Linear Dependence of Steering Vectors Associated with Tripole Arrays

Kwok-Chiang Ho, Kah-Chye Tan, *Senior Member, IEEE*, and B. T. G. Tan

***Abstract*—We are concerned with the linear independence of steering vectors associated with tripoles, each of which provides measurements of the three components of electric field induced by electromagnetic signals. We first establish that for a single tripole, any steering vector is linearly dependent on at least one other steering vector corresponding to a different direction-of-arrival (DOA) for a general problem where signals may arrive from anywhere in a three-dimensional (3-D) space, but every two steering vectors with distinct DOA's are linearly independent if the signals are nonlinearly polarized and arrive from a strictly hemispherical space. We then obtain a series of upper bounds for the number of linearly independent steering vectors associated with a tripole array with general sensor configurations. We also show that for applications where signals are known to be linearly polarized in the same direction, the ability to estimate DOA's using a tripole array is identical to that using a scalar-sensor array if both of them have identical sensor configurations.**

***Index Terms*— Array signal processing, direction of arrival estimation.**

I. Introduction

One main objective of array signal processing is estimating the directions-of-arrival (DOA's) of narrow-band electromagnetic (EM) waves. As a matter of fact, many existing DOA-estimation systems are developed based on an array of scalar sensors, each of which provides measurements of only one component of the electric field induced at the sensor. For such systems, it is the phase delays of signals received at the sensors that provide the necessary information for DOA estimation.

In recent years, researchers have proposed the use of sensors that provide measurements of more than one component of electric/magnetic field, for example, EM vector sensors [1]–[8], for DOA estimation. An EM vector sensor provides measurements of the three components of electric field and three components of magnetic field. The measurements obtained with such sensors contain polarization information of the signals impinging on the array in addition to phase delay information. Since the DOA and polarization of an EM signal (assuming a planewave) are closely related, one can expect a better DOA estimation performance with the use of such sensors. Indeed, Nehorai and Paldi [1] have demonstrated, via an explicit evaluation of the Cramér–Rao bound, that superior DOA estimates can be obtained with EM vector sensors. Moreover, Tan *et al*. [2] (see also [3]–[5]) have established that with just one EM vector sensor, one can determine uniquely the DOA's of two uncorrelated EM signals, in general, or three uncorrelated signals if they are skywaves. In comparison, one would need four appropriately spaced scalar sensors to determine uniquely the DOA of even one signal.

A shortcoming of EM vector sensor is its high implementation cost and complexity. Indeed, it requires appreciable design effort to ensure that the measurements of the electric field and those of magnetic field are effectively independent of one another. In addition, it is a nontrivial task to develop EM vector sensors with adequate sensitivity for sufficient long-range applications.

A good compromise between scalar sensor and EM vector sensor is one which provides measurements of only the three components of electric field, commonly referred to as tripole. As a matter of fact, tripole measurements provide some polarization information that scalar sensor measurements lack. In addition, considerable reduction in implementation cost/complexity can be expected because the complications due to simultaneously measuring the electric and magnetic fields are absent. Therefore, it is of both theoretical interest and practical importance to investigate DOA estimation using tripoles. In this connection, we are aware of the work carried out by Compton [11] on the use of tripoles for interference rejection, a subject related to DOA estimation. We shall discuss his findings in relation to ours in more detail.

In this work, we focus on the linear independence of steering vectors associated with a tripole array, an issue very closely related to that of the number of signals whose DOA's are uniquely determinable. We shall not discuss the relationship here, but refer interested readers to [4],[5], [9], and [10]. We first establish that for a single tripole, any steering vector is linearly dependent on at least one other steering vector corresponding to a different DOA in the general case where signals may arrive from anywhere in a three-dimensional (3-D) space, but every two steering vectors with distinct DOA's are linearly independent if the signals are nonlinearly polarized and arrive from a strictly hemispherical space. We then obtain a series of upper bounds for the number of linearly independent steering vectors associated with a tripole array with general sensor configurations. We also show that for applications where signals are known to be linearly polarized in the same direction, the ability to estimate DOA's using a tripole array is identical to that using a scalar-sensor array if both of them have identical sensor configurations. (The main idea of the

Manuscript received September 26, 1996.

K.-C. Ho and K.-C. Tan are with the Centre for Signal Processing, School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore, 639798 Republic of Singapore

B. T. G. Tan is with the Faculty of Science, National University of Singapore, Singapore, 119260 Republic of Singapore.

Publisher Item Identifier S 0018-926X(98)08893-0.

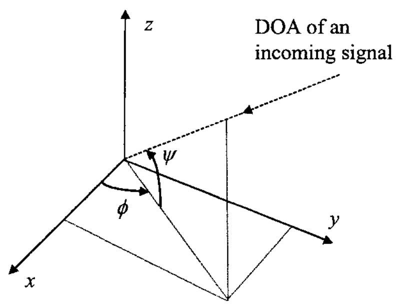

Fig. 1. The azimuth ϕ and the elevation ψ of the DOA of a signal in the Cartesian coordinate system.

paper has been presented in [12] and the detailed derivations are provided here.)

II. Data Model and Preliminary Discussion

Consider n narrow-band plane EM waves impinging on an array of m tripoles and assume that the waves are completely polarized and travel in a nonconductive, homogeneous, and isotropic medium. We set up a Cartesian coordinate system with the origin colocated with the reference sensor and each of the axes coinciding with a component of the electric field being measurable by the tripoles. Let $\mathbf{r}_l = (x_l, y_l, z_l)$ for $l = 2, \cdots m$, be the coordinate of the lth sensor (the coordinate of the first sensor is $(0,0,0)$), w the frequency of the signals, ϕ_k, and ψ_k, respectively, the azimuth and elevation of the kth signal (see Fig. 1), and α_k and β_k the polarization parameters commonly referred to as the orientation and ellipticity angles (see [1] for a more detailed description). Then the phasor measurement of the array of m tripoles is given by

$$\mathbf{y}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t) \in C^{3m\times 1}$$

where $\mathbf{y}(t)$ is a $3m \times 1$ complex vector containing measurements recorded with the tripole array at time t and $\mathbf{n}(t)$ is the corresponding noise in the measurement

$$\begin{aligned}
\mathbf{y}(t) &= \left[\mathbf{y}_e^{(1)}(t), \mathbf{y}_e^{(2)}(t), \cdots, \mathbf{y}_e^{(m)}(t)\right]^T \in C^{3m\times 1}\\
\mathbf{n}(t) &= \left[\mathbf{n}_e^{(1)}(t), \mathbf{n}_e^{(2)}(t), \cdots, \mathbf{n}_e^{(m)}(t)\right]^T \in C^{3m\times 1}\\
\mathbf{A} &= [\mathbf{a}(\theta_1), \cdots, \mathbf{a}(\theta_n)] \in C^{3m\times n}\\
\mathbf{a}(\theta_k) &= \mathbf{e}(\phi_k, \psi_k) \otimes \mathbf{B}(\phi_k, \psi_k)\mathbf{Q}(\alpha_k)\mathbf{w}(\beta_k) \in C^{3m\times 1}\\
\theta_k &= [\phi_k, \psi_k, \alpha_k, \beta_k]\\
\mathbf{e}(\phi_k, \psi_k) &= [e^{-jw\tau_{l,k}}, \cdots, e^{-jw\tau_{m,k}}]^T\\
\tau_{l,k} &= -\mathbf{u}(\phi_k, \psi_k) \bullet \mathbf{r}_l / c\\
\mathbf{u}(\phi_k, \psi_k) &= [\cos\phi_k \cos\psi_k \quad \sin\phi_k \cos\psi_k \quad \sin\psi_k]^T\\
\mathbf{B}(\phi_k, \psi_k) &= \begin{pmatrix} -\sin\phi_k & -\cos\phi_k \sin\psi_k \\ \cos\phi_k & -\sin\phi_k \sin\psi_k \\ 0 & \cos\psi_k \end{pmatrix}\\
\mathbf{Q}(\alpha_k) &= \begin{pmatrix} \cos\alpha_k & \sin\alpha_k \\ -\sin\alpha_k & \cos\alpha_k \end{pmatrix}, \quad \mathbf{w}(\beta_k) = \begin{pmatrix} \cos\beta_k \\ j\sin\beta_k \end{pmatrix}\\
\mathbf{s}(t) &= [s_1(t), \cdots, s_n(t)]^T
\end{aligned}$$

$\otimes$, $\bullet$, and "^{T}" are, respectively, the Kronecker product operator, the dot product operator, and the transpose operator and $\mathbf{y}_e^{(l)}$, $\mathbf{n}_e^{(l)}(t) \in C^{1\times 3}$ for $l = 1, \cdots, m$. Note that $\mathbf{y}_e^{(l)}(t)$ and $\mathbf{n}_e^{(l)}(t)$ are, respectively, the three-component measurements of the electric field and the corresponding noise components at the lth sensor at time t. The symbol $\tau_{l,k}$ is the differential delay of the kth signal at the lth sensor with respect to the reference sensor, c is the velocity of wave propagation, and the kth entry of the vector $\mathbf{s}(t)$ is the complex envelope of the kth signal at time t. The vector $\mathbf{a}(\theta_k)$ is commonly referred to as the *steering vector* (corresponding to the kth signal with DOA-cum-polarization parameter θ_k) of the *tripole* array. It is beneficial to note that if one replaces each of the tripoles with a scalar sensor, then $\mathbf{e}(\phi_k, \psi_k)$ will be the steering vector of the scalar-sensor array. The two columns of $\mathbf{B}(\phi_k, \psi_k)$ are orthogonal vectors that span the same plane as the electric and magnetic field vectors of the kth signal and the vector $\mathbf{u}(\phi_k, \psi_k)$ is the unit vector pointing toward the DOA (ϕ_k, ψ_k). Here, we shall consider ϕ_k falling within $(-\pi, \pi]$ and ψ_k within $[-\pi/2, \pi/2]$, meaning that the signals can come from any direction in a 3-D space. The ranges of the polarization parameters are $\alpha_k \in (-\pi/2, \pi/2]$, $\beta_k \in [-\pi/4, \pi/4]$.

III. Linear Independence of Steering Vectors of a Single Tripole

A relevant result has been obtained by Compton [11], who investigated the performance of a single tripole in rejecting an interference in the presence of a desired signal. He showed that the performance is dependent on a measure called signal-to-interference-plus-noise ratio (SINR):

$$\mathit{SINR} = \xi_d \left[1 - \frac{|\mathbf{a}^H(\theta_d)\mathbf{a}(\theta_i)|^2}{\xi_i^{-1} + 1}\right] \tag{1}$$

where "^{H}" is the Hermitian operator $\xi_d = c_d^2/\sigma^2$, $\xi_i = c_i^2/\sigma^2$, c_d, and c_i are some nonnegative constants, σ^2 is the strength of noise, Θ_d, and θ_i denote the DOA-cum-polarization parameters of the desired and interference signals, respectively, and $\mathbf{a}(\theta) = \mathbf{B}(\phi, \psi)\mathbf{Q}(\alpha)\mathbf{w}(\beta)$ is the steering vector (of a single tripole) corresponding to θ. The ability to reject interference is poorest when the value of SINR is at its minimum.

Interestingly enough, the value of *SINR* is closely dependent on the linear dependence of $\mathbf{a}(\theta_d)$, the steering vector associated with the desired signal, and $\mathbf{a}(\theta_i)$, that of the interference. Indeed, the following lemma yields one of these relationships.

Lemma 1: The value of SINR (as defined in (1)) is at its minimum if and only if the steering vectors $\mathbf{a}(\theta_d) = \mathbf{B}(\phi_d, \psi_d)\mathbf{Q}(\alpha_d)\mathbf{w}(\beta_d)$ and $\mathbf{a}(\theta_i) = \mathbf{B}(\phi_i, \psi_i)\mathbf{Q}(\alpha_i)\mathbf{w}(\beta_i)$ associated with a single tripole are linearly dependent.

Proof: See Appendix A. □

Remark: For ease of comparison with our results, we shall state Compton's results in terms of linear dependence of $\mathbf{a}(\theta_d)$ and $\mathbf{a}(\theta_i)$ instead of his original statement of SINR attaining its minimum value.

Before discussing Compton's results, we shall state three relevant definitions.

Definition 1 (R. T. Compton [11]): Let (α_1, β_1) and (α_2, β_2) be the polarization parameters of two signals. Then the two signals are said to have *conjugate polarizations* if the following two conditions are satisfied:

1) $\beta_1 = -\beta_2$;
2) $\alpha_1 = -\alpha_2$ if $\beta_1, \beta_2 \notin \{-\pi/4, \pi/4\}$.

Definition 2: Let $(\phi_1, \psi_1, \alpha_1, \beta_1)$ and $(\phi_2, \psi_2, \alpha_2, \beta_2)$ be the DOA-cum-polarization parameters of two signals. Then the two signals are said to be *linearly polarized with parallel electric field* if the following two conditions are satisfied:

1) $\beta_1 = \beta_2 = 0$;
2) $\mathbf{B}(\phi_1, \psi_1)\mathbf{Q}(\alpha_1)\mathbf{w}(0) = \pm\mathbf{B}(\phi_2, \psi_2)\mathbf{Q}(\alpha_2)\mathbf{w}(0)$.

Remark: Physically, the first condition indicates that the two signals are linearly polarized (see [1] for a detailed description of ellipticity angle). Note that it can be shown that for a linearly polarized signal with DOA-cum-polarization parameter $(\phi, \psi, \alpha, 0)$, the vector $\mathbf{B}(\phi, \psi)\mathbf{Q}(\alpha)\mathbf{w}(0)$ is pointing in the same direction as the electric field vector of the signal. Thus, the second condition means that the electric fields induced by the two linearly polarized signals at the tripole array are parallel.

Definition 3: Let (ϕ_1, ψ_1) and (ϕ_2, ψ_2) be the DOA's of two signals. Then the two signals are said to be of *opposite* DOA's if $\mathbf{u}(\phi_1, \psi_1) = -\mathbf{u}(\phi_2, \psi_2)$.

Remark: Note that $\mathbf{u}(\phi_1, \psi_1) = -\mathbf{u}(\phi_2, \psi_2)$ if and only if the following two conditions are satisfied:

1) $\psi_1 = -\psi_2$;
2) $\phi_1 = \phi_2 + \pi$ if $\psi_1,\ \psi_2 \notin \{-\pi/2, \pi/2\}$.

Theorem 1 (Compton [11]): Consider a desired signal with steering vector $\mathbf{a}(\theta_d) = \mathbf{B}(\phi_d, \psi_d)\mathbf{Q}(\alpha_d)\mathbf{w}(\beta_d)$ and an interference signal with steering vector $\mathbf{a}(\theta_i) = \mathbf{B}(\phi_i, \psi_i)\mathbf{Q}(\alpha_i)$ $\mathbf{w}(\beta_i)$, impinging on a single tripole and suppose the DOA's of the signals are distinct (i.e., $\mathbf{u}(\phi_d, \psi_d) \neq \mathbf{u}(\phi_i, \psi_i)$). Then $\mathbf{a}(\theta_d)$ and $\mathbf{a}(\theta_i)$ are linearly dependent if any one of the following conditions is satisfied

1) the two signals are of opposite DOA's with conjugate polarizations;
2) the two signals are linearly polarized with parallel electric field.

Remarks:

1) An interpretation of Theorem 1 is that if there is no constraint on the DOA's of signals (i.e., $\phi_k \in (-\pi, \pi]$ and $\psi_k \in [-\pi/2, \pi/2]$), then with a single tripole, a steering vector with any DOA-cum-polarization parameter is linearly dependent on at least one other steering vector corresponding to a different DOA. This implies that one cannot determine uniquely the DOA of even one signal with a single tripole, regardless of the DOA and polarization of the signal.
2) Since the earth is approximately a conductor, groundwaves are practically linearly polarized with electric fields perpendicular to the earth surface. Consequently, an implication of Theorem 1 is that it is never possible to determine uniquely the DOA of a groundwave.

Conditions 1) and 2) of Theorem 1 are sufficient conditions for two steering vectors (with distinct DOA's) to be linearly dependent. The immediate question of concern is whether there are other conditions that can lead to $\mathbf{a}(\theta_d)$ being linearly dependent on $\mathbf{a}(\theta_i)$. If there are many more, then the applicability of a single tripole will be limited. In this connection, we establish the following theorem.

Theorem 2: Consider a desired signal with steering vector $\mathbf{a}(\theta_d) = \mathbf{B}(\phi_d, \psi_d)\mathbf{Q}(\alpha_d)\mathbf{w}(\beta_d)$ and an interference signal with steering vector $\mathbf{a}(\theta_i) = \mathbf{B}(\phi_i, \psi_i)\mathbf{Q}(\alpha_i)\mathbf{w}(\beta_i)$ impinging on a single tripole and suppose the DOA's of the signals are distinct. Then $\mathbf{a}(\theta_d)$ and $\mathbf{a}(\theta_i)$ are linearly dependent *if and only if* any one of the following conditions is satisfied:

1) the two signals are of opposite DOA's with conjugate polarizations;
2) the two signals are linearly polarized with parallel electric field.

Proof: See Appendix B. □

Remarks:

1) Theorem 2 means that the sufficient conditions established in [11] are, in fact, necessary. This implies that it is generally rare to encounter situations where $\mathbf{a}(\theta_d)$ and $\mathbf{a}(\theta_i)$ are linearly dependent.
2) Consider estimating the DOA's of skywaves with a ground-based tripole. Since the signals strictly arrive from the upper hemisphere of the ground plane containing the tripole, the allowable range of the DOA's are $\phi_k \in (-\pi, \pi]$ and $\psi_k \in (0, \pi/2]$. This implies that Condition 1) of Theorem 2 will not be satisfied. Moreover, skywave is unlikely to be linearly polarized since each reflection from the ionosphere will cause a change in polarization. Consequently, Condition 2) is satisfied only for some sets of DOA-cum-polarization parameters with measure zero. Effectively, the above two arguments imply that every two steering vectors are linearly independent, except for some sets of DOA-cum-polarization parameters with measure zero and, thus, one can determine uniquely the DOA of one skywave for virtually all cases in practice.

IV. Upper Bounds for the Number of Linearly Independent Steering Vectors of Tripole Arrays

Here we shall establish some upper bounds for the number of linearly independent steering vectors of tripole arrays with general sensor configurations. We first establish a theorem which relates linear dependence of steering vectors of a tripole array to those of a scalar-sensor array having the same sensor configuration as that of the tripole array.

Theorem 3: Consider an m-tripole array $(m > 1)$ and n distinct DOA's $(\phi_1, \psi_1), (\phi_2, \psi_2), \cdots, (\phi_n, \psi_n)$. If the n signals are all linearly polarized with parallel electric field, then

$$\begin{aligned}&\text{rank}[\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), \cdots, \mathbf{a}(\theta_n)]\\&\quad = \text{rank}[\mathbf{e}(\phi_1, \psi_1), \mathbf{e}(\phi_2, \psi_2), \cdots, \mathbf{e}(\phi_n, \psi_n)].\end{aligned}$$

Proof: See Appendix C. □

Remarks:

1) $\mathbf{a}(\theta_k)$ is the steering vector of the tripole array for the signal arriving from (ϕ_k, ψ_k) and by replacing each of the tripoles with a scalar sensor, the steering vector will become $\mathbf{e}(\phi_k, \psi_k)$.

2) Since groundwaves are all linearly polarized in the direction normal to the ground plane, Theorem 3 means that the identifiability of a tripole array for groundwaves is identical to that of a scalar-sensor array if both of them have identical sensor configurations. On the other hand, since skywaves are unlikely to be linearly polarized, this problem does not exist for scenarios involving skywaves.

Corollary 1: Consider an m-tripole array, where $m \geq 2$. Then any $m+1$ steering vectors corresponding to $m+1$ signals, which are linearly polarized with parallel electric field, are linearly dependent.

Corollary 2: Consider an m-tripole array, where $m \geq 2$. Then any m steering vectors, corresponding to m linearly polarized signals with electric fields all parallel to the line joining two of the sensors, are linearly dependent.

Proof of Corollary 2: See Appendix D. □

Remarks of the Corollaries:

1) It is immediate from Corollary 1 that one cannot determine uniquely the DOA's of m uncorrelated groundwaves with an m-tripole array, regardless of the sensor configuration, although the array actually provides $3m$-dimensional measurements.
2) Corollary 2 suggests that on estimating the DOA's of groundwaves, one should not arrange the sensors such that there exist two sensors with the line joining them being perpendicular to the ground plane.

Next, we shall establish a theorem that yields a hint as to the identifiability limit of a tripole array.

Theorem 4: Given any $(3m-1)$ steering vectors of an m-tripole array, then for any DOA there exists a steering vector which is linearly dependent on the $(3m-1)$ steering vectors.

Proof: See Appendix E. □

Remark: It follows from Theorem 4 that when there are $(3m-1)$ signals impinging on a m-tripole array, there exists a steering vector corresponding to an arbitrary DOA that intersects the signal subspace (the space spanned by the steering vectors of the $(3m-1)$ signals). Thus, estimating the DOA's of $(3m-1)$ signals with an m-tripole array, using subspace methods such as MUSIC [13], is impossible. In particular, for a single tripole (i.e., $m=1$), it is always impossible to estimate the DOA's of two signals using subspace methods. (It is interesting to note that it has been established in [2] that with a single EM vector sensor, one can determine uniquely the DOA's of three uncorrelated skywaves.)

Next, we shall establish a theorem which provides an insight into estimation of $\lfloor 3m/2 \rfloor$ signals with m-tripole arrays, where $\lfloor x \rfloor$ denotes the integer part of x.

Theorem 5: Consider an m-tripole array and $n = \lfloor 3m/2 \rfloor + 1$ signals with *arbitrary* DOA's $(\phi_1, \psi_1), (\phi_2, \psi_2), \cdots, (\phi_n, \psi_n)$ that are distinct. Then, there is a set of polarizations $(\alpha_1, \beta_1), (\alpha_2, \beta_2), \cdots, (\alpha_n, \beta_n)$ associated with the signals such that the n steering vectors $\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), \cdots, \mathbf{a}(\theta_n)$ are linearly dependent.

Proof: See Appendix F. □

Remark: It follows from Theorem 5 that with an m-tripole array, it is not always possible to estimate uniquely the DOA's of $\lfloor 3m/2 \rfloor$ signals.

Before we end this section, we shall highlight the main differences among the four upper bounds that we have established in this section. The two bounds given by Corollaries 1 and 2 of Theorem 3 are applicable to cases where the DOA's of the signals are all linearly polarized with parallel electric fields. One practical application is estimating the DOA's of groundwaves with ground-based tripole array. On the other hand, the bound given by Theorem 4 is applicable for signals with arbitrary DOA's and polarizations, whereas that by Theorem 5 is applicable for signals with arbitrary DOA's but specific polarizations.

V. Concluding Remarks

We have shown that the sufficient conditions established by Compton [11] for two steering vectors of a single tripole to be linearly dependent are, in fact, necessary. We have also shown that any steering vector is linearly dependent on at least one other steering vector corresponding to a different DOA for a general problem where signals may arrive from anywhere in a 3-D space, but every two steering vectors with distinct DOA's are linearly independent if the signals are nonlinearly polarized and arrive from a strictly hemispherical space. This implies that for a single tripole, the DOA of a signal can be uniquely determined if the signals are nonlinearly polarized and arrive from a strictly hemispherical space, but not from anywhere in a 3-D space. In addition, we have shown that it is impossible to determine uniquely the DOA's of two signals with a single tripole.

We have also obtained four upper bounds for the number of linearly independent steering vectors associated with a tripole array with general sensor configurations. These bounds are potentially useful for determining the maximum number of signals whose DOA's can be uniquely identified with such an array. Moreover, for scenarios involving the estimation of the DOA's of linearly polarized signals with parallel electric field, we have established that the ability to identify DOA's using a tripole array is identical to that using a scalar-sensor array if both of them have identical sensor configurations. This then enables one to obtain more insight into the identifiability of tripole arrays using the results that have been established for scalar-sensor arrays (see [14]–[17]).

Appendix A
Proof of Lemma 1

It is easy to see from (1) that SINR reaches its minimum if and only if $|\mathbf{a}^H(\theta_d)\mathbf{a}(\theta_i)|^2$ reaches its maximum. Therefore, what we need to show here is that $|\mathbf{a}^H(\theta_d)\mathbf{a}(\theta_i)|^2$ reaches its maximum if and only if $\mathbf{a}(\theta_i) = e^{j\delta}\mathbf{a}(\theta_d)$ for some constant δ.

Let $\mathbf{g}_1$ and $\mathbf{g}_2$ be two orthonormal vectors lying in the null space of $\mathbf{a}(\theta_d)$. Then the vectors $\mathbf{a}(\theta_d)$, $\mathbf{g}_1$ and $\mathbf{g}_2$ are mutually orthonormal vectors that span the 3-D space for which $\mathbf{a}(\theta_i)$ is in. Consequently, we can write

$$\mathbf{a}(\theta_i) = c_1\mathbf{g}_1 + c_2\mathbf{g}_2 + c_3\mathbf{a}(\theta_d) \tag{A.1}$$

for some complex constants c_1, c_2, and c_3. Since $\mathbf{a}(\theta_d)$, $\mathbf{g}_1$, and $\mathbf{g}_2$ are mutually orthogonal vectors, we obtain from (A.1) that

$$|\mathbf{a}(\theta_i)|^2 = |c_1|^2 + |c_2|^2 + |c_3\mathbf{a}(\theta_d)|^2 = |c_1|^2 + |c_2|^2 + |c_3|^2. \tag{A.2}$$

Since $|\mathbf{a}(\theta_i)|^2 = 1$, we obtain from (A.2) that $|c_3|^2 \leq 1$. Now, premultiplying (A.1) by $\mathbf{a}^H(\theta_d)$, we have

$$\mathbf{a}^H(\theta_d)\mathbf{a}(\theta_i) = c_3. \tag{A.3}$$

Since $|c_3|^2 \leq 1$, it follows from (A.3) that $|\mathbf{a}^H(\theta_d)\mathbf{a}(\theta_i)|^2$ reaches its maximum if and only if $|c_3|^2 = 1$. From (A.1), it can be easily shown that $|c_3|^2 = 1$ if and only if $c_1 = c_2 = 0$ and $c_3 = e^{j\delta}$ for some constant δ. Thus, we conclude from (A.1) that $|\mathbf{a}^H(\theta_d)\mathbf{a}(\theta_i)|^2$ reaches its maximum (i.e., one) if and only if $\mathbf{a}(\theta_i) = e^{j\delta}\mathbf{a}(\theta_d)$ for some constant δ. □

APPENDIX B
PROOF OF THEOREM 2

We shall first state a lemma derived in [1] and then establish another lemma.

Lemma 2 (Nehorai & Paldi [1]): Every vector $\mathbf{v} = (v_1, v_2)^T \in C^{2\times 1}$ has the representation

$$\mathbf{v} = ||\mathbf{v}||e^{j\gamma}\mathbf{Q}(\alpha)\mathbf{w}(\beta)$$

where $\gamma \in (-\pi, \pi]$, $\alpha \in (-\pi/2, \pi/2]$, and $\beta \in [-\pi/4, \pi/4]$. Moreover, the parameters $\mathbf{v}$, γ, α, and β in the above equation are uniquely determined if and only if $v_1^2 + v_2^2 \neq 0$.

Remark: It is further established by Nehorai and Paldi that if $||\mathbf{v}|| \neq 0$, then $v_1^2 + v_2^2 = 0$ if and only if $\beta = \pm\pi/4$. Furthermore, if $v_1^2 + v_2^2 = 0$ and $||\mathbf{v}|| \neq 0$, then $||\mathbf{v}||$, β but not γ, α are unique.

Lemma 3: Let $\mathbf{a}(\theta_1) = \mathbf{B}(\phi_1, \psi_1)\mathbf{Q}(\alpha_1)\mathbf{w}(\beta_1)$ and $\mathbf{a}(\theta_2) = \mathbf{B}(\phi_2, \psi_2)\mathbf{Q}(\alpha_2)\mathbf{w}(\beta_2)$ be two linearly dependent steering vectors of a single tripole with distinct DOA's (ϕ_1, ψ_1) and (ϕ_2, ψ_2). Then β_1 equals zero if and only if β_2 equals zero.

Remark: Note that a signal is linearly polarized if and only if its ellipticity angle β is equal to zero. Therefore, Lemma 3 implies that two steering vectors with distinct DOA's of a single tripole are linearly dependent only if they are both linearly polarized or both nonlinearly polarized.

Proof of Lemma 3: Since $\mathbf{a}(\theta_1) = \mathbf{B}(\phi_1, \psi_1)\mathbf{Q}(\alpha_1)\mathbf{w}(\beta_1)$ and $\mathbf{a}(\theta_2) = \mathbf{B}(\phi_2, \psi_2)\mathbf{Q}(\alpha_2)\mathbf{w}(\beta_2)$ are linearly dependent steering vectors of a single tripole with distinct DOA's, we have

$$\mathbf{a}(\theta_1) = c\mathbf{a}(\theta_2) \tag{B.1}$$

for some complex number c. Since the norm of $\mathbf{a}(\theta_1)$ and $\mathbf{a}(\theta_2)$ are the same, we have $c = e^{j\gamma}$ for some $\gamma \in (-\pi, \pi]$.

Now, suppose $\beta_1 = 0$ and we shall show that $\beta_2 = 0$. Premultiplying (B.1) by $\mathbf{Q}^H(\alpha_2)\mathbf{B}^H(\phi_2, \psi_2)$, we obtain

$$\mathbf{Q}^H(\alpha_2)\mathbf{B}^H(\phi_2, \psi_2)\mathbf{B}(\phi_1, \psi_1)\mathbf{Q}(\alpha_1)\mathbf{w}(\beta_1) = e^{j\gamma}\mathbf{w}(\beta_2). \tag{B.2}$$

Since $\beta_1 = 0$, $\mathbf{w}(\beta_1)$ is real. This, together with the fact that $\mathbf{Q}(\alpha_2)$, $\mathbf{B}(\phi_2, \psi_2)$, $\mathbf{B}(\phi_1, \psi_1)$, and $\mathbf{Q}(\alpha_1)$ are real, ensure that the left-hand side of (B.2) is real. Therefore, the right-hand side (RHS) of (B.2), which can be expressed as $(e^{j\gamma}\cos\beta_2, je^{j\gamma}\sin\beta_2)^T$, must also be a real vector. Since $\cos\beta_2 \neq 0$, $e^{j\gamma}$ must be a real number (i.e., either -1 or 1). Now, since $je^{j\gamma}\sin\beta_2$ and $e^{j\gamma}$ are both real numbers, $\sin\beta_2$ must be zero, implying that $\beta_2 = 0$. Therefore, we have shown that $\beta_2 = 0$ if $\beta_1 = 0$.

Using the same strategy, one can show that $\beta_1 = 0$ if $\beta_2 = 0$. □

Proof of Theorem 2: Although a proof for the sufficiency part is available in [11], we shall provide another version.

Sufficiency part: Let θ_1 and θ_2 be the DOA-cum-polarization parameters of two signals with distinct DOA's and consider a single tripole.

First, suppose θ_1 and θ_2 correspond to two signals with opposite DOA's and conjugate polarizations. Then, we have $\theta_2 = (\phi_1 + \pi, -\psi_1, -\alpha_1, -\beta_1)$. It can be verified that $\mathbf{a}(\theta_1) = -\mathbf{a}(\theta_2)$ and this establishes Condition 1 of the sufficiency part.

Next, suppose θ_1 and θ_2 correspond to two linearly polarized signals (i.e., $\beta_1 = \beta_2 = 0$) whose electric fields are parallel, then, by Definition 2, we immediately obtain $\mathbf{B}(\phi_1, \psi_1)\mathbf{Q}(\alpha_1)\mathbf{w}(0) = \pm\mathbf{B}(\phi_2, \psi_2)\mathbf{Q}(\alpha_2)\mathbf{w}(0)$. Therefore, we have $\mathbf{a}(\theta_1) = \pm\mathbf{a}(\theta_2)$. This establishes Condition 2 of the sufficiency part.

Necessity part: Let $\mathbf{a}(\theta_1)$ and $\mathbf{a}(\theta_2)$ be linearly dependent steering vectors of one tripole that correspond to distinct DOA's. It follows from Lemma 3 that the ellipticity angles of the signals satisfy either: 1) $\beta_1 = \beta_2 = 0$ or 2) $\beta_1, \beta_2 \neq 0$.

Case 1: ($\beta_1 = \beta_2 = 0$). We have established in the proof of Lemma 3 that if $\beta_1 = 0$, then $\mathbf{a}(\theta_1) = \pm\mathbf{a}(\theta_2)$. By Definition 2, the electric fields of the two linearly polarized signals are parallel, which leads to Theorem 2, Condition 2.

Case 2: ($\beta_1, \beta_2 \neq 0$). We have established in the proof of Lemma 3 that

$$\mathbf{a}(\theta_1) = e^{j\gamma}\mathbf{a}(\theta_2) \tag{B.3}$$

where $\gamma \in (-\pi, \pi]$. Premultiplying both sides of (B.3) by $\mathbf{u}^H(\phi_1, \psi_1) = [\cos\phi_k\cos\psi_k \;\; \sin\phi_k\cos\psi_k \;\; \sin\psi_k]$, we obtain

$$0 = \cos\beta_2(p\cos\alpha_2 - q\sin\alpha_2) + j\sin\beta_2(p\sin\alpha_2 + q\cos\alpha_2)$$

where $[p \;\; q] = \mathbf{u}^H(\phi_1, \psi_1)\mathbf{B}(\phi_2, \psi_2)$. Equating the real and imaginary parts of the above equation and using the fact that $\cos\beta_2 \neq 0$ and $\sin\beta_2 \neq 0$ (since $\beta_2 \in [-\pi/4, \pi/4]$ and $\beta_2 \neq 0$), we obtain

$$p\cos\alpha_2 - q\sin\alpha_2 = 0 \quad \text{and} \quad p\sin\alpha_2 + q\cos\alpha_2 = 0.$$

Solving the above two equations, we obtain $p = q = 0$. Therefore, we have $\mathbf{u}^H(\phi_1, \psi_1)\mathbf{B}(\phi_2, \psi_2) = [0 \;\; 0]$. This, together with the fact that $\mathbf{u}^H(\phi_2, \psi_2)\mathbf{B}(\phi_2, \psi_2) = [0 \;\; 0]$, imply that $\mathbf{u}(\phi_1, \psi_1)$ and $\mathbf{u}(\phi_2, \psi_2)$ both lie in the orthogonal complement of the column space of $\mathbf{B}(\phi_2, \psi_2)$, which is of dimension one. Since $\mathbf{u}(\phi_1, \psi_1)$ and $\mathbf{u}(\phi_2, \psi_2)$ are both real unit vectors and (ϕ_1, ψ_1) and (ϕ_2, ψ_2) correspond to distinct DOA's, $\mathbf{u}(\phi_1, \psi_1)$ must be equal to $-\mathbf{u}(\phi_2, \psi_2)$. Thus, (ϕ_1, ψ_1) and (ϕ_2, ψ_2) correspond to opposite DOA's and, hence

$$(\phi_2, \psi_2) = (\phi_1 + \pi, -\psi_1). \tag{B.4}$$

Now, substituting $(\phi_2, \psi_2) = (\phi_1 + \pi, -\psi_1)$ into (B.3) and then premultiplying it by $\mathbf{B}^H(\phi_1, \psi_1)$, we obtain

$$\mathbf{Q}(\alpha_1)\mathbf{w}(\beta_1) = e^{j\gamma}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{Q}(\alpha_2)\mathbf{w}(\beta_2).$$

The RHS of the above equation can be simplified to $-e^{j\gamma}\mathbf{Q}(-\alpha_2)\mathbf{w}(-\beta_2)$. It then follows immediately from Lemma 2 and its remark that $\beta_1 = -\beta_2$. Moreover, we have $\alpha_1 = -\alpha_2$ if $\beta_1, \beta_2 \neq \pm\pi/4$. Therefore, (α_1, β_1) and (α_2, β_2) correspond to conjugate polarizations. This, together with (B.4), lead to Theorem 2, Condition 1.

Combining the results for both cases, we obtain the necessity part of Theorem 2. □

Appendix C
Proof of Theorem 3

Let $\mathbf{a}(\theta_1)$, $\mathbf{a}(\theta_2), \cdots, \mathbf{a}(\theta_n)$, where $\mathbf{a}(\theta_k) = \mathbf{e}(\phi_k, \psi_k) \otimes \mathbf{B}(\phi_k, \psi_k)\mathbf{Q}(\alpha_k)\mathbf{w}(\beta_k)$, be the steering vectors (of a multitripole array) associated with n linearly polarized signals with distinct DOA's and parallel electric fields. By Condition 2 of Theorem 2, $\mathbf{B}(\phi_1, \psi_1)\mathbf{Q}(\alpha_1)\mathbf{w}(\beta_1)$, $\mathbf{B}(\phi_2, \psi_2)\mathbf{Q}(\alpha_2)\mathbf{w}(\beta_2)$, $\cdots$, $\mathbf{B}(\phi_n, \psi_n)\mathbf{Q}(\alpha_n)\mathbf{w}(\beta_n)$, the steering vectors associated with a single tripole (notice that "$\mathbf{e}(\phi_k, \psi_k)\otimes$" has been removed) are pairwise linearly dependent. Thus, each row of $\Gamma = [\mathbf{B}(\phi_1, \psi_1)\mathbf{Q}(\alpha_1)\mathbf{w}(\beta_1), \cdots, \mathbf{B}(\phi_n, \psi_n)\mathbf{Q}(\alpha_n)\mathbf{w}(\beta_n)]$ is linearly dependent on one other row of Γ. Since $\mathbf{B}(\phi_1, \psi_1)\mathbf{Q}(\alpha_1)\mathbf{w}(\beta_1)$ is a nonzero vector, at least one of the entries of $\mathbf{B}(\phi_1, \psi_1)\mathbf{Q}(\alpha_1)\mathbf{w}(\beta_1)$ is nonzero. Now assuming that the lth entry of $\mathbf{B}(\phi_1, \psi_1)\mathbf{Q}(\alpha_1)\mathbf{w}(\beta_1)$ is nonzero where $l \in \{1, 2, 3\}$. Since $\mathbf{a}(\theta_k) = \mathbf{e}(\phi_k, \psi_k) \otimes\mathbf{B}(\phi_k, \psi_k)\mathbf{Q}(\alpha_k)\mathbf{w}(\beta_k)$ and that the lth entry of $\mathbf{B}(\phi_1, \psi_1)\mathbf{Q}(\alpha_1)\mathbf{w}(\beta_1)$ is nonzero, the $(3k - p)$th row of $\mathbf{A}$ is linearly dependent on the $(3k - l)$th row for $p \in \{1, 2, 3\}\backslash\{l\}$ and $k = 1, \cdots, m$ where

$$\mathbf{A} = [\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), \cdots, \mathbf{a}(\theta_n)].$$

Consequently, we have $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}[\mathbf{e}(\phi_1, \psi_1), \mathbf{e}(\phi_2, \psi_2), \cdots, \mathbf{e}(\phi_n, \psi_n)]$. □

Appendix D
Proof of Corollary 2 of Theorem 3

Consider an m-tripole array and m steering vectors $\mathbf{a}(\theta_1)$, $\mathbf{a}(\theta_2), \cdots, \mathbf{a}(\theta_m)$ that correspond to m linearly polarized signals with electric fields all parallel to the line joining two sensors. Since the electric fields of all signals are parallel, by Theorem 3 we have

$$\begin{aligned}&\operatorname{rank}[\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), \cdots, \mathbf{a}(\theta_m)]\\ &\quad= \operatorname{rank}[\mathbf{e}(\phi_1, \psi_1), \mathbf{e}(\phi_2, \psi_2), \cdots, \mathbf{e}(\phi_m, \psi_m)].\end{aligned} \tag{D.1}$$

We shall show that rank $[\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), \cdots, \mathbf{a}(\theta_m)] < m$. Without loss of generality, let the electric fields of all the signals be parallel to the line joining the first two sensors. Then the DOA's of the signals are all normal to the line joining the first two sensors. Consequently, the phase delay of the kth signal at the second sensor with respect to the first sensor is zero for $k = 1, \cdots, m$. This leads to

$$\begin{aligned}&\operatorname{rank}[\mathbf{e}(\phi_1, \psi_1), \mathbf{e}(\phi_2, \psi_2), \cdots, \mathbf{e}(\phi_m, \psi_m)]\\ &= \operatorname{rank}\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ e^{-jw\tau_{3,1}} & e^{-jw\tau_{3,2}} & & e^{-jw\tau_{3,m}} \\ \vdots & \vdots & & \vdots \\ e^{-jw\tau_{m,1}} & e^{-jw\tau_{m,2}} & & e^{-jw\tau_{m,m}} \end{bmatrix} < m.\end{aligned} \tag{D.2}$$

Thus, it follows from (D.1) and (D.2) that $\operatorname{rank}[\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), \cdots, \mathbf{a}(\theta_m)] < m$. □

Appendix E
Proof of Theorem 4

Consider $(3m - 1)$ steering vectors $\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), \cdots, \mathbf{a}(\theta_{3m-1})$ and a DOA (ϕ_{3m}, ψ_{3m}). Let $[\mathbf{p}, \mathbf{q}] = \mathbf{e}(\phi_{3m}, \psi_{3m}) \otimes \mathbf{B}(\phi_{3m}, \psi_{3m})$. Then the $(3m + 1)$ vectors $\mathbf{a}(\theta_1)$, $\mathbf{a}(\theta_2), \cdots, \mathbf{a}(\theta_{3m-1}), \mathbf{p}$ and $\mathbf{q}$ are linearly dependent since they are $3m\times 1$ complex vectors. Therefore, there exists $(c_1, c_2, \cdots, c_{3m+1})^T \in C^{(3m+1)\times 1}\backslash\{\mathbf{0}_{(3m+1)\times 1}\}$ that satisfies

$$\begin{aligned}&c_1\mathbf{a}(\theta_1) + c_2\mathbf{a}(\theta_2) + \cdots + c_{3m-1}\mathbf{a}(\theta_{3m-1})\\ &\quad= c_{3m}\mathbf{p} + c_{3m+1}\mathbf{q}.\end{aligned} \tag{E.1}$$

By Lemma 2, we can write

$$(c_{3m}, c_{3m+1})^T = \|(c_{3m}, c_{3m+1})^T\|e^{j\gamma}\mathbf{Q}(\alpha_{3m})\mathbf{w}(\beta_{3m})$$

for some $\gamma \in (-\pi, \pi]$, $\alpha_{3m} \in (-\pi/2, \pi/2]$ and $\beta_{3m} \in [-\pi/4, \pi/4]$. Therefore, we have

$$\begin{aligned}c_{3m}\mathbf{p} + c_{3m+1}\mathbf{q} &= \|(c_{3m}, c_{3m+1})^T\|e^{j\gamma}\mathbf{e}(\phi_{3m}, \psi_{3m})\\ &\quad\otimes \mathbf{B}(\phi_{3m}, \psi_{3m})\mathbf{Q}(\alpha_{3m})\mathbf{w}(\beta_{3m})\\ &= \|(c_{3m}, c_{3m+1})^T\|e^{j\gamma}\mathbf{a}(\theta_{3m}).\end{aligned} \tag{E.2}$$

It follows from (E.1) and (E.2) that

$$\begin{aligned}&c_1\mathbf{a}(\theta_1) + c_2\mathbf{a}(\theta_2) + \cdots + c_{3m-1}\mathbf{a}(\theta_{3m-1})\\ &\quad= \|(c_{3m}, c_{3m+1})^T\|e^{j\gamma}\mathbf{a}(\theta_{3m}).\end{aligned}$$

Therefore, the steering vectors $\mathbf{a}(\theta_1)$, $\mathbf{a}(\theta_2), \cdots, \mathbf{a}(\theta_{3m})$ are linearly dependent. □

Appendix F
Proof of Theorem 5

Consider an m-tripole array. Let $(\phi_1, \psi_1), \cdots, (\phi_n, \psi_n)$ be distinct DOA's associated with n signals and

$$\begin{aligned}\Gamma = [&\mathbf{e}(\phi_1, \psi_1) \otimes \mathbf{B}(\phi_1, \psi_1), \mathbf{e}(\phi_2, \psi_2)\\ &\otimes \mathbf{B}(\phi_2, \psi_2), \cdots, \mathbf{e}(\phi_n, \psi_n) \otimes \mathbf{B}(\phi_n, \psi_n)] \in C^{3m\times 2n}\end{aligned}$$

where $n = \lfloor 3m/2\rfloor + 1$. Then the columns of Γ are linearly dependent since the number of columns of Γ is more than the number of rows. Therefore, there exists $(\mathbf{z}_1^T, \cdots, \mathbf{z}_n^T)^T \in C^{2n\times 1}\backslash\{\mathbf{0}_{2n\times 1}\}$ (with each $\mathbf{z}_l \in C^{2\times 1}$) such that

$$\Gamma\big(\mathbf{z}_1^T, \cdots, \mathbf{z}_n^T\big)^T = \mathbf{0}_{3m\times 1}. \tag{F.1}$$

By Lemma 2, we may write for $k = 1, \cdots, n$, $\mathbf{z}_k = \|\mathbf{z}_k\| e^{j\gamma_k} \mathbf{Q}(\alpha_k)\mathbf{w}(\beta_k)$ for some $\gamma_k \in (-\pi, \pi]$, $\alpha_k \in (-\pi/2, \pi/2]$, and $\beta_k \in [-\pi/4, \pi/4]$. Hence, we may write (F.1)

$$\sum_{k=1}^{n} c_k \mathbf{e}(\phi_k, \psi_k) \otimes \mathbf{B}(\phi_k, \psi_k)\mathbf{Q}(\alpha_k)\mathbf{w}(\beta_k) = \sum_{k=1}^{n} c_k \mathbf{a}(\theta_k) = \mathbf{0}_{3m \times 1}$$

where $c_k = \|\mathbf{z}_k\| e^{j\gamma k}$ and $\mathbf{a}(\theta_k) = \mathbf{e}(\phi_k, \psi_k) \otimes \mathbf{B}(\phi_k, \psi_k) \mathbf{Q}(\alpha_k)\mathbf{w}(\beta_k)$. Since $(\mathbf{z}_1^T, \cdots, \mathbf{z}_n^T)^T \neq \mathbf{0}_{2n \times 1}$, we have $(c_1, \cdots, c_n)^T \neq \mathbf{0}_{n \times 1}$. This implies that $\mathbf{a}(\theta_1), \cdots, \mathbf{a}(\theta_n)$ are linearly dependent. [Note that the polarization parameter associated with the kth signal is (α_k, β_k)]. □

REFERENCES

[1] A. Nehorai and E. Paldi, "Vector sensor array processing for electromagnetic source localization," *IEEE Trans. Signal Processing*, vol. 42, pp. 376–398, Feb. 1994.

[2] K.-C. Tan, K.-C. Ho, and A. Nehorai, "Linear independence of steering vectors of an electromagnetic vector sensor," *IEEE Trans. Signal Processing*, vol. 44, pp. 3099–3107, Dec. 1996.

[3] K.-C. Ho, K.-C. Tan, and W. Ser, "A study of the resolvability of vector-sensor array in direction-of-arrival estimation," in *Proc. Int. Conf. Commun. Syst.'94*, Singapore, Nov. 1994, vol. 1, pp. 35–39.

[4] B. Hochwald and A. Nehorai, "Identifiability in array processing models with vector-sensor applications," *IEEE Trans. Signal Processing*, vol. 44, pp. 83–95, Jan. 1996.

[5] K.-C. Ho, K.-C. Tan, and W. Ser, "An investigation on number of signals whose directions-of-arrival are uniquely determinable with an electromagnetic vector sensor," *Signal Processing*, vol. 47, no. 1, pp. 41–54, 1995.

[6] J. Li, "Direction and polarization estimation using arrays with small loops and short dipoles," *IEEE Trans. Antenna Propagation*, vol. 41, pp. 379–387, Feb. 1993.

[7] K. T. Wong and M. D. Zoltowski, "High accuracy 2-D angle estimation with extended aperture vector sensor arrays," in *Proc. Int. Conf. Acoust., Speech, Signal Processing*, Atlanta, GA, May 1996, vol. 5, pp. 2789–2792.

[8] K.-C. Ho, K.-C. Tan, and B. T. G. Tan, "Estimation of directions-of-arrival of partially polarized signals with electromagnetic vector sensors," in *Proc. Int. Conf. Acoust., Speech, Signal Processing*, Atlanta, GA, May 1996, vol. 5, pp. 2900–2903.

[9] M. Wax and I. Ziskind, "On unique localization of multiple sources by passive sensor arrays," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 37, pp. 996–1000, July 1989.

[10] A. Nehorai, D. Starer, and P. Stoica, "Direction-of-arrival estimation in applications with multipath and few snapshots," *Circuits Syst. Signal Processing*, vol. 10, no. 3, pp. 327–342, 1991.

[11] R. T. Compton Jr., "The tripole antenna: An adaptive array with full polarization flexibility," *IEEE Trans. Antennas Propagat,*, vol. AP-29, pp. 944–952, June 1981.

[12] K.-C. Ho, K.-C. Tan, and B. T. G. Tan, "A study of the resolvability of tripole arrays," in *Proc. Int. Conf. Commun. Syst.*, Singapore, Nov. 1996, vol. 1, pp. 302–306.

[13] R. O. Schmidt, "A signal subspace approach to multiple emitter location and spectral estimation," Ph.D. dissertation, Dept. Elect. Eng., Stanford Univ., Stanford, CA, 1981.

[14] K.-C. Tan, G.-L. Oh, and M. H. Er, "A study of the uniqueness of steering vectors in array processing," *Signal Processing*, vol. 34, no. 3, pp. 245–256, 1993.

[15] K.-C. Tan and Z. Goh, "A detailed derivation of arrays free of higher rank ambiguities," *IEEE Trans. Signal Processing*, vol. 44, no. 2, pp. 351–359, Feb. 1996.

[16] K.-C. Tan, S.-S. Goh, and E.-C. Tan, "A study of the rank ambiguity issues in direction-of-arrival estimation," *IEEE Trans. Signal Processing*, vol. 44, no. 4, pp. 880–887, Apr. 1996.

[17] K.-C. Tan, E.-C. Tan, and S.-S. Goh, "Counter-examples to a conjecture for characterizing higher rank ambiguities," *IEEE Trans. Signal Processing*, vol. 44, no. 4, pp. 1028–1029, Apr. 1996.

Kwok-Chiang Ho was born in Singapore in 1968. He received the B.Sc. (honors) degree in mathematics from the National University of Singapore, in 1993.

From 1993 to 1996, he worked at the Defence Science Organization of Ministry of Defence, Singapore, as an Analyst. Since August 1996 he has been working at the Centre for Signal Processing, Nanyang Technological University, Singapore, as a Project Officer. His research interests are in array signal processing, speech processing, and digital communications.

Kah-Chye Tan (M'98–SM'98) was born in Singapore in 1961. He received the B.Sc. (honors) and Ph.D. degrees, both in physics, from the National University of Singapore, in 1985 and 1992, respectively.

From 1985 to 1996, he worked at the Defence Science Organization of Ministry of Defence, Singapore, as a Principal Analyst. In August 1996 he joined Centre for Signal Processing, Nanyang Technological University, Singapore, as the Research and Development Manager. He was appointed as Deputy Director of the Centre in June 1998. His research interests are in array signal processing, speech/audio processing, image/video processing, and digital communications.

Dr. Tan was the recipient of the Defence Technology Prize (individual) in 1993 and the Institute of Physics Medal in 1985.

B. T. G. Tan received the B.Sc. degree (honors) in physics from the University of Singapore, in 1965, and the D.Phil. degree from Oxford University, Oxford, U.K., in 1968.

Since 1968, he has taught at the National University of Singapore where he is now an Associate Professor in physics. His research interests include the microwave properties of semiconductors and dielectrics, surface physics, image processing, and digital sound synthesis.

Dr. Tan is a chartered engineer and a member of the Institution of Electrical Engineers.