28	0	0		0	0	
(0,4a,0)	-0.123393	-0.12500	-1.30	1.02056	1.00000	2.01	0	0		0	0	
(2a,4a,0)	-0.0364791	-0.0357771	1.92	1.07840	1.07331	0.47	0.106794	0.107331	-0.50	0.339494	0.357771	-5.38
(0,4a,2a)	-0.0890090	-0.0894427	-0.49	0.909651	0.894427	1.67	0	0		0	0	
(4a,2a,0)	0.125488	0.125220	0.21	1.58921	1.60997	-1.31	0.106794	0.107331	-0.50	0.339494	0.357771	-5.38
(4a,4a,0)	0.0220746	0.0220971	-0.10	1.05696	1.06066	-0.35	0.0663782	0.0662213	0.13	0.342649	0.353553	-3.18
(0,4a,4a)	-0.0441491	-0.0441942	-0.10	0.714651	0.707107	1.06	0	0		0	0	
(2a,4a,4a)	-0.0247274	-0.0246914	0.15	0.744909	0.740741	0.55	0.0246916	0.0246914	0.11	0.144020	0.148148	-2.87
(4a,4a.4a)	0	0		0.769854	0.769800	0.007	0.0240817	0.0240563	0.11	0.188487	0.192450	-2.10
(6a,0,0)	0.0738788	0.0740741	-0.26	1.32057	1.33333	-097	0	0		0	0	
(0,6a,0)	-0.0369394	-0.0370370	-0.26	0.672814	0.666667	0.91	0	0		0	0	

over satisfactorily. This, of course, is a manifestation of the fact that in the exterior near field the influence of shape is very important and that the far-field (dipole) expressions do not yield acceptable field values at points close to v.

2) The percent difference of the less dominant k^2a^2 terms is consistently much smaller (even for adjacent cells), while for the k^3a^3 terms it was shown to be zero everywhere.
3) In particular, in directions for which $x^2 = r^2/3$, the dominant constant term of E_x in the far field is zero, as seen from (26b). The same term from the near-field expression (22), is zero for cubes and very small for more elongated v; here, dominant are the other field components E_y, E_z (see Tables I and II).
4) Another proof of this relationship between near- and far-field expressions is based on the diminishing percent difference between their corresponding terms as the ratio r/a increases. The table results verify this remark for point x, y, z for which r/a is large (particularly for the dominant component of $\overline{E}$) and there is a simple theoretical explanation for this behavior. As r/a increases, the percent difference between corresponding terms in near- and far-field expressions essentially reflects the same difference between the exact integral $I_{Rn} = \iiint_v R^n\,dV$ and its far-field approximation $I_{rn} = \iiint_v r^n\,dV = 8abcr^n$ for $n = -1, 0, 1, 2, \cdots$. As r/a increases, the obvious inequalities

$$r - \delta a \le R \le r + \delta a, \quad \delta = \sqrt{1 + \left(\frac{b}{a}\right)^2 + \left(\frac{c}{a}\right)^2}$$
$$1 \le \delta \le \sqrt{3} \tag{28}$$

lead to $(r - \delta a)^n 8abc \le I_{Rn} \le (r + \delta a)^n abc$ and, finally, to

$$\left|\frac{I_{Rn} - I_{rn}}{I_{rn}}\right| \le |n|\frac{a}{r} \tag{29}$$

which, for the first few values of n, verifies the preceding statement.

It remains to answer how large ka (with a the larger of a, b, c) can be taken for a prescribed accuracy for $\overline{e}$ and at what r/a value the simple (complete) far-field values (26a) and (27a) [not the restricted (26b) and (27b) ones] can take over satisfactorily from the near-field ones. To this end, we have calculated and plotted in Fig. 1 the total field $|\overline{e}| = [|e_x|^2 + |e_y|^2 + |e_z|^2]^{1/2} = |(4\pi\omega\varepsilon/J_x)\overline{E}|$ from the above two sets of expressions, namely $|\overline{e}_n|$, $|\overline{e}_f|$ in the exterior of v for about $2.5 \le r/a \le 12$, in three typical directions $r = x(y = z = 0)$, $r = y(x = z = 0)$, $x = y = z = r/\sqrt{3}$, in half of the first octant and for cubical $v(a = b = c)$ with $ka = 0.1$. These plots differ unnoticeably for other values of ka, owing to the scales and to the small difference between $|\overline{e}_n|$ and $|\overline{e}_f|$. In order to select ka and transition-r/a value for a prescribed accuracy, more useful are the plots of the percent difference (PCD) $100(|\overline{e}_n| - |\overline{e}_f|)/|\overline{e}_n|$, which is shown in Figs. 2 and 3, corresponding to the configurations of Tables I and II, respectively, with ka as a parameter. We further notice at this point, that the first two dominant terms of the $\overline{e}$-components are real, while the third, the imaginary one, is the same for both near- and far-field expressions; this implies that the phase of $\overline{e}$ is well taken into account in the overlapping region (small PCD), even when the comparison is limited between magnitude $|\overline{e}|$ values. Looking at the plots of the PCD in Figs. 2 and 3, we can easily conclude that the y and x directions are the worst as far as the PCD in the

TABLE II
SAME AS TABLE I FOR $b = c = a/2$ ($a = \alpha$)

1	2	3	4	5	6	7	8	9	10	11	12	13
point (x,y,z)	Constant term of e_x		% diff. from col.2	Coeff. of $(ka)^2$-term of e_x		% diff. from col.5	Constant term of e_y		% diff. from col.8	Coeff. of $(ka)^2$-term of e_y		% diff. from col.11
	From (22)	From (26b)		From (22)	From (26b)		From (16),(17) (23)	From (27b)		From (16),(17) (23)	From (27b)	
(0,0,0)	-1.61086			2.65702			0			0		
(0,a,0)	-0.719668	-2.00000	-177.9	1.086122	1.000000	7.93	0	0		0	0	
(0,2a,0)	-0.187671	-0.250000	-33.21	0.521037	0.500000	4.04	0	0		0	0	
(0,6a,0)	-0.0089727	-0.0092593	-3.19	0.167610	0.166667	0.56	0	0		0	0	
(0,10a,0)	-0.00197758	-0.00200000	-1.13	0.100207	0.100000	0.21	0	0		0	0	
(0,12a,0)	-0.0011484	-0.0011574	-0.79	0.0834532	0.0833333	0.14	0	0		0	0	
(2a,0,0)	0.697310	0.500000	28.30	1.033551	1.00000	3.25	0	0		0	0	
(4a,0,0)	0.0685167	0.0625000	8.78	0.505014	0.500000	0.99	0	0		0	0	
(10a,0,0)	0.00406028	0.0040000	1.48	0.200332	0.200000	0.17	0	0		0	0	
(2a,a,0)	0.266719	0.250440	6.10	0.804288	0.804985	-0.09	0.288905	0.214663	25.70	0.179957	0.178885	0.60
(2a,2a,0)	0.0301294	0.0441942	-46.68	0.522066	0.530330	-1.58	0.136866	0.132583	3.13	0.166532	0.176777	-6.15
(2a,6a,0)	-0.00550683	-0.00553399	-0.49	0.174131	0.173925	0.12	0.0068610	0.0071151	-3.70	0.0460443	0.0474342	-3.02
(2a,a,a)	0.122278	0.136083	-11.29	0.672250	0.680414	-1.21	0.160023	0.136083	14.96	0.131833	0.136083	-3.22
(2a,2a,2a)	-0.00454901	0	∞	0.381215	0.384900	-0.97	0.0463656	0.0481125	-3.77	0.0902708	0.0962250	-6.60
(8a,6a,0)	0.0018259	0.0018400	-0.77	0.163857	0.164000	-0.09	0.0029066	0.0028800	0.92	0.479142	0.480000	-0.18
(6a,8a,0)	0.00013555	0.00016000	-18.04	0.135823	0.136000	-0.13	0.0028713	0.0028800	-0.30	0.0476641	0.0480000	-0.70
(4a,10a,4a)	-0.00083965	-0.00083922	0.05	0.0976048	0.0975890	0.02	0.00118670	0.00119888	-1.03	0.0261470	0.0263754	-0.87

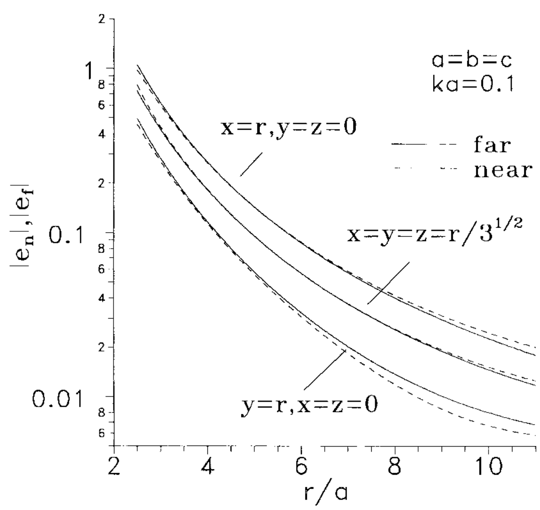

Fig. 1. Cube: $a = b = c$. Exterior field values $|\overline{e}_n|$, $|\overline{e}_f|$ versus r/a for $ka = 0.1$.

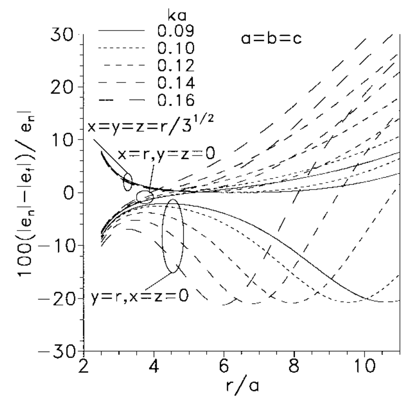

Fig. 2. Cube: $a = b = c$. PCD of $|\overline{e}_n|$, $|\overline{e}_f|$ versus r/a for various ka.

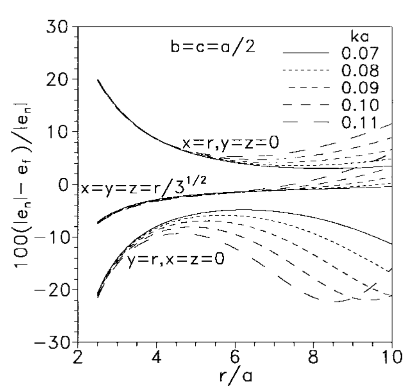

Fig. 3. Same as Fig. 2 for $b = c = a/2$.

overlapping region is concerned. This helps make the choice of the optimum ka and transition r/a values by just restricting the calculations in either the x or y directions or both, at the most.

The following conclusions may then be drawn: For cubes $a = b = c$, Figs. 1 and 2 show that $ka = 0.12$ leads to a PCD of less than 4% and transition r/a around 3.8, $ka = 0.1$ to a PCD of less than 2.7%, and transition r/a around 4.1, and so on. The overlapping region is quite wide, as could be expected, while the resulting minimum PCD for each ka in the overlapping region insures an even higher accuracy when the near/far-field expressions are used for points closer to/farther from v than the above transition-r/a value. Higher values of ka are possible at the expense of PCD and accuracy (for instance, $ka = 0.16$ lowers the optimum PCD to 7% and transition r/a to 3.2). It should always be remembered in this connection that larger values of ka, although desirable when solving an EFIDE, are actually restricted from the start by two fundamental assumptions of this approach: 1) that $\overline{J}$ can be maintained constant throughout v in an actual situation and 2) that the assumption $R \cong r$ in the phase factors $\exp(ikR) \cong \exp(ikr)$, as opposed to $1/R \cong 1/r$, requires definitely small ka; otherwise the far-field expressions (26a) and (27a) are useless, regardless of how large r/a is taken. This latter restriction does not apply to the near-field expressions, which can be extended to quite higher ka values by including terms of order higher than k^3r^3 in the integration, something not prohibitive as far as this author is concerned. This appears quite unnecessary, however, as far as solving an EFIDE with near cubical v is concerned.

For elongated shapes smaller ka are required for the same PCD; for instance, $ka = 0.08$ gives PCD of about 6% around $r/a \cong 6.5$ (Fig. 3 for $b = c = a/2$), while for thin v, $b = c = a/5$, it may not be possible to lower PCD below 5% even with $ka = 0.06$. That is another case, where terms of order higher than k^3r^3 may be necessary for lower PCD and larger ka.

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