Electrostatic Solution for Three-Dimensional Arbitrarily Shaped Conducting Bodies Using Finite Element and Measured Equation of Invariance

John H. Henderson and Sadasiva M. Rao

***Abstract*—Differential equation techniques such as finite element (FE) and finite difference (FD) have the advantage of sparse system matrices that have relatively small memory requirements for storage and relatively short central processing unit (CPU) time requirements for solving. However, these techniques do not lend themselves as readily for use in open-region problems as the method of moments (MoM) because they require the discretization of the space surrounding the object where MoM only requires discretization of the surface of the object. In this work, a relatively new mesh truncation method known as the measured equation of invariance (MEI) is investigated augmenting the FE method for the solution of electrostatic problems involving three-dimensional (3-D) arbitrarily shaped conducting objects. This technique allows truncation of the mesh as close as two node layers from the object. MEI views sparse-matrix numerical techniques as methods of determining weighting coefficients between neighboring nodes and finds those weights for nodes on the boundary of the mesh by assuming viable charge distributions on the surface of the object and using Green's function to measure the potentials at the nodes. Problems in the implementation of FE/MEI are discussed and the method is compared against MoM for a cube and a sphere.**

***Index Terms*—Electrostatic analysis, finite-element methods.**

I. Introduction

There exist a number of numerical methods that may be used to solve for the fields surrounding or charge induced upon a conducting body held at a fixed potential. Two of the most popular methods are, viz., the method of moments (MoM) [1], and the finite-element method (FE) [2]. Each of these methods have associated advantages and disadvantages for solving the open-region electrostatic problem. The MoM lends itself well to the open-region problem in that it involves the discretization of only the surface of the object. However, this technique leads to a full-system matrix, which requires extensive computer resources to store and solve. On the other hand, the FE method leads to a very sparse system matrix that is quickly solved using iterative techniques and is less demanding on computer storage. Without some method to truncate the problem space, the FE technique would require gridding of the infinite space surrounding the conducting body. Many methods have been proposed and used to truncate the problem space and involve either making an assumption on the form of the field at a distance [3] or using an integral method similar to MoM [4], [5], on the boundary. The disadvantage of the first technique is that it does not take the geometry of the object into account and typically requires truncating the problem space far from the object for good results, thus leading to a large system matrix. The disadvantage of the second method is that it results in full rows in the system matrix for the boundary nodes, thereby destroying the sparsity of the matrix and, therefore, one of the main advantages of FE techniques.

In contrast, the measured equation of invariance (MEI) [6]–[16] is a mesh truncation technique that takes the geometry of the object into account and maintains the sparsity of the system matrix. Thus, this new method appears to retain the advantages of both the MoM and FE techniques.

In this paper, the open-region electrostatic conducting body problem is solved with the mesh truncated at a close distance, in fact, only two layers from the object. The node-based FE method is augmented with the MEI which results in a relatively small, sparse matrix. The application of the FE method along with MEI truncation condition is the subject matter of this work.

This paper is organized as follows. In the next section, a detailed explanation of the MEI truncation method as applied to the electrostatic problem is provided. Since the application of FE methods to electromagnetic field problems is well known [2], the analysis of the FE method is excluded in this presentation. Section III discusses the numerical results obtained for a conducting sphere and a cube which are compared with other methods. Furthermore, a detailed account of the required computer resources is provided in this section. Finally, in Section IV, some important conclusions drawn from this study are presented.

II. Implementation of the Measured Equation of Invariance

Sparse system methods such as FE and finite difference (FD) can be thought of as solving a system of equations of the form

$$\alpha_0\phi_0 + \alpha_1\phi_1 + \alpha_2\phi_2 + \alpha_3\phi_3 + \alpha_4\phi_4 = 0 \qquad (1)$$

Manuscript received February 11, 1998; revised July 17, 1998. This work was supported by the NASA Graduate Student Researchers Program, Electromagnetics Research Branch, Langley Research Center, Langley, VA 23665 USA.

J. H. Henderson is with the Harris Corporation, Melbourne, FL 32902 USA.

S. M. Rao is with the Department of Electrical Engineering, Auburn University, Auburn, AL 36849 USA.

Publisher Item Identifier S 0018-926X(98)08886-3.

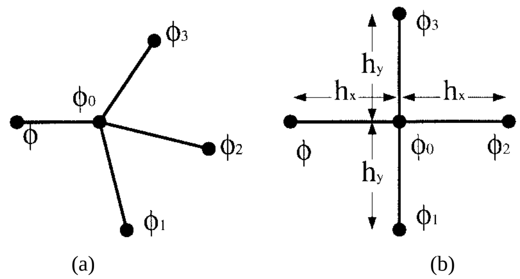

Fig. 1. (a) Generic 2-D problem grid. (b) Two-dimensional FD grid.

where ϕ_j is the potential at node j as shown in Fig. 1(a) and there is one such equation for each node. The potentials are the unknowns and the α's are weighting coefficients, which may be known or unknown depending on the location of the given node. In the FE method, the weighting coefficients are derived from Laplace's equation

$$\nabla^2\phi = 0 \tag{2}$$

through the minimization of a functional. For the FD method, these weights are determined by replacing the derivatives with difference approximations. For example, for the two-dimensional (2-D) FD grid in Fig. 1(b), if $h_x = h_y$, the result will be

$$-4\phi_0 + \phi_1 + \phi_2 + \phi_3 + \phi_4 = 0. \tag{3}$$

Therefore, relating this to the generic difference equation (1), $\alpha_0 = -4$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 1$, and $\alpha_4 = 1$.

Difficulty occurs when it is necessary to truncate the mesh and one of the nodes in the kernel in Fig. 1 is missing. It is impossible to determine the weighting coefficients using FE or FD scheme alone. The MEI is a method to determine the weights for any node, which is explained as follows.

First of all, by choosing one of the weights arbitrarily, e.g., $\alpha_0 = -1$, (1) may be rewritten as

$$\alpha_1\phi_1 + \alpha_2\phi_2 + \alpha_3\phi_3 + \alpha_4\phi_4 = \phi_0 \tag{4}$$

Now, referring to Fig. 1, assume that four linearly independent solutions of Laplace's equation (potential functions) are known for a given geometry. These independent potential functions may be easily generated by assuming independent source distributions, known as *metrons*, on the structure and using Green's function methods. By substituting these potential functions in (4), we obtain a system of equations given by

$$\begin{bmatrix}\phi_{11} & \phi_{12} & \phi_{13} & \phi_{14}\\ \phi_{21} & \phi_{22} & \phi_{23} & \phi_{24}\\ \phi_{31} & \phi_{32} & \phi_{33} & \phi_{34}\\ \phi_{41} & \phi_{42} & \phi_{43} & \phi_{44}\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix} = \begin{bmatrix}\phi_{10}\\ \phi_{20}\\ \phi_{30}\\ \phi_{40}\end{bmatrix} \tag{5}$$

where ϕ_{ij} is the potential at node j of solution i. (5) may be solved to obtain the remaining weighting coefficients.

It may be easily seen that the weights are geometry dependent since the potential functions are obtained by integrating the source functions over the structure geometry. Furthermore, since the Green's functions techniques automatically satisfy the radiation condition, the weights obtained by MEI, when applied to boundary nodes, adaptively generate a correct absorbing condition. Thus, MEI allows the truncation of the the mesh close to the object in open-region problems.

To apply FE and MEI to the electrostatic conducting body problem, a narrow layer outside of the object is discretized into two layers of tetrahedrons. This results in three layers of nodes. The first layer is on the object and their potential is fixed by the potential of the body. The nodes on the second layer are connected via the tetrahedrons to other nodes on all three layers. The weights relating these nodes to their neighbors are determined using the FE method. Although one may use MEI to obtain these weighting coefficients, it is numerically more efficient to use FE methods. Nodes on the last layer are connected only to nodes on the second layer and to other nodes on the same layer. The weighting coefficients for the nodes on the outer layer are determined by selecting the five nearest neighbors connected to each node and applying the MEI. Only five neighbors are chosen to reduce the time required to apply the MEI technique and to maintain sparsity of the system matrix. We have determined that using more neighboring nodes does not result in an appreciable increase in accuracy.

The most challenging aspect to the MEI, as expressed by many researchers, is the selection of the metron functions. Although the theory of MEI allows any set of linearly independent functions as metrons, in practice, one must find a set that does not result in a poorly conditioned matrix in (5) when trying to determine the weighting coefficients. So far, in the application of the MEI method, the source functions were defined over entire range of the object (entire domain functions). At least, this was true for 2-D objects [6], [7], [11], [12]. However, for three-dimensional (3-D) objects of general shape it is quite difficult to define entire domain functions unless some complicated mapping to a spherical or rectangular surface is carried out. In this work, we overcome this difficulty by selecting subdomain pulse functions and solving an overdetermined system for the weighting coefficients. However, we did not experiment with other subdomain or entire domain functions since the results are satisfactory in the present scheme as demonstrated later.

A drawing showing the MEI kernel and the 3-D configuration is shown in Fig. 2. As is depicted by the figure, the tetrahedral discretization of the space surrounding the object automatically leads to a triangularization of the surface of the object. Each source function is assumed to be constant over a given triangle and zero elsewhere. Using these source distributions, which are clearly linearly independent, and using the free-space Green's function, one can obtain N linearly independent potential functions where N is the number of triangular patches on the conductor surface.

Using these potential functions and evaluating them at the boundary nodes and the nodes connecting them, one can obtain the system of equations to determine the weighting coefficients

$$\begin{bmatrix}\phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} & \phi_{15}\\ \phi_{21} & \phi_{22} & \phi_{23} & \phi_{24} & \phi_{25}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \phi_{N1} & \phi_{N2} & \phi_{N3} & \phi_{N4} & \phi_{N5}\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\\ \alpha_5\end{bmatrix} = \begin{bmatrix}\phi_{10}\\ \phi_{20}\\ \vdots\\ \phi_{N0}\end{bmatrix} \tag{6}$$

where ϕ_{ij} is the potential due to surface triangle j at local

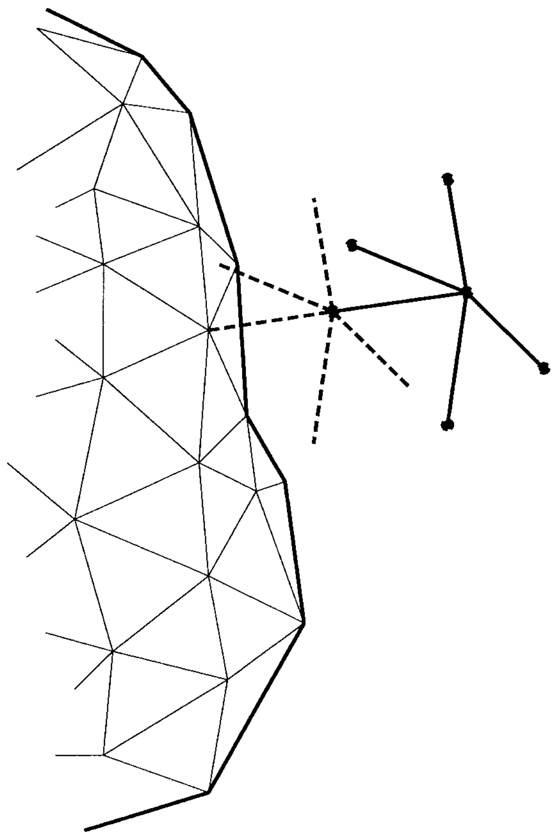

Fig. 2. Geometry used for applying FE and MEI to a 3-D electrostatic conducting body problem.

node i and local node zero is the center node of the kernel. The weights are then obtained by using a least-squares solving routine that minimizes the norm of the result. It is important that the norm of the result be minimized because certain possible symmetries in the geometry might generate linearly dependent columns in the matrix.

The weights are stored in the system matrix using a modified compressed sparse row technique [17]. This method is one of the most efficient and elegant sparse matrix storage schemes using only one real and one integer location for each diagonal entry and each off-diagonal nonzero entry and one additional real and integer value to simplify indexing. Once all of the weighting coefficients are found by either FE or MEI, the system is solved for the node potentials using the conjugate gradient method [17]–[19].

The main subtasks of the FE/MEI program are the FE matrix fill, the MEI matrix fill, and the conjugate gradients solving. A plot of the breakdown of execution times by each subtask for the method so far discussed for a sample geometry are shown in Fig. 3. It is shown that the majority of the central processing unit (CPU) time is required by the MEI matrix fill subtask. This demonstrates why it is not desirable to solve for the weights of all of the nodes using MEI and why FE was used for nonboundary nodes. Furthermore, to improve the efficiency, the potential function is approximated as follows:

$$\phi = \iint_T \frac{\rho_s(\vec{r}\,')}{4\pi\epsilon|\vec{r}-\vec{r}\,'|}\,dT \approx \rho_s(\vec{r}_c)\frac{A}{4\pi\epsilon|\vec{r}-\vec{r}_c|}, \quad \vec{r}\,' \in T \tag{7}$$

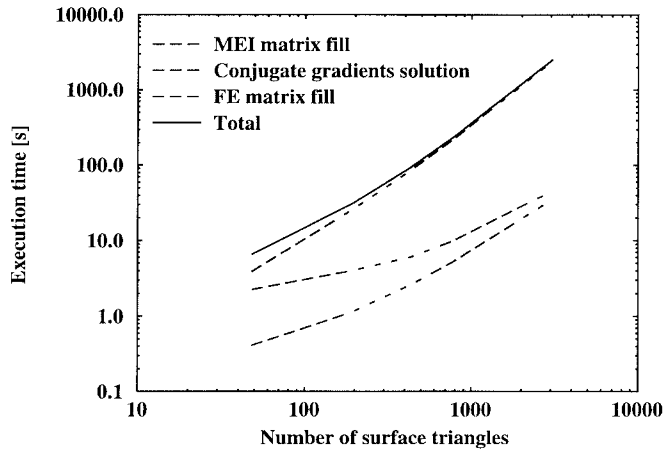

Fig. 3. Breakdown of execucution times for electrostatic cube and sphere problems by major subtasks.

where T represents the source triangle, $\vec{r}$ is the observation point, $\vec{r}_c$ is the centroid of the triangle, and A is the area of the source triangle T. This approximation is reasonable since the observation point $\vec{r}$ is at least two edge lengths (i.e., two tetrahedrons) away from the source triangle. This approximation was incorporated into the program and the results are presented in the following section.

III. Numerical Results

In this section, we present the numerical results obtained from the FE/MEI scheme described so far. The geometries considered are: 1) a conducting sphere of 1-m diameter and 2) a conducting cube of 1-m side, each held to a potential of 100 V. The sphere results are compared with the analytical solution and the conducting cube results are compared to the well-known MoM solution [20]. For both geometries, in the FE/MEI scheme the thickness of each layer of tetrahedrons is fixed at 0.1 m. In addition, the sphere was also run with a 0.15-m layer thickness. It may be noted that this procedure generates some what skewed tetrahedrons, particularly at large discretizations. Still, we retained this grid scheme because the final unknown quantity (charge distribution) is obtained via numerical differentiation. The tetrahedra model for the cube problem is generated by first dividing each side of the cube into certain number of divisions to obtain subcubes and then dividing each subcube into five tetrahedrons. To create the sphere, the discretization for the cube was "inflated." This is important to keep in mind, as the tables and plots refer to "number of edge discretizations." This was the number of discretizations per edge before the cube was morphed into a sphere. This morphing also explains the same numbers of unknowns and the same run times for the two geometries.

All programs were run on a SUN SPARC server 1000 running 50 MHz SuperSPARC processors. Execution times were found using the execution profiling commands of the operating system.

The storage requirements for the various discretizations are shown in Table I. Execution times are presented in Table II. The capacitance for the sphere is given Fig. 4 and the potential outside the sphere is plotted in Fig. 5. The capacitance for the

TABLE I
STORAGE REQUIREMENTS FOR 3-D ELECTROSTATIC CUBE AND SPHERE PROBLEMS COMPARING METHOD OF MOMENTS TO FINITE ELEMENT WITH MEASURED EQUATION OF INVARIANCE MESH TRUNCATION

	Method of Moments		FE/MEI	
Number of edge discretization	Number of unknowns	Storage requirements	Number of unknowns	Storage requirements
2	48	2304 real	316	2416 integer 2416 real
4	192	36,864 real	604	4696 integer 4696 real
6	432	186,624 real	988	7744 integer 7744 real
8	768	589,824 real	1468	11,560 integer 11,560 real
16	3072	9,437,184 real	4348	34,504 integer 34,504 real
32	12,288	150,994,944 real	14,716	117,256 integer 117,256 real

TABLE II
EXECUTION TIMES FOR 3-D ELECTROSTATIC CUBE AND SPHERE PROBLEM COMPARING METHOD OF MOMENTS TO FINITE ELEMENT WITH MEASURED EQUATION OF INVARIANCE MESH TRUNCATION

Number of edge discretizations	MoM	FE/MEI
2	0.68 s	3.61 s
4	15.46 s	11.20 s
6	128.78 s	28.56 s
8	688.22 s	64.49 s
16	28,797.26 s	685.83 s
32		7502.05 s

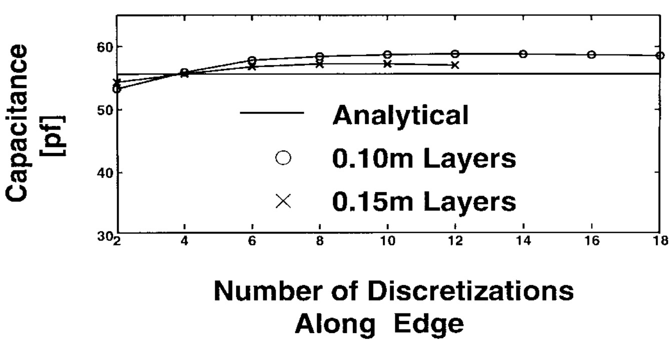

Fig. 4. Capacitance of 1-m PEC sphere at various discretizations comparing analytical to FE/MEI.

cube is given in Fig. 6. In the following, we present a short discussion of the numerical results.

First of all, it should be noted that integral equation methods such as MoM generate more accurate solutions than differential equation methods for a given discretization. Also, the charge calculated from the FE/MEI method is actually the charge on a surface removed from the surface of the object introducing some error, where in MoM, charge is calculated by the method itself. Further, it should be noted that MoM and FE/MEI have different quantities as unknowns. For MoM, the unknowns are the surface charge densities. For FE, the unknowns are potentials at the nodes. As discussed previously, finding the surface charge density from FE requires a FD approximation, which introduces error into the charge results. Further, the charge thus obtained using this procedure is not really residing on the body, but at a surface somewhere in the first layer of tetrahedra. This must be kept in mind when comparing the results from FE/MEI to those from MoM.

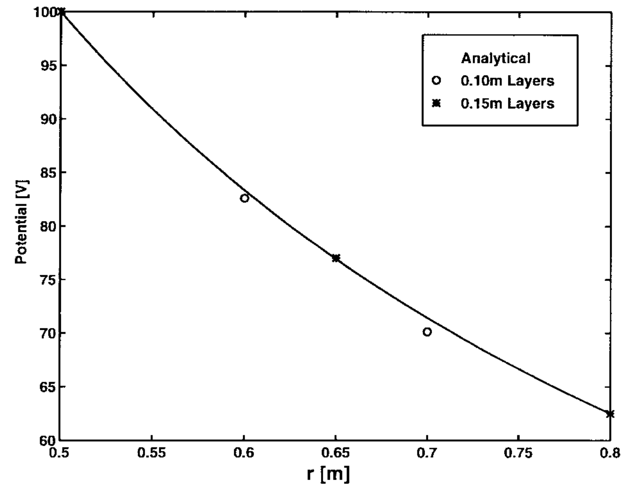

Fig. 5. Potential outside 1-m-diameter PEC sphere held to 100 V.

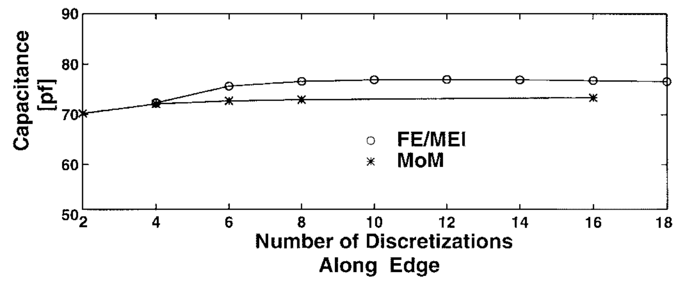

Fig. 6. Capacitance of 1-m PEC cube at various discretizations comparing MoM to FE/MEI.

As a first comparison, let us compare FE/MEI against the analytical solution of the potential for the sphere. The results are shown in Fig. 5. The FE/MEI method gives good agreement to the analytical solution. In Fig. 4, the capacitance for the same problem is presented and compared with the analytical solution. It is quite evident from the figure that the capacitance converged at around eight divisions along the edge and further increase in the discretization did not alter the result. The error encountered in the present scheme is around 6% for the 0.1-m layer thickness. Lastly, the results for the conducting cube show a similar behavior.

IV. CONCLUSION

This paper has demonstrated the use of the FE method to solve open-region electrostatic problems involving 3-D, arbitrarily shaped conducting bodies. FE was augmented by a relatively new technique—the MEI—that allows the problem space mesh to be truncated as close as two node layers from the body. This method was compared to results of the well-established MoM by investigating a cube and against the analytical solution for a sphere. Because FE/MEI maintains the sparsity of the system matrix, this method was shown to greatly reduce memory requirements and computer execution times below that of MoM while returning good results. MoM still seems to produce more accurate results at smaller discretizations where the surface charge or capacitance is desired, so we do not propose replacing MoM by FE/MEI in all applications, but rather offer an alternative method to solving static problems.

Presently, work is being done to extend these techniques to the dynamic electromagnetic scattering problem and we hope that this new method offers distinct advantages involving objects involving objects so large that the initial discretization would preclude the use of MoM due to limitations in computer resources.

REFERENCES

[1] R. F. Harrington, *Field Computation by Moment Methods*. New York: Macmillan, 1968.

[2] J. Jin, *The Finite Element Method in Electromagnetics*. New York: Wiley, 1993.

[3] A. Bayliss, M. Gunzburger, and E. Turkel, "Boundary conditions for the numerical solution of elliptic equations in exterior regions," *SIAM J. Appl. Math.*, vol. 42, pp. 430–451, Apr. 1982.

[4] P. P. Silvester and M. S. Hsieh, "Finite element solution of two-dimensional exterior field problems," *Proc. Inst. Elect. Eng.*, vol. 118, pp. 1743–1747, Dec. 1971.

[5] B. H. McDonald and A. Wexler, "Finite element solution of unbounded field problems," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-20, pp. 841–847, Dec. 1972.

[6] K. K. Mei, R. Pous, Z. Chen, Y. W. Liu, and M. D. Prouty, "Measured equation of invariance: A new concept in field computation," *IEEE Trans. Antennas Propagat.*, vol. 42, pp. 320–327, Mar. 1994.

[7] R. Pous, *The Measured Equation of Inavariance: A News Concept in Field Computation*, Ph.D. dissertation, Univ. California, Berkeley, 1992.

[8] A. C. Cangellaris and D. B. Wright, "Application of the measured equation of invariance to electromagnetic scattering by penetrable bodies," *IEEE Trans. Magn.*, vol. 29, pp. 1628–1631, Mar. 1993.

[9] J. O. Jetvic and R. Lee, "A theoretical and numerical analysis of the measured equation of invariance," *IEEE Trans. Antennas Propagat.*, vol. 42, pp. 1097–1105, Aug. 1994.

[10] J. O. Jetvic, "An analysis of measured equation of invariance," Master's thesis, Ohio State Univ., Columbus, 1994.

[11] G. K. Gothard, S. M. Rao, T. K. Sarkar, and M. S. Palma, "Finite-element solution of open region electrostatic problems incorporating the measured equation of invariance," *IEEE Microwave Guided Wave Lett.*, vol. 5, pp. 252–254, Aug. 1995.

[12] G. K. Gothard, "The two-dimensional finite integral technique combined with the measured equation of invariance applied to open region problems," Ph.D. dissertation, Auburn Univ., Auburn, AL, Dec. 1995.

[13] J. O. Jetvic and R. Lee, "An analytical characterization of the error in the measured equation of invariance," *IEEE Trans. Antennas Propagat.*, vol. 43, pp. 1109–1115, Oct. 1995.

[14] ——, "How invariant is the measured equation of invariance?," *IEEE Microwave Guided Wave Lett.*, vol. 5, pp. 45–47, Feb. 1995.

[15] K. K. Mei, "Comments on "How invariant is the measured equation of invariance?," *IEEE Microwave Guided Wave Lett.*, vol. 5, pp. 417, Nov. 1995.

[16] K. K. Mei, Y. Liu, J. O. Jetvic, and R. Lee, "Comments on 'A theoretical and numerical analysis of the measured equation of invariance,'" *IEEE Trans. Antennas Propagat.*, vol. 43, pp. 1168–1171, Oct. 1995.

[17] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes in C: The Art of Scientific Computing*. Cambridge, U.K.: Cambridge Univ. Press, 1992.

[18] G. H. Golub and C, F. van Loan, *Matrix Computations*. Baltimore, MD: Johns Hopkins Univ. Press, 1989.

[19] D. R. Wilton and J. E. Wheeler, "Comparison of convergence rates of the conjugate gradients method applied to various integral equation formulations," Tech. Rep. 87-16, Texas Instruments, Dallas, TX, 1987.

[20] S. M. Rao, A. W. Glisson, D. R. Wilton, and B. S. Vidula, "A simple numerical solution procedure for statics problems involving arbitrary shaped surfaces," *IEEE Trans. Antennas Propagat.*, vol. AP-27, pp. 604–608, Sept. 1979.

John H. Henderson received the B.E.E., M.S. and Ph.D. degrees in electrical engineering from Auburn University, Auburn, AL, in 1990, 1991, and 1997, respectively.

While working toward the Ph.D. degree, he was supported by NASA Langley Research Center through the NASA Graduate Student Researchers Fellowhip Program. He is currently employed as an Antenna Engineer and Electromagnetic Analyst in the Government Aerospace Systems Division of the Harris Corporation, Palm Bay, FL.

Dr. Henderson is a member of Eta Kappa Nu, Tau Beta Pi, Sigma Pi Sigma, and an Associate Member of Sigma Xi.+

Sadasiva M. Rao received the B.S. degree in electrical communication engineering from Osmania University, Hyderabad, India, in 1974, the M.S. degree in microwave engineering from the Indian Institute of Sciences, Bangalore, India, in 1976, and the Ph.D. degree with specialization in electromagnetic theory from University of Mississippi, University, MS, in 1980.

He served as a Research Assistant at the University of Mississippi and Syracuse University, Syracuse, NY, from 1976 to 1980 and as an Assistant Professor in the Department of Electrical Engineering, Rochester Institute of Technology, Rochester, NY, from 1980 to 1985. From 1985 to 1987 he was a Senior Scientist at Osmania University, Hyderabad, India, and from 1987 to 1988 he was a Visiting Associate Professor in the Department of Electrical Engineering, University of Houston, TX. He is currently a Professor in the Electrical Engineering Department, Auburn University, Auburn, AL. His research interest is in the area of numerical methods applied to antennas and scattering.