Plane Wave Diffraction by a Thick-Walled Parallel-Plate Impedance Waveguide

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***Abstract*— The diffraction of E-polarized plane waves by a thick-walled parallel-plate impedance waveguide is investigated rigorously by using the Fourier transform technique in conjunction with the mode-matching method. This mixed method of formulation gives rise to a *scalar* Wiener-Hopf equation of the second kind, the solution of which contains infinitely many constants satisfying an infinite system of linear algebraic equations. A numerical solution of this system is obtained for various values of the plate impedances, plate thickness, and the distance between the plates through which the effect of these parameters on the diffraction phenomenon are studied.**

***Index Terms*— Electromagnetic diffraction, parallel-plate waveguides.**

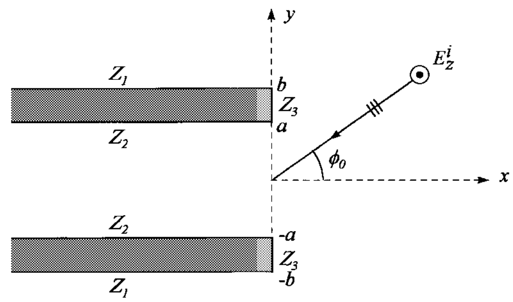

Fig. 1. Parallel-plate waveguide with thick impedance.

I. Introduction

The diffraction and radiation by open-ended parallel-plate waveguides is a classical problem, which, up until now, has been subjected to numerious investigations. Exact closed-form solutions are available only in few cases and have been obtained through the Wiener–Hopf technique (see, for example, [1] and [2]). All these considerations were based upon the basic assumptions of infinitely thin waveguide walls and of perfect conductivity. In practice, walls of a waveguide are neither infinitely thin nor perfectly conducting. It is, therefore, desirable to discuss the diffraction characteristics of a parallel-plate waveguide with certain wall thickness and satisfying impedance boundary conditions. Note that the diffraction by an open-ended waveguide with infinitely thin impedance walls were treated in [3] by formulating the problem in terms of four coupled Wiener–Hopf equations.

In the present work, the diffraction of E_z-polarized plane waves by a parallel plate waveguide with thick impedance walls will be analyzed rigorously by using the Wiener–Hopf technique in conjunction with the mode-matching method. By using the classical Fourier transform technique, the related boundary value problem can generally be reduced into a modified *matrix* Wiener–Hopf equation. Except for a very restricted class of matrices, no general method exists to accomplish the Wiener–Hopf factorization of an arbitrary matrix including the one related to the present problem. This mixed method of formulation, which is based on expanding the diffracted field into a series of normal modes in the waveguide region and using the Fourier transform technique elsewhere, gives rise to a *scalar* modified Wiener–Hopf equation. Note that a variant of this method was used by Matsiu [4] and then by Ando [5] for the problem of diffraction of sound waves by a semi-infinite cylindrical rigid tube of certain wall thickness, and by Yoshidomi and Aoki [6] in treating the scattering of an E-polarized plane wave by two parallel rectangular impedance cylinders. The solution contains infinitely many constants satisfying an infinite system of linear algebraic equations. A numerical solution of this system is obtained for various values of the plate impedances, plate thicknesses, and the distance between the plates, through which the effect of these parameters on the diffraction phenomenon are studied.

A time factor e^{-iwt} with w being the angular frequency is assumed and suppressed throughout the paper.

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II. Analysis

We consider the diffraction of an E_z-polarized plane wave by a waveguide formed by two thick semi-infinite impedance plates defined by $\mathcal{S}_1 = \{(x,y,z);\ x \in (-\infty,0),\ y \in (a,b),\ z \in (-\infty,\infty)\}$, and $\mathcal{S}_2 = \{(x,y,z);\ x \in (-\infty,0),\ y \in (-b,-a),\ z \in (-\infty,\infty)\}$, respectively, as depicted in Fig. 1. The surface impedances of the horizontal walls $y = \pm b,\ x < 0$ and $y = \pm a,\ x < 0$ are denoted by $Z_1 = \eta_1\ Z_0$ and $Z_2 = \eta_2\ Z_0$, respectively, while the impedance of the vertical walls $x = 0,\ y \in (a,b)$ and $x = 0,\ y \in (-b,-a)$ is $Z_3 = \eta_3\ Z_0$, with Z_0 being the characteristic impedance of the free-space.

In order to determine the scattered field, one can proceed by decomposing the incident wave into even and odd excitations as indicated in Fig. 2(a) and (b). Relying upon the image bisection principle, it can be shown that the configurations shown in Fig. 2(a) and (b) are equivalent to those depicted in Fig. 2(c) and (d), respectively. In what follows, the even and odd excitations will be treated separately.

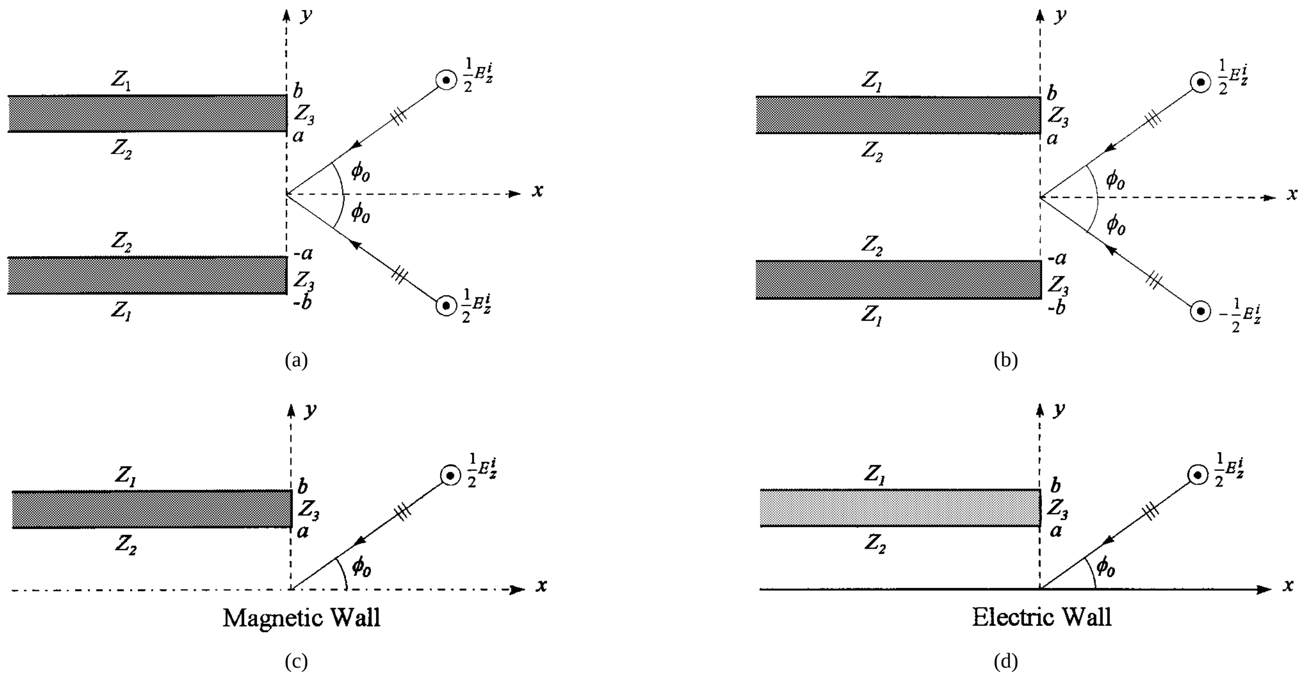

Fig. 2. Equivalent problems. (a) Symmetric (even) excitation. (b) Asymmetric (odd) excitation. (c) Equivalence to (a). (d) Equivalence to (b).

A. Even Excitation

Let us consider first the configuration depicted in Fig. 2(c), which is equivalent to the even excitation case. Since in this case the field is symmetrical about the plane $y=0$, the normal derivative of the total electric field must vanish for $y=0$, $x\in(-\infty,\infty)$ (magnetic wall).

For analysis purposes, it is convenient to express the total field as follows:

$$u_T^{(e)}(x,y)=\begin{cases}u^i+u^r+u_1^{(e)}, & y>b\\ u_2^{(e)}, & 0<y<a,\quad x<0\\ u_3^{(e)}, & 0<y<b,\quad x>0.\end{cases}\tag{1a}$$

Here, u^i is the incident field given by

$$E_z^i=u^i(x,y)=\exp\{-ik[x\cos\phi_0+y\sin\phi_0]\}\tag{1b}$$

while u^r denotes the field reflected from the plane $y=b$, namely

$$u^r(x,y)=\frac{\eta_1\sin\phi_0-1}{\eta_1\sin\phi_0+1}\exp\{-ik[x\cos\phi_0-(y-2b)\sin\phi_0]\}\tag{1c}$$

with k being the free-space wave number. $u_j^{(e)}$, $j=1,2,3$, which satisfy the Helmholtz equation, are to be determined with the aid of the following boundary and continuity relations:

$$\left(1+\frac{\eta_1}{ik}\frac{\partial}{\partial y}\right)u_1^{(e)}(x,b)=0,\quad x<0\tag{2a}$$

$$\left(1-\frac{\eta_2}{ik}\frac{\partial}{\partial y}\right)u_2^{(e)}(x,a)=0,\quad x<0\tag{2b}$$

$$\frac{\partial}{\partial y}u_2^{(e)}(x,0)=0,\quad x<0\tag{2c}$$

$$\frac{\partial}{\partial y}u_3^{(e)}(x,0)=0,\quad x>0\tag{2d}$$

$$u_1^{(e)}(x,b)-u_3^{(e)}(x,b)=-\frac{2\eta_1\sin\phi_0}{1+\eta_1\sin\phi_0}e^{-ikb\sin\phi_0}\,e^{-ikx\cos\phi_0},\quad x>0\tag{2e}$$

$$\frac{\partial}{\partial y}u_1^{(e)}(x,b)-\frac{\partial}{\partial y}u_3^{(e)}(x,b)=\frac{2ik\sin\phi_0}{1+\eta_1\sin\phi_0}e^{-ikb\sin\phi_0}e^{-ikx\cos\phi_0},\quad x>0\tag{2f}$$

$$u_2^{(e)}(0,y)=u_3^{(e)}(0,y),\quad 0<y<a\tag{2g}$$

$$\frac{\partial}{\partial x}u_2^{(e)}(0,y)=\frac{\partial}{\partial x}u_3^{(e)}(0,y),\quad 0<y<a\tag{2h}$$

$$\left(1+\frac{\eta_3}{ik}\frac{\partial}{\partial x}\right)u_3^{(e)}(0,y)=0,\quad a<y<b.\tag{2i}$$

Since $u_1^{(e)}(x,y)$ satisfies the Helmholtz equation in the range $x\in(-\infty,\infty)$, its Fourier transform with respect to x gives

$$\left[\frac{d^2}{dy^2}+(k^2-\alpha^2)\right]F^{(e)}(\alpha,y)=0\tag{3a}$$

with

$$F^{(e)}(\alpha,y)=F_+^{(e)}(\alpha,y)+F_-^{(e)}(\alpha,y)\tag{3b}$$

where

$$F_\pm^{(e)}(\alpha,y)=\pm\int_0^{\pm\infty}u_1^{(e)}(x,y)e^{i\alpha x}\,dx.\tag{3c}$$

By taking into account the following asymptotic behaviors of $u_1^{(e)}$ for $x\to\pm\infty$

$$u_1^{(e)}(x,y)=\begin{cases}O(e^{-ikx}), & x\to-\infty\\ O(e^{-ikx\cos\phi_0}), & x\to\infty\end{cases}\tag{4}$$

one can show that $F_+^{(e)}(\alpha,y)$ and $F_-^{(e)}(\alpha,y)$ are regular functions of α in the half-planes $\Im m(\alpha)>\Im m(k\cos\phi_0)$ and

$\Im m(\alpha) < \Im m(k)$, respectively. The general solution of (3a), satisfying the radiation condition for $y \to \infty$, reads

$$F_{+}^{(e)}(\alpha, y) + F_{-}^{(e)}(\alpha, y) = A^{(e)}(\alpha)\, e^{iK(\alpha)(y-b)} \tag{5a}$$

with

$$K(\alpha) = \sqrt{k^2 - \alpha^2}. \tag{5b}$$

The square-root function is defined in the complex α plane cut along $\alpha = k$ to $\alpha = k + i\infty$ and $\alpha = -k$ to $\alpha = -k - i\infty$, such that $K(0) = k$.

In the Fourier transform domain (2a) takes the form

$$F_{-}^{(e)}(\alpha, b) + \frac{\eta_1}{ik}\dot{F}_{-}^{(e)}(\alpha, b) = 0 \tag{6}$$

where the dot specifies the derivative with respect to y. By using (5a) (its derivative with respect to y) and (6), we get

$$R_{+}^{(e)}(\alpha) = \frac{K(\alpha)}{k\chi(\alpha)} A^{(e)}(\alpha) \tag{7a}$$

where

$$R_{+}^{(e)}(\alpha) = F_{+}^{(e)}(\alpha, b) + \frac{\eta_1}{ik}\dot{F}_{+}^{(e)}(\alpha, b) \tag{7b}$$

and

$$\chi(\alpha) = \left[\eta_1 + \frac{k}{K(\alpha)}\right]^{-1}. \tag{7c}$$

In the region $0 < y < b$, $u_3^{(e)}(x, y)$ satisfies the Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right) u_3^{(e)}(x, y) = 0 \tag{8}$$

in the range $x > 0$. The half-range Fourier transform of (8) yields

$$\left[\frac{d^2}{dy^2} + K^2(\alpha)\right] G_{+}^{(e)}(\alpha, y) = f^{(e)}(y) + \alpha g^{(e)}(y) \tag{9a}$$

with

$$f^{(e)}(y) = \frac{\partial}{\partial x} u_3^{(e)}(0, y), \quad g^{(e)}(y) = -i u_3^{(e)}(0, y). \tag{9b, c}$$

$G_{+}^{(e)}(\alpha, y)$, which is defined by

$$G_{+}^{(e)}(\alpha, y) = \int_0^\infty u_3^{(e)}(x, y) e^{i\alpha x}\, dx \tag{10}$$

is a function regular in the half-plane $\Im m(\alpha) > \Im m(-k)$. The general solution of (9a) satisfying the Neumann boundary condition at $y = 0$ reads

$$G_{+}^{(e)}(\alpha, y) = B^{(e)}(\alpha)\cos[Ky] + \frac{1}{K(\alpha)}\int_0^y \left[f^{(e)}(t) + \alpha g^{(e)}(t)\right] \sin[K(y-t)]\, dt. \tag{11}$$

Combining (2e) and (2f), we get

$$R_{+}^{(e)}(\alpha) = G_{+}^{(e)}(\alpha, b) + \frac{\eta_1}{ik}\dot{G}_{+}^{(e)}(\alpha, b) \tag{12}$$

and $B^{(e)}(\alpha)$ can be solved uniquely to give

$$M^{(e)}(\alpha)B^{(e)}(\alpha) = R_{+}^{(e)}(\alpha) - \int_0^b [f^{(e)}(t) + \alpha g^{(e)}(t)] \times \left[\frac{\sin[K(b-t)]}{K} + \frac{\eta_1}{ik}\cos[K(b-t)]\right] dt \tag{13a}$$

with

$$M^{(e)}(\alpha) = \cos[Kb] - \frac{\eta_1}{ik} K \sin[Kb]. \tag{13b}$$

Replacing (13a) into (11) we get

$$G_{+}^{(e)}(\alpha, y) = \frac{\cos[Ky]}{M^{(e)}(\alpha)}\left\{R_{+}^{(e)}(\alpha) - \int_0^b [f^{(e)}(t) + \alpha g^{(e)}(t)] \times \left[\frac{\sin[K(b-t)]}{K} + \frac{\eta_1}{ik}\cos[K(b-t)]\right] dt\right\} + \frac{1}{K}\int_0^y [f^{(e)}(t) + \alpha g^{(e)}(t)] \sin[K(y-t)]\, dt. \tag{14}$$

Although the left-hand side of (14) is regular in the upper half-plane $\Im m(\alpha) > \Im m(-k)$, the regularity of the right-hand side is violated by the presence of simple poles occurring at the zeros of $M^{(e)}(\alpha)$, namely at $\alpha = \alpha_m^e$ satisfying

$$M^{(e)}(\alpha_m^e) = 0, \quad \Im m(\alpha_m^e) > \Im m(k), \quad m = 1, 2, \cdots. \tag{15}$$

These poles can be eliminated by imposing that their residues are zero. This gives

$$R_{+}^{(e)}(\alpha_m^e) = \frac{\sin[K_m^e b]}{2K_m^e}\left[1 - \frac{\eta_1^2}{k^2}(K_m^e)^2\right]\nu_m^e\left[f_m^e + \alpha_m^e g_m^e\right] \tag{16a}$$

where K_m^e, ν_m^e, f_m^e, and g_m^e specify

$$K_m^e = K(\alpha_m^e) \tag{16b}$$

$$\nu_m^e = b + \frac{\eta_1}{ik}\sin^2[K_m^e b] \tag{16c}$$

$$\begin{bmatrix} f_m^e \\ g_m^e \end{bmatrix} = \frac{2}{\nu_m^e}\int_0^b \begin{bmatrix} f^{(e)}(t) \\ g^{(e)}(t) \end{bmatrix} \cos[K_m^e t]\, dt. \tag{16d}$$

Consider now the waveguide region $0 < y < a$, $x < 0$ where the total field can be expressed in terms of Fourier Cosine series as

$$u_2^{(e)}(x, y) = \sum_{n=1}^{\infty} c_n^e \cos[\xi_n^e y] e^{-i\beta_n^e x} \tag{17a}$$

with

$$\cos[\xi_n^e a] + \frac{\eta_2}{ik}\xi_n^e \sin[\xi_n^e a] = 0, \quad n = 1, 2, \cdots \tag{17b}$$

and

$$\beta_n^e = \sqrt{k^2 - \left(\xi_n^e\right)^2}. \tag{17c}$$

From the continuity relations (2g), (2h), and (9b), (9c), we get

$$u_2^{(e)}(0,y) = g^{(e)}(y), \quad \frac{\partial}{\partial x} u_2^{(e)}(0,y) = f^{(e)}(y), \quad 0 < y < a. \tag{18a, b}$$

Using (2g), (2h), and (2i) we may write

$$\left(1 + \frac{\eta_3}{ik}\frac{\partial}{\partial x}\right) u_3^{(e)}(0,y) = \begin{cases} \left(1 + \frac{\eta_3}{ik}\frac{\partial}{\partial x}\right) u_2^{(e)}(0,y), & 0 < y < a \\ 0, & a < y < b. \end{cases} \tag{19}$$

Hence, we get

$$\frac{\eta_3}{ik} f^{(e)}(y) + i g^{(e)}(y) = \begin{cases} \left(1 + \frac{\eta_3}{ik}\frac{\partial}{\partial x}\right) u_2^{(e)}(0,y), & 0 < y < a \\ 0, & a < y < b. \end{cases} \tag{20}$$

Owing to (16d), $f^{(e)}(y)$ and $g^{(e)}(y)$ can be expanded into Fourier Cosine series as follows:

$$\begin{bmatrix} f^{(e)}(y) \\ g^{(e)}(y) \end{bmatrix} = \sum_{m=1}^{\infty} \begin{bmatrix} f_m^e \\ g_m^e \end{bmatrix} \cos\left[K_m^e y\right]. \tag{21}$$

Substituting (17a) and (21) into (18) and (20), we obtain

$$\sum_{m=1}^{\infty} f_m^e \cos\left[K_m^e y\right] = -i \sum_{n=1}^{\infty} c_n^e \beta_n^e \cos\left[\xi_n^e y\right], \quad 0 < y < a \tag{22}$$

and

$$i \sum_{m=1}^{\infty} \left(g_m^e - \frac{\eta_3}{k} f_m^e\right) \cos\left[K_m^e y\right] = \begin{cases} \sum_{n=1}^{\infty} c_n^e \left(1 - \frac{\eta_3}{k}\beta_n^e\right) \cos\left[\xi_n^e y\right], & 0 < y < a \\ 0, & a < y < b. \end{cases} \tag{23}$$

Let us multiply both sides of (22) by $\cos[\xi_\ell^e y]$ and integrate from $y = 0$ to $y = a$ to get

$$c_\ell^e = -\frac{2i\xi_\ell^e}{\beta_\ell^e \mu_\ell^e} \sin\left[\xi_\ell^e a\right] \sum_{m=1}^{\infty} \frac{\Omega_m^e}{\vartheta_{m\ell}^e} f_m^e, \quad \ell = 1, 2, \cdots. \tag{24a}$$

with μ_ℓ^e, Ω_m^e, and $\vartheta_{m\ell}^e$ being defined by

$$\mu_\ell^e = a - \frac{\eta_2}{ik} \sin^2\left[\xi_\ell^e a\right] \tag{24b}$$

$$\Omega_m^e = \cos\left[K_m^e a\right] + \frac{\eta_2}{ik} K_m^e \sin\left[K_m^e a\right] \tag{24c}$$

$$\vartheta_{m\ell}^e = \left(K_m^e\right)^2 - \left(\xi_\ell^e\right)^2. \tag{24d}$$

Similarly, the multiplication of both sides of (23) by $\cos[K_\ell^e y]$ and its integration from $y = 0$ to $y = b$ yields

$$g_\ell^e - \frac{\eta_3}{k} f_\ell^e = \frac{2i\Omega_\ell^e}{\nu_\ell^e} \sum_{n=1}^{\infty} \frac{\xi_n^e \sin\left[\xi_n^e a\right]\left(1 - \frac{\eta_3}{k}\beta_n^e\right)}{\vartheta_{\ell n}^e} c_n^e, \quad \ell = 1, 2, \cdots. \tag{25a}$$

with ν_ℓ^e being defined by

$$\nu_\ell^e = b + \frac{\eta_1}{ik} K_\ell^e \sin^2\left[K_\ell^e b\right]. \tag{25b}$$

Consider the continuity relation (2f) which reads, in the Fourier transform domain

$$\dot{F}_+^{(e)}(\alpha, b) - \dot{G}_+^{(e)}(\alpha, b) = -\frac{2k\sin\phi_0}{1 + \eta_1 \sin\phi_0} \frac{e^{-ikb\sin\phi_0}}{\alpha - k\cos\phi_0}. \tag{26}$$

Taking into account (5a), (7a), and (14) one obtains

$$ik\frac{e^{-iKb}\chi(\alpha)}{M^{(e)}(\alpha)} R_+^{(e)}(\alpha) - \dot{F}_-^{(e)}(\alpha, b) = -\frac{2k\sin\phi_0}{1 + \eta_1\sin\phi_0} \frac{e^{-ikb\sin\phi_0}}{\alpha - k\cos\phi_0} + \frac{1}{M^{(e)}(\alpha)} \int_0^b \left[f^{(e)}(t) + \alpha g^{(e)}(t)\right] \cos[Kt]\, dt. \tag{27}$$

Substituting (21) in (27) and evaluating the resultant integral, one obtains the following modified Wiener–Hopf equation of the second kind valid in the strip $\Im m(k\cos\phi_0) < \Im m(\alpha) < \Im m(k)$

$$ik\frac{\chi(\alpha)}{N^{(e)}(\alpha)} R_+^{(e)}(\alpha) - \dot{F}_-^{(e)}(\alpha, b) = -\frac{2k\sin\phi_0}{1 + \eta_1\sin\phi_0} \frac{e^{-ikb\sin\phi_0}}{\alpha - k\cos\phi_0} + \sum_{m=1}^{\infty} \frac{K_m^e \sin\left[K_m^e b\right]}{\left[\alpha^2 - \left(\alpha_m^e\right)^2\right]} \left(f_m^e + \alpha g_m^e\right) \tag{28a}$$

with

$$N^{(e)}(\alpha) = e^{iK(\alpha)b} M^{(e)}(\alpha). \tag{28b}$$

The formal solution of (28a) can easily be obtained through the classical Wiener–Hopf procedure. The result is

$$ik\frac{\chi_+(\alpha)}{N_+^{(e)}(\alpha)} R_+^{(e)}(\alpha) = -\frac{2k\sin\phi_0}{1 + \eta_1\sin\phi_0} \frac{N_-^{(e)}(k\cos\phi_0)}{\chi_-(k\cos\phi_0)} \frac{e^{-ikb\sin\phi_0}}{\alpha - k\cos\phi_0} - \sum_{m=1}^{\infty} \frac{K_m^e \sin\left[K_m^e b\right]}{2\alpha_m^e} \frac{N_+^{(e)}(\alpha_m^e)}{\chi_+(\alpha_m^e)} \frac{\left(f_m^e - \alpha_m^e g_m^e\right)}{\alpha + \alpha_m^e}. \tag{29a}$$

Here, $N_+^{(e)}(\alpha)$, $\chi_+(\alpha)$, and $N_-^{(e)}(\alpha)$, $\chi_-(\alpha)$ are the split functions, regular and free of zeros in the half-planes $\Im m(\alpha) > \Im m(-k)$ and $\Im m(\alpha) < \Im m(k)$, respectively, resulting from the Wiener–Hopf factorization of the kernel function $\chi(\alpha)/N^{(e)}(\alpha)$ as

$$\frac{\chi(\alpha)}{N^{(e)}(\alpha)} = \frac{\chi_+(\alpha)}{N_+^{(e)}(\alpha)} \frac{\chi_-(\alpha)}{N_-^{(e)}(\alpha)}. \tag{29b}$$

The explicit expression of $N_{\pm}^{(e)}(\alpha)$ can be obtained by following the procedure outlined in [2]:

$$N_{+}^{(e)}(\alpha)=\left[\cos[kb]-\frac{\eta_1}{i}\sin[kb]\right]^{1/2}\times\exp\left\{\frac{Kb}{\pi}\ln\left(\frac{\alpha+iK}{k}\right)\right\}\times\exp\left\{\frac{i\alpha b}{\pi}\left(1-\mathcal{C}+\ln\left(\frac{2\pi}{kb}\right)+i\frac{\pi}{2}\right)\right\}\times\prod_{m=1}^{\infty}\left(1+\frac{\alpha}{\alpha_m^e}\right)\exp\left(\frac{i\alpha b}{m\pi}\right) \tag{30a}$$

$$N_{-}^{(e)}(\alpha)=N_{+}^{(e)}(-\alpha). \tag{30b}$$

In (30a) $\mathcal{C}$ is the Euler's constant given by $\mathcal{C}=0.577\,21\cdots$. As to the split functions $\chi_{\pm}(\alpha)$, they can be expressed explicitly in terms of the Maluizhinets function [7] as follows:

$$\begin{aligned}\chi_{+}(k\cos\phi)&=2^{3/2}\sqrt{\frac{2}{\eta_1}}\sin\frac{\phi}{2}\times\left\{\frac{\mathcal{M}_\pi(3\pi/2-\phi-\theta)\mathcal{M}_\pi(\pi/2-\phi+\theta)}{\mathcal{M}_\pi^2(\pi/2)}\right\}^2\\&\times\left\{\left[1+\sqrt{2}\cos\left(\frac{\pi/2-\phi-\theta}{2}\right)\right]\times\left[1+\sqrt{2}\cos\left(\frac{3\pi/2-\phi-\theta}{2}\right)\right]\right\}^{-1}\end{aligned} \tag{31a}$$

with

$$\sin\theta=\frac{1}{\eta_1} \tag{31b}$$

and

$$\mathcal{M}_\pi(z)=\exp\left\{-\frac{1}{8\pi}\int_0^z\frac{\pi\sin u-2\sqrt{2}\pi\sin(u/2)+2u}{\cos u}\,du\right\}. \tag{31c}$$

Substituting $\alpha=\alpha_1^e,\alpha_2^e,\cdots$ in (29a) and using (16a), yields the following equations for f_r^e and g_r^e:

$$\begin{aligned}&\frac{ik\nu_r^e}{2}\left[1-\frac{\eta_1^2}{k^2}(K_r^e)^2\right]\frac{\sin[K_r^e b]}{K_r^e}\frac{\chi_+(\alpha_r^e)}{N_+^{(e)}(\alpha_r^e)}[f_r^e+\alpha_r^e g_r^e]\\&=-\frac{2k\sin\phi_0}{1+\eta_1\sin\phi_0}\frac{N_-^{(e)}(k\cos\phi_0)}{\chi_-(k\cos\phi_0)}\frac{e^{-ikb\sin\phi_0}}{\alpha_r^e-k\cos\phi_0}\\&-\sum_{m=1}^{\infty}\frac{K_m^e\sin[K_m^e b]}{2\alpha_m^e}\frac{N_+^{(e)}(\alpha_m^e)}{\chi_+(\alpha_m^e)}\frac{(f_m^e-\alpha_m^e g_m^e)}{\alpha_r^e+\alpha_m^e},\\&r=1,2,\cdots.\end{aligned} \tag{32}$$

Replacing (24a) in (25a), g_r^e can be expressed in terms of f_r^e as

$$g_r^e=\frac{\eta_3}{k}f_r^e+4\frac{\Omega_r^e}{\nu_r^e}\sum_{m=1}^{\infty}f_m^e\Omega_m^e\times\sum_{n=1}^{\infty}\frac{(\xi_n^e)^2\sin^2[\xi_n^e a](1-\frac{\eta_3}{k}\beta_n^e)}{\beta_n^e\mu_n^e\vartheta_{mn}^e\vartheta_{rn}^e}. \tag{33}$$

Substituting (33) in (32) we get infinitely many equations in infinite number of unknowns that yield the constants f_r^e as follows:

$$\begin{aligned}&\left[\frac{ik\nu_r^e}{2}\left(1+\frac{\eta_3}{k}\alpha_r^e\right)\left[1-\frac{\eta_1^2}{k^2}(K_r^e)^2\right]\times\frac{\sin[K_r^e b]}{K_r^e}\frac{\chi_+(\alpha_r^e)}{N_+^{(e)}(\alpha_r^e)}+C_r^e(\alpha_r^e)\right]f_r^e+\sum_{\substack{m=1\\m\neq r}}C_m^e(\alpha_r^e)f_m^e\\&=-\frac{2k\sin\phi_0}{1+\eta_1\sin\phi_0}\frac{N_-^{(e)}(k\cos\phi_0)}{\chi_-(k\cos\phi_0)}\frac{e^{-ikb\sin\phi_0}}{\alpha_r^e-k\cos\phi_0}\end{aligned} \tag{34a}$$

with

$$\begin{aligned}C_m^e(\alpha_r^e)&=\left(1-\frac{\eta_3}{k}\alpha_m^e\right)\frac{K_m^e\sin[K_m^e b]}{2\alpha_m^e(\alpha_r^e+\alpha_m^e)}\frac{N_+^{(e)}(\alpha_m^e)}{\chi_+(\alpha_m^e)}\\&+\Omega_m^e\sum_{n=1}^{\infty}\frac{(\xi_n^e)^2\sin^2[\xi_n^e a](1-\frac{\eta_3}{k}\beta_n^e)}{\beta_n^e\mu_n^e\vartheta_{mn}^e}\\&\times\left\{ik\left[1-\frac{\eta_1^2}{k^2}(K_r^e)^2\right]\frac{\alpha_r^e\Omega_r^e}{\vartheta_{rn}^e}\frac{\sin[K_r^e b]}{K_r^e}\frac{\chi_+(\alpha_r^e)}{N_+^{(e)}(\alpha_r^e)}\right.\\&\left.-\sum_{s=1}^{\infty}\frac{2\Omega_s^e}{\nu_s^e\vartheta_{sn}^e}\frac{K_s^e\sin[K_s^e b]}{(\alpha_r^e+\alpha_s^e)}\frac{N_+^{(e)}(\alpha_s^e)}{\chi_+(\alpha_s^e)}\right\}.\end{aligned} \tag{34b}$$

B. Odd Excitation

The solution for odd excitation is similar to that of even excitation. Indeed, by assuming a representation similar to (1a) with the superscript (e) being replaced by (o); it can be seen that all the boundary and continuity relations in (2a)–(2i) remain valid for the odd excitation case also, except (2c) and (2d), which are to be changed as

$$u_2^{(o)}(x,0)=0,\quad x<0 \tag{35a}$$

$$u_3^{(o)}(x,0)=0,\quad x>0. \tag{35b}$$

In this case, the Wiener–Hopf equation reads

$$\begin{aligned}&k\frac{\chi(\alpha)}{N^{(o)}(\alpha)}R_+^{(o)}(\alpha)+\dot{F}_-^{(o)}(\alpha,b)\\&=\frac{2k\sin\phi_0}{1+\eta_1\sin\phi_0}\frac{e^{-ikb\sin\phi_0}}{\alpha-k\cos\phi_0}\\&+\sum_{m=1}^{\infty}\frac{K_m^o\cos[K_m^o b]}{[\alpha^2-(\alpha_m^o)^2]}(f_m^o+\alpha g_m^o)\end{aligned} \tag{36a}$$

with

$$N^{(o)}(\alpha)=e^{iKb}\left[\sin[Kb]+\frac{\eta_1}{ik}K\cos[Kb]\right] \tag{36b}$$

$$K_m^o=\sqrt{k^2-(\alpha_m^o)^2} \tag{36c}$$

$$\begin{bmatrix}f_m^o\\g_m^o\end{bmatrix}=\frac{2}{\nu_m^o}\int_0^b\begin{bmatrix}f^{(o)}(t)\\g^{(o)}(t)\end{bmatrix}\sin[K_m^o t]\,dt,\quad \nu_m^o=b+\frac{\eta_1}{ik}\cos^2[K_m^o b] \tag{36d}$$

where α_m^o are the roots of

$$\sin[Kb]+\frac{\eta_1}{ik}K\cos[Kb]=0,\quad \alpha=\alpha_m^o,\quad \Im m(\alpha_m^o)>\Im m(k). \tag{36e}$$

The application of the Wiener–Hopf procedure to (36a) yields

$$k\frac{\chi_+(\alpha)}{N_+^{(o)}(\alpha)}R_+^{(o)}(\alpha) = \frac{2k\sin\phi_0}{1+\eta_1\sin\phi_0}\frac{N_-^{(o)}(k\cos\phi_0)}{\chi_-(k\cos\phi_0)}\frac{e^{-ikb\sin\phi_0}}{\alpha-k\cos\phi_0} - \sum_{m=1}^{\infty}\frac{K_m^o\cos\left[K_m^o b\right]}{2\alpha_m^o}\frac{N_+^{(o)}(\alpha_m^o)}{\chi_+(\alpha_m^o)}\frac{(f_m^o-\alpha_m^o g_m^o)}{\alpha+\alpha_m^o}. \quad (37)$$

$N_+^{(o)}(\alpha)$ and $N_-^{(o)}(\alpha)$ are the split functions resulting from the Wiener–Hopf factorization of (36b) as

$$N^{(o)}(\alpha) = N_+^{(o)}(\alpha)N_-^{(o)}(\alpha). \quad (38a)$$

The explicit expressions of $N_\pm^{(o)}(\alpha)$ are [2]

$$N_+^{(o)}(\alpha) = \sqrt{\alpha+k}\left[\frac{\sin[kb]}{k}+\frac{\eta_1}{ik}\cos[kb]\right]^{1/2} \times\exp\left\{\frac{Kb}{\pi}\ln\left(\frac{\alpha+iK}{k}\right)\right\} \times\exp\left\{\frac{i\alpha b}{\pi}\left(1-\mathcal{C}+\ln\left(\frac{2\pi}{kb}\right)+i\frac{\pi}{2}\right)\right\} \times\prod_{m=1}^{\infty}\left(1+\frac{\alpha}{\alpha_m^o}\right)\exp\left(\frac{i\alpha b}{m\pi}\right) \quad (38b)$$

$$N_-^{(o)}(\alpha) = N_+^{(o)}(-\alpha). \quad (38c)$$

By using the continuity relations at the aperture $0<y<b$, $x=0$ we get infinitely many equations in infinite number of unknowns which yield the constants f_r^o as follows:

$$\left[-\frac{k\nu_r^o}{2}\left(1+\frac{\eta_3}{k}\alpha_r^o\right)\left[1-\frac{\eta_1^2}{k^2}(K_r^o)^2\right] \times\frac{\cos[K_r^o b]}{K_r^o}\frac{\chi_+(\alpha_r^o)}{N_+^{(o)}(\alpha_r^o)}+C_r^o(\alpha_r^o)\right]f_r^o+\sum_{\substack{m=1\\m\neq r}}C_m^o(\alpha_r^o)f_m^o = \frac{2k\sin\phi_0}{1+\eta_1\sin\phi_0}\frac{N_-^{(o)}(k\cos\phi_0)}{\chi_-(k\cos\phi_0)}\frac{e^{-ikb\sin\phi_0}}{\alpha_r^o-k\cos\phi_0} \quad (39a)$$

with

$$C_m^o(\alpha_r^o) = \left(1-\frac{\eta_3}{k}\alpha_m^o\right)\frac{K_m^o\cos\left[K_m^o b\right]}{2\alpha_m^o(\alpha_r^o+\alpha_m^o)}\frac{N_+^{(o)}(\alpha_m^o)}{\chi_+(\alpha_m^o)} - \Omega_m^o\sum_{n=1}^{\infty}\frac{(\xi_n^o)^2\cos^2\left[\xi_n^o a\right]\left(1-\frac{\eta_3}{k}\beta_n^o\right)}{\beta_n^o\mu_n^o\vartheta_{mn}^o} \times\left\{k\left[1-\frac{\eta_1^2}{k^2}(K_r^o)^2\right]\frac{\alpha_r^o\Omega_r^o}{\vartheta_{rn}^o}\frac{\cos\left[K_r^o b\right]}{K_r^o}\frac{\chi_+(\alpha_r^o)}{N_+^{(o)}(\alpha_r^o)} +\sum_{s=1}^{\infty}\frac{2\Omega_s^o}{\nu_s^o\vartheta_{sn}^o}\frac{K_s^o\cos\left[K_s^o b\right]}{(\alpha_r^o+\alpha_s^o)}\frac{N_+^{(o)}(\alpha_s^o)}{\chi_+(\alpha_s^o)}\right\}. \quad (39b)$$

Here, β_n^o, μ_n^o, Ω_m^o, ν_s^o, and ϑ_{mn}^o stand for

$$\beta_n^o = \sqrt{k^2-(\xi_n^o)^2} \quad (39c)$$

$$\mu_n^o = a-\frac{\eta_2}{ik}\cos^2\left[\xi_n^o a\right] \quad (39d)$$

$$\Omega_m^o = \sin\left[K_m^o a\right]-\frac{\eta_2}{ik}K_m^o\cos\left[K_m^o a\right] \quad (39e)$$

$$\nu_s^o = b+\frac{\eta_1}{ik}\cos^2\left[K_s^o b\right] \quad (39f)$$

$$\vartheta_{mn}^o = (K_m^o)^2-(\xi_n^o)^2 \quad (39g)$$

with ξ_n^o being the roots of

$$\sin\left[\xi_n^o a\right]-\frac{\eta_2}{ik}\xi_n^o\cos\left[\xi_n^o\right]=0, \quad n=1,2,\cdots \quad (39h)$$

g_r^o can be expressed in terms of f_r^o as

$$g_r^o = \frac{\eta_3}{k}f_r^o+4\frac{\Omega_r^o}{\nu_r^o}\sum_{m=1}^{\infty}f_m^o\Omega_m^o \times\sum_{n=1}^{\infty}\frac{(\xi_n^o)^2\cos^2\left[\xi_n^o a\right]\left(1-\frac{\eta_3}{k}\beta_n^o\right)}{\beta_n^o\mu_n^o\vartheta_{mn}^o\vartheta_{rn}^o}. \quad (40)$$

III. Analysis of the Diffracted Field

The scattered field in the region $y>b$ for even and odd excitations can be obtained by taking the inverse Fourier transform of $F^{(e)}(\alpha,y)$ and $F^{(o)}(\alpha,y)$, respectively

$$u_1^{(e)}(x,y) = \frac{1}{2\pi}\int_{\mathcal{L}}F^{(e)}(\alpha,b)\,e^{iK(\alpha)(y-b)}\,e^{-i\alpha x}\,d\alpha \quad (41a)$$

$$u_1^{(o)}(x,y) = \frac{1}{2\pi}\int_{\mathcal{L}}F^{(o)}(\alpha,b)\,e^{iK(\alpha)(y-b)}\,e^{-i\alpha x}\,d\alpha. \quad (41b)$$

Here $\mathcal{L}$ is a straight line parallel to the real α axis lying in the strip $\Im m(k\cos\phi_0)<\Im m(\alpha)<\Im m(k)$. The asymptotic evaluation of the integrals in (41a) and (41b) through the saddle-point technique enables us to write for the diffracted field

$$u_1(\rho,\phi) = \frac{u_1^{(e)}(\rho,\phi)+u_1^{(o)}(\rho,\phi)}{2} \quad (42a)$$

with

$$u_1^{(e)}(\rho,\phi) \sim \left\{u_0 D^{(e)}(\phi,\phi_0)+\frac{e^{i\pi/4}}{\sqrt{2\pi}}\frac{\sin\phi}{1+\eta_1\sin\phi}\frac{N_-^{(e)}(k\cos\phi)}{\chi_-(k\cos\phi)} \times\sum_{m=1}^{\infty}\frac{K_m^e\sin\left[K_m^e b\right]}{2\alpha_m^e}\frac{N_+^{(e)}(\alpha_m^e)}{\chi_+(\alpha_m^e)}\frac{f_m^e-\alpha_m^e g_m^e}{\alpha_m^e-k\cos\phi}\right\}\frac{e^{ik\rho}}{\sqrt{k\rho}} \quad (42b)$$

$$u_1^{(o)}(\rho,\phi) \sim \left\{u_0 D^{(o)}(\phi,\phi_0)+\frac{e^{i3\pi/4}}{\sqrt{2\pi}}\frac{\sin\phi}{1+\eta_1\sin\phi}\frac{N_-^{(o)}(k\cos\phi)}{\chi_-(k\cos\phi)} \times\sum_{m=1}^{\infty}\frac{K_m^o\cos\left[K_m^o b\right]}{2\alpha_m}\frac{N_+^{(o)}(\alpha_m^o)}{\chi_+(\alpha_m^o)}\frac{f_m^o-\alpha_m^o g_m^o}{\alpha_m^o-k\cos\phi}\right\}\frac{e^{ik\rho}}{\sqrt{k\rho}} \quad (42c)$$

$$u_0 = e^{-ikb\sin\phi_0} \quad (42d)$$

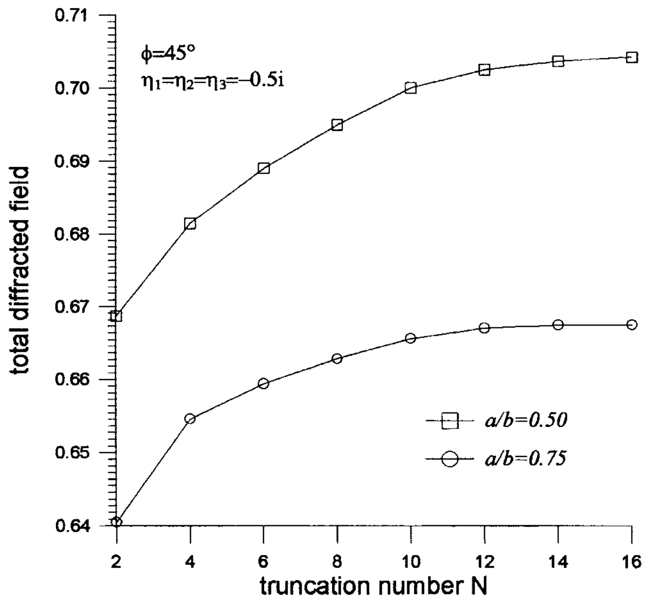

Fig. 3. The diffracted field versus the truncation number N.

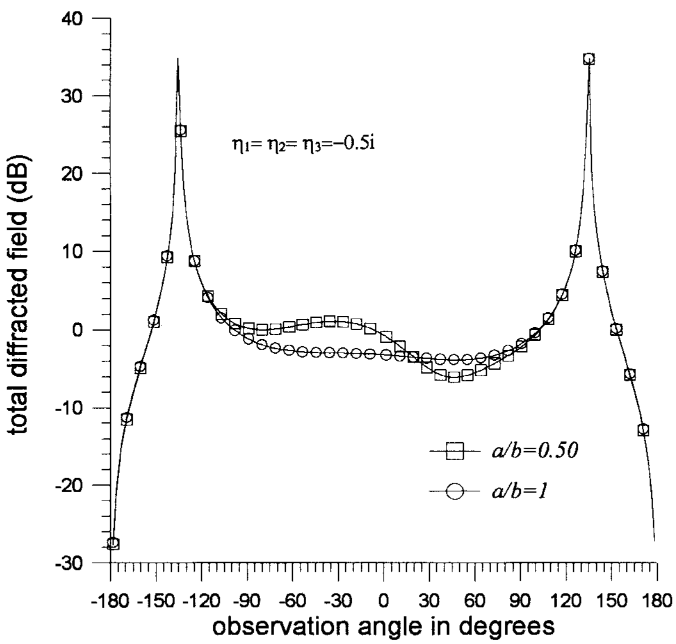

Fig. 4. The diffracted field versus the observation angle ϕ, for different values of a/b.

$$D^{(e)}(\phi,\phi_0)=e^{-i3\pi/4}\sqrt{\frac{2}{\pi}}\frac{\sin\phi_0}{1+\eta_1\sin\phi_0}\frac{\sin\phi}{1+\eta_1\sin\phi}\times\frac{N_-^{(e)}(k\cos\phi_0)N_-^{(e)}(k\cos\phi)}{\chi_-(k\cos\phi_0)\chi_-(k\cos\phi)}\frac{1}{\cos\phi_0+\cos\phi} \tag{42e}$$

$$D^{(o)}(\phi,\phi_0)=e^{i3\pi/4}\sqrt{\frac{2}{\pi}}\frac{\sin\phi_0}{1+\eta_1\sin\phi_0}\frac{\sin\phi}{1+\eta_1\sin\phi}\times\frac{N_-^{(o)}(k\cos\phi_0)N_-^{(o)}(k\cos\phi)}{\chi_-(k\cos\phi_0)\chi_-(k\cos\phi)}\frac{1}{\cos\phi_0+\cos\phi} \tag{42f}$$

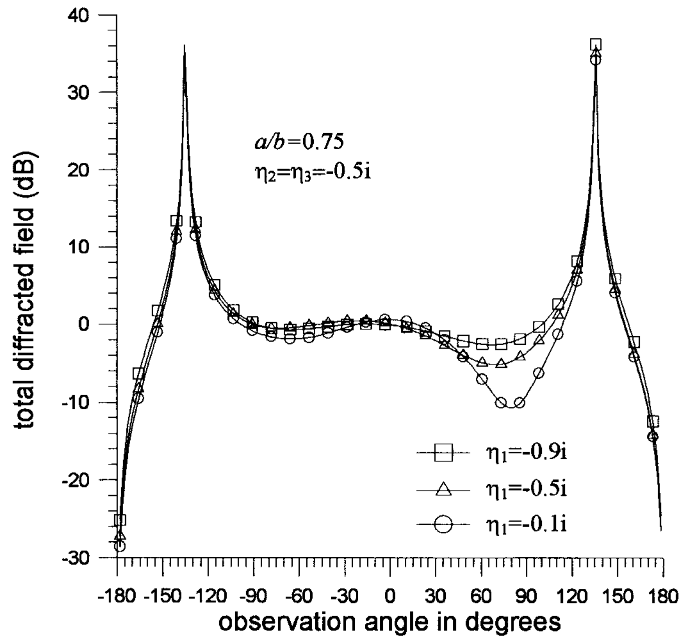

Fig. 5. The diffracted field versus the observation angle ϕ,for different values of η_1.

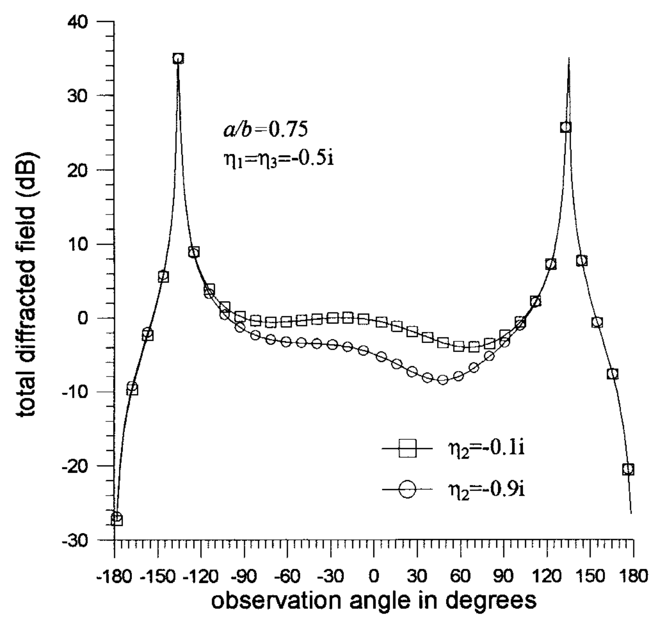

Fig. 6. The diffracted field versus the observation angle ϕ, for different values of η_2.

where (ρ,ϕ) are the cylindrical polar coordinates defined by

$$x=\rho\cos\phi,\qquad y-b=\rho\sin\phi$$

and u_0 is the expression of the incident field at $y=b$, $x=0$.

For the special case $Z_1+Z_2=0$ $(Z_1=iX, X\in \mathbb{R})$, $a=b$, we get

$$\alpha_m^e=\beta_m^e,\qquad K_m^e=\xi_m^e\quad m=1,2,\cdots.$$

Therefore, after putting $\eta_3=0$ and $a=b$ at every step, (22) and (23) are identically satisfied for

$$f_m^e=-ic_m^e\beta_m^e\quad\text{and}\quad ig_m^e=c_m^e.$$

This gives

$$f_m^e = \alpha_m^e g_m^e$$

implying the series contribution to the total diffracted field given in (42b) be zero. Similar considerations are also valid for the odd excitation case. Hence, $D^{(e),(o)}(\phi,\phi_0)$ given in (42e) and (42f) correspond to the "diffraction coefficients" related to the edge $y=b$, $x=0$ for even and odd excitation cases, respectively.

For the perfectly conducting case $X=0$, (42e) and (42f) reduces to

$$D^{(e)}(\phi,\phi_0) = e^{-i3\pi/4}\sqrt{\frac{2}{\pi}}\frac{2\cos\frac{\phi_0}{2}\cos\frac{\phi}{2}}{\cos\phi_0+\cos\phi}N_-^{(e)}(k\cos\phi_0) \times N_-^{(e)}(k\cos\phi) \tag{43a}$$

$$D^{(o)}(\phi,\phi_0) = e^{i3\pi/4}\sqrt{\frac{2}{\pi}}\frac{2\cos\frac{\phi_0}{2}\cos\frac{\phi}{2}}{\cos\phi_0+\cos\phi}N_-^{(o)}(k\cos\phi_0) \times N_-^{(o)}(k\cos\phi) \tag{43b}$$

which are nothing but the well-known results related to a perfectly conducting parallel-plate waveguide [1], [2]. Furthermore, in the case when $a=b=0$, we get

$$N^{(e)}(\alpha)=1, \quad N^{(o)}(\alpha)=0$$

and u_1 given in (42a) coincides with the perfectly conducting half-plane solution [2].

In order to show the influence of the values of the wall thickness and the surface impedances on the diffraction phenomenon, some numerical results showing the variation of the diffracted field $(20\log|u_d\times\sqrt{k\rho}|)$ with the observation angle are presented. In all the graphical solutions that follow, we take $ka=1$ rad and $\phi_0=45°$. The surface impedances η_i, $i=1,2,3$ are taken as negative imaginary (inductive case). In Fig. 3, we show the variation of the diffacted field with the truncation number N at a fixed point. It is seen that the diffracted field becomes insensitive to the truncation number N for $N\geq 10$, $a/b=0.50$, $\eta_1=\eta_2=\eta_3=-0.5i$. We require a smaller N for increasing values of a/b as expected. For the numerical examples that follow, N is chosen by taking into account this criterion. Fig. 4 shows the variation of the diffracted field with the observation angle, for different values of the wall thickness. As expected, the diffracted field increases with the increasing values of the wall thickness in the range $\phi\in(-120°,20°)$. Fig. 5 shows the variation of the diffracted field with the observation angle, for different values of the exterior surface impedance η_1. The diffracted field increases with the increasing values of $|\eta_1|$. Fig. 6 shows the variation of the diffracted field with the observation angle, for different values of the interior surface impedance η_2. The diffracted field decreases with the increasing values of $|\eta_2|$. No noticeable variation of the diffracted field with η_3 could be observed, even for very small values of the incidence angle ϕ_0.

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