Integral Equations and Discretizations for Waveguide Apertures

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***Abstract*—We present integral equations and their discretizations for calculating the fields radiated from arbitrarily shaped antennas fed by cylindrical waveguides of arbitrary cross sections. We give results for scalar fields in two dimensions with Dirichlet and Neumann boundary conditions and for (vector) electric and magnetic fields in three dimensions. The discretized forms of the equations are cast in identical format for all four cases. Feed modes can be TM, TE, or transverse electromagnetic (TEM). A method for numerically computing the modes of an arbitrarily shaped, cylindrical waveguide aperture is also given.**

***Index Terms*—Aperture antennas, integral equations.**

I. Introduction

NUMERICAL simulation of the electromagnetic performance of antennas using integral equations requires a mathematical model of the driving sources. In contrast to scattering cross-section computations where a distant source creates a plane wave in the vicinity of the scatterer, construction of an accurate source model for an antenna is nontrivial. If a simple approach, such as a "delta-gap" excitation [1] is used, the accuracy of some important antenna parameters, such as input impedance, gain, and reflection can be seriously compromised, even for cases in which the far-field pattern is obtained accurately.

The purpose of this paper is twofold. First, we develop integral equations representing exact specification of the field emanating from an aperture of arbitrary shape with the field entering the aperture left unconstrained and to be determined. The exact definition of the "emanating" field is accomplished by analysis of a translationally invariant waveguide that has the cross section of the given aperture. In the context of a generalized scattering problem such as a waveguide-fed antenna, such an integral equation may serve as a boundary condition that must be obeyed inside the waveguide on any plane normal to its axis. Second, we derive discretized forms of the integral equations[1] (using the method of moments) that are suitable for numerical computation. As part of this development, we give a useful interpretation of the kernel that appears in the "waveguide integral equation."

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[1] An equivalent formulation of the feed model for the electromagnetic case has been used previously by McGrath and Pyati [2] We, however, try to clarify the intent, development, and use of this formulation in the context of a generalized method of moments discretization.

Our development is based on the assumption that the waveguide is:

- translationally invariant in the half-space behind the aperture along the axis normal to the aperture;
- terminated by a perfect absorber or is so long as to be practically nonreflecting;
- filled with a linear, isotropic, homogeneous medium;
- enclosed by walls that are infinitely hard or infinitely soft in the scalar scattering case or perfectly conducting in the electromagnetic scattering case.

The first section is devoted to finding continuous and discretized forms of the waveguide integral equations for scalar waves and then applying them to more general scattering problems. These equations apply to acoustic scattering in two or three dimensions as well as the two-dimensional (2-D) analogues of three-dimensional (3-D) electromagnetic scattering (which apply to scatterers with translational symmetry in a direction orthogonal to the axis of the waveguide). In the second section, we do the same for 3-D electromagnetic scattering. The two treatments are entirely analogous. Formulas for the power flow out of (due to the given excitation) and into (due to back scattering) the waveguide are also given in each section. In the third section, we show how the waveguide integral equations can be extended to more general circumstances. Prescriptions for numerically computing the modes of cylindrical waveguides with arbitrary cross sections may be found in the Appendix.

II. Scalar Waveguide Equations

A. Modes

An arbitrary field $\psi(\mathbf{x})$ that satisfies the scalar Helmholtz equation

$$(\nabla^2 + k^2)\psi(\mathbf{x}) = 0 \tag{1}$$

inside a waveguide aligned with the $\mathbf{z}$ axis, can be written as a sum of modal components[2] traveling in the $+\hat{\mathbf{z}}$ and $-\hat{\mathbf{z}}$ directions [3]

$$\psi(\mathbf{x}_\perp, z) = \sum_n (a_n e^{i\beta_n z} + b_n e^{-i\beta_n z}) u_n(\mathbf{x}_\perp). \tag{2}$$

[2] For simplicity, we will assume that no cutoff modes (i.e., those with $\beta = 0$) are present. It is straightforward to amend the development to handle such modes.

Likewise, the longitudinal derivative of the field may be written as

$$\frac{\partial\psi(\mathbf{x}_\perp,z)}{\partial z}=\sum_n(a_n e^{i\beta_n z}-b_n e^{-i\beta_n z})\frac{ik}{Z_n}u_n(\mathbf{x}_\perp) \quad (3)$$

where

$$Z_n=\frac{k}{\beta_n} \quad (4)$$

is the modal impedance. In these equations, an implicit $e^{-i\omega t}$ time dependence is assumed for the fields, $k=\omega/c$ is the free-space propagation constant and β_n and $u_n(\mathbf{x}_\perp)$ are, respectively, the propagation constant and transverse field distribution of the nth mode inside the guide. The modes are eigensolutions to the scalar wave equation

$$(\nabla_\perp^2+k^2-\beta_n^2)u_n(\mathbf{x}_\perp)=0 \quad (5)$$

for $\mathbf{x}_\perp$ inside the waveguide aperture W and the $u_n(\mathbf{x}_\perp)$ are constrained to satisfy the boundary conditions of the waveguide walls when $\mathbf{x}_\perp$ is on the boundary of the aperture ∂W. With proper normalization, the modes form a complete and orthonormal set of functions over W, i.e.,

$$\sum_n u_n(\mathbf{x}_\perp)u_n(\mathbf{x}'_\perp)=\delta(\mathbf{x}_\perp-\mathbf{x}'_\perp) \quad \text{Completeness} \quad (6)$$

and

$$\int_W d\mathbf{x}_\perp u_m(\mathbf{x}_\perp)u_n(\mathbf{x}_\perp)=\delta_{mn} \quad \text{Orthonormality.} \quad (7)$$

B. Waveguide Integral Equation

Let $\psi^{\text{out}}(\mathbf{x}_\perp,z)$ denote a specified outgoing wave, $z=0$ correspond to the plane of the waveguide aperture, and the rest of the waveguide be located in the half-space with $z<0$. Using the modal expansions and the completeness relation for the modes, we can write the following expression for $\psi^{\text{out}}(\mathbf{x}_\perp,0)$ in terms of the field and its longitudinal derivative on W:

$$\begin{aligned}\psi^{\text{out}}(\mathbf{x}_\perp,0)&=\sum_n a_n u_n(\mathbf{x}_\perp)\\&=\frac{1}{2}\sum_n(a_n+b_n)u_n(\mathbf{x}_\perp)\\&\quad+\frac{1}{2}\sum_n\frac{Z_n}{ik}(a_n-b_n)\frac{ik}{Z_n}u_n(\mathbf{x}_\perp)\\&=\frac{1}{2}\psi(\mathbf{x}_\perp,0)+\frac{1}{2}\int_W d\mathbf{x}'_\perp H(\mathbf{x}_\perp,\mathbf{x}'_\perp)\\&\quad\times\left.\frac{\partial\psi(\mathbf{x}'_\perp,z')}{\partial z'}\right|_{z'=0}\end{aligned} \quad (8)$$

where

$$H(\mathbf{x}_\perp,\mathbf{x}'_\perp)=\sum_n\frac{Z_n}{ik}u_n(\mathbf{x}_\perp)u_n(\mathbf{x}'_\perp). \quad (9)$$

For any point $\mathbf{x}$ on a general surface S, we may define an independent surface field quantity

$$\sigma(\mathbf{x})\equiv-\lim_{\mathbf{x}'\to\mathbf{x}}\hat{\mathbf{n}}(\mathbf{x})\cdot\nabla'\psi(\mathbf{x}'); \quad \mathbf{x} \text{ on } S \quad (10)$$

where $\hat{\mathbf{n}}(\mathbf{x})$ is the outward unit normal to S at $\mathbf{x}$. In the case of a waveguide aperture, σ simplifies to

$$\sigma(\mathbf{x}_\perp,0)\equiv-\left.\frac{\partial\psi(\mathbf{x}'_\perp,z')}{\partial z'}\right|_{z'=0}; \quad \mathbf{x}_\perp \text{ on } W. \quad (11)$$

Inserting this into (8) and dropping the spatial coordinate z, we obtain the following integral equation on the waveguide aperture that relates the field, its longitudinal derivative, and the specified waveguide excitation on W:

$$2\psi^{\text{out}}(\mathbf{x}_\perp)=\psi(\mathbf{x}_\perp)-\int_W d\mathbf{x}'_\perp H(\mathbf{x}_\perp,\mathbf{x}'_\perp)\sigma(\mathbf{x}'_\perp). \quad (12)$$

$H(\mathbf{x}_\perp,\mathbf{x}'_\perp)$ is the kernel of the "square root" of the transverse wave operator in the sense that

$$\int_W d\mathbf{x}'_\perp H(\mathbf{x}_\perp,\mathbf{x}'_\perp)H(\mathbf{x}'_\perp,\mathbf{x}''_\perp)=\hat{G}_\perp(\mathbf{x}_\perp,\mathbf{x}''_\perp) \quad (13)$$

where $\hat{G}_\perp$ obeys

$$(\nabla_\perp^2+k^2)\hat{G}_\perp(\mathbf{x}_\perp,\mathbf{x}''_\perp)=-\delta(\mathbf{x}_\perp-\mathbf{x}''_\perp) \quad (14)$$

inside the waveguide and satisfies the boundary conditions on the waveguide walls.

A different relation between ψ, σ, and the outgoing wave is obtained if we specify $\partial\psi^{\text{out}}(\mathbf{x}_\perp,z)/\partial z$ instead of $\psi^{\text{out}}(\mathbf{x}_\perp,z)$ to write

$$\begin{aligned}\frac{\partial\psi^{\text{out}}(\mathbf{x}_\perp,0)}{\partial z}&=\sum_n a_n\frac{ik}{Z_n}u_n(\mathbf{x}_\perp)\\&=\frac{1}{2}\sum_n(a_n+b_n)\frac{ik}{Z_n}u_n(\mathbf{x}_\perp)\\&\quad+\frac{1}{2}\sum_n\frac{ik}{Z_n}(a_n-b_n)u_n(\mathbf{x}_\perp)\\&=\frac{1}{2}\left.\frac{\partial\psi(\mathbf{x}'_\perp,z')}{\partial z'}\right|_{z'=0}+\frac{1}{2}\int_W d\mathbf{x}'_\perp\tilde{H}(\mathbf{x}_\perp,\mathbf{x}'_\perp)\\&\quad\times\psi(\mathbf{x}_\perp,0)\end{aligned} \quad (15)$$

where[3]

$$\tilde{H}(\mathbf{x}_\perp,\mathbf{x}'_\perp)=\sum_n\frac{ik}{Z_n}u_n(\mathbf{x}_\perp)u_n(\mathbf{x}'_\perp). \quad (16)$$

Dropping the spatial coordinate z and defining σ as before, we get an alternative form for the waveguide integral equation

$$2\frac{\partial\psi^{\text{out}}(\mathbf{x}_\perp)}{\partial z}=\frac{\partial\psi(\mathbf{x}_\perp)}{\partial z}+\int_W d\mathbf{x}'_\perp\tilde{H}(\mathbf{x}_\perp,\mathbf{x}'_\perp)\psi(\mathbf{x}'_\perp) \quad (17)$$

or

$$-2\frac{\partial\psi^{\text{out}}(\mathbf{x}_\perp)}{\partial z}=\sigma(\mathbf{x}_\perp)-\int_W d\mathbf{x}'_\perp\tilde{H}(\mathbf{x}_\perp,\mathbf{x}'_\perp)\psi(\mathbf{x}'_\perp). \quad (18)$$

$\tilde{H}(\mathbf{x}_\perp,\mathbf{x}'_\perp)$ and $H(\mathbf{x}_\perp,\mathbf{x}'_\perp)$ are "inverse operators" in the sense that

$$\int_W d\mathbf{x}'_\perp H(\mathbf{x}_\perp,\mathbf{x}'_\perp)\tilde{H}(\mathbf{x}'_\perp,\mathbf{x}''_\perp)=\delta(\mathbf{x}_\perp-\mathbf{x}''_\perp). \quad (19)$$

[3]Note that $\tilde{H}(\mathbf{x}_\perp,\mathbf{x}'_\perp)$ is not a *function* since the sum over all n does not converge. Rather, like the Dirac delta "function" $\delta(\mathbf{x}_\perp,\mathbf{x}'_\perp)$, it is a *distribution*, which, when convolved with a suitably smooth function, produces a well-defined value.

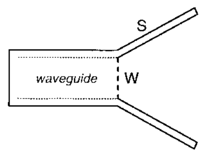

Fig. 1. Antenna system composed of waveguide aperture W and antenna surface S.

Using (12) and (18) on the waveguide aperture W, we can derive boundary integral equations that apply to more general scattering cases. For example, we can write coupled boundary integral equations for the case of a waveguide aperture connected to a general scatterer. This is demonstrated in the next subsection for the special cases in which the scattering surface obeys either Dirichlet or Neumann boundary conditions. In both cases, it is assumed that the union of the scatterer S and waveguide aperture W forms a closed surface, as indicated in Fig. 1.

C. Coupled Integral Equations

In this section, we derive integral equations relating the known field emanating from the waveguide aperture to an unknown surface field (either ψ or σ) for the generic closed antenna system shown in Fig. 1. For Dirichlet (Neumann) boundary conditions on S, the unknown surface field on both S and W is chosen to be $\sigma(\psi)$.

1) Dirichlet Boundary Conditions on S: The integral equation for the field (in the absence of an explicit incident wave) is [4]

$$\frac{1}{2}\psi(\mathbf{x}) = \oint_{S\ominus W} ds'\{[\hat{\mathbf{n}}(\mathbf{x}')\cdot\nabla' G(\mathbf{x},\mathbf{x}')]\psi(\mathbf{x}') + G(\mathbf{x},\mathbf{x}')\sigma(\mathbf{x}')\} \tag{20}$$

for $\mathbf{x}$ on $S\oplus W$. The Helmholtz kernel $G(\mathbf{x},\mathbf{x}')$ is given by

$$G(\mathbf{x},\mathbf{x}') = \begin{cases} \frac{i}{4}H_0^{(1)}(k|\mathbf{x}-\mathbf{x}'|) & \text{in } 2d \\ \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} & \text{in } 3d \end{cases} \tag{21}$$

where $H_0^{(1)}$ is the zeroth-order Hankel function of the first kind. For Dirichlet boundary conditions on S (i.e., $\psi(\mathbf{x} \text{ on } S) = 0$) we have

$$0 = \oint_{S\ominus W} ds' G(\mathbf{x},\mathbf{x}')\sigma(\mathbf{x}') + \int_W ds'[\hat{\mathbf{n}}(\mathbf{x}')\cdot\nabla' G(\mathbf{x},\mathbf{x}')]\psi(\mathbf{x}') \tag{22}$$

for $\mathbf{x}$ on S and

$$\frac{1}{2}\psi(\mathbf{x}) = \oint_{S\ominus W} ds' G(\mathbf{x},\mathbf{x}')\sigma(\mathbf{x}') + \int_W ds'[\hat{\mathbf{n}}(\mathbf{x}')\cdot\nabla' G(\mathbf{x},\mathbf{x}')]\psi(\mathbf{x}') \tag{23}$$

for $\mathbf{x}$ on W. Equations (22) and (23) along with either (12) or (18) form a set of coupled integral equations to be solved for $\psi(\mathbf{x})$ on W and $\sigma(\mathbf{x})$ on $S\oplus W$. Using (12) we can eliminate ψ, putting the known field $\psi^{\text{out}}(\mathbf{x})$ on the left and the unknown quantity $\sigma(\mathbf{x})$ on the right

$$\begin{aligned} -2\oint_W ds'(\hat{\mathbf{n}}'\cdot\nabla' G(\mathbf{x},\mathbf{x}'))\psi^{\text{out}}(\mathbf{x}') &= \oint_S ds' G(\mathbf{x},\mathbf{x}')\sigma(\mathbf{x}') + \oint_w ds' G(\mathbf{x},\mathbf{x}')\sigma(\mathbf{x}') \\ &\quad + \oint_W ds'(\hat{\mathbf{n}}'\cdot\nabla' G(\mathbf{x},\mathbf{x}'))\int_W ds'' H(\mathbf{x}',\mathbf{x}'')\sigma(\mathbf{x}'') \end{aligned} \tag{24}$$

for x on S and

$$\begin{aligned} \psi^{\text{out}}(\mathbf{x}) - 2\oint_W ds'(\hat{\mathbf{n}}'\cdot\nabla' G(\mathbf{x},\mathbf{x}'))\psi^{\text{out}}(\mathbf{x}') &= \oint_S ds' G(\mathbf{x},\mathbf{x}')\sigma(\mathbf{x}') \\ &\quad + \oint_W ds'[G(\mathbf{x},\mathbf{x}')\sigma(\mathbf{x}') - \frac{1}{2}H(\mathbf{x},\mathbf{x}')\sigma(\mathbf{x}')] \\ &\quad + \oint_W ds'(\hat{\mathbf{n}}'\cdot\nabla' G(\mathbf{x},\mathbf{x}'))\int_W ds'' H(\mathbf{x}',\mathbf{x}'')\sigma(\mathbf{x}'') \end{aligned} \tag{25}$$

for x on W.

2) Neumann Boundary Conditions on S: The integral equation for σ (i.e. the normal derivative of the field) may be written as [4]

$$\frac{1}{2}\sigma(\mathbf{x}) = -(\hat{\mathbf{n}}(\mathbf{x})\cdot\nabla)\oint_{S\ominus W} ds'\{[\hat{\mathbf{n}}(\mathbf{x}')\cdot\nabla' G(\mathbf{x},\mathbf{x}')]\psi(\mathbf{x}') + G(\mathbf{x},\mathbf{x}')\sigma(\mathbf{x}')\} \tag{26}$$

or

$$\begin{aligned} \frac{1}{2}\sigma(\mathbf{x}) = \oint_{S\ominus W} ds'\{&[\hat{\mathbf{n}}(\mathbf{x})\times\nabla G(\mathbf{x},\mathbf{x}')]\cdot[\hat{\mathbf{n}}(\mathbf{x}')\times\nabla'\psi(\mathbf{x}')] \\ &- k^2(\hat{\mathbf{n}}(\mathbf{x})\cdot\hat{\mathbf{n}}(\mathbf{x}'))G(\mathbf{x},\mathbf{x}')\psi(\mathbf{x}') \\ &- \hat{\mathbf{n}}(\mathbf{x})\nabla G(\mathbf{x},\mathbf{x}')\sigma(\mathbf{x}')\} \end{aligned} \tag{27}$$

for $\mathbf{x}$ on $S\oplus W$. The first form is more compact (and for that reason is employed below), the second more convenient for numerical computation. For Neumann boundary conditions on S (i.e., $\sigma(\mathbf{x} \text{ on } S) = 0$), we have

$$\begin{aligned} 0 = &-(\hat{\mathbf{n}}(\mathbf{x})\cdot\nabla)\int_S ds'[\hat{\mathbf{n}}(\mathbf{x}')\cdot\nabla' G(\mathbf{x},\mathbf{x}')]\psi(\mathbf{x}') \\ &-(\hat{\mathbf{n}}(\mathbf{x})\cdot\nabla)\int_W ds'\{[\hat{\mathbf{n}}(\mathbf{x}')\cdot\nabla' G(\mathbf{x},\mathbf{x}')]\psi(\mathbf{x}') \\ &+ G(\mathbf{x},\mathbf{x}')\sigma(\mathbf{x}')\} \end{aligned} \tag{28}$$

for $\mathbf{x}$ on S and

$$\begin{aligned} \frac{1}{2}\sigma(\mathbf{x}) = &-(\hat{\mathbf{n}}(\mathbf{x})\cdot\nabla)\int_S ds'[\hat{\mathbf{n}}(\mathbf{x}')\cdot\nabla' G(\mathbf{x},\mathbf{x}')]\psi(\mathbf{x}') \\ &-(\hat{\mathbf{n}}(\mathbf{x})\cdot\nabla)\int_W ds'\{[\hat{\mathbf{n}}(\mathbf{x}')\cdot\nabla' G(\mathbf{x},\mathbf{x}')]\psi(\mathbf{x}') \\ &+ G(\mathbf{x},\mathbf{x}')\sigma(\mathbf{x}')\} \end{aligned} \tag{29}$$

for $\mathbf{x}$ on W. Combining (28) and (29) with (18), we can eliminate σ and write the following integral equations for $\psi(\mathbf{x})$

in terms of the known quantity $\partial\psi^{\rm out}(\mathbf{x})/\partial z$:

$$\begin{aligned}2\int_W ds'(\hat{\mathbf{n}}\cdot\nabla G(\mathbf{x},\mathbf{x}'))\frac{\partial\psi^{\rm out}}{\partial z}(\mathbf{x}') &= (\hat{\mathbf{n}}\cdot\nabla)\int_S ds'(\hat{\mathbf{n}}'\cdot\nabla' G(\mathbf{x},\mathbf{x}'))\psi(\mathbf{x}')\\ &+(\hat{\mathbf{n}}\cdot\nabla)\int_W ds'(\hat{\mathbf{n}}'\cdot\nabla' G(\mathbf{x},\mathbf{x}'))\psi(\mathbf{x}')\\ &+\int_W ds'(\hat{\mathbf{n}}\cdot\nabla G(\mathbf{x},\mathbf{x}'))\int_W ds''\tilde{H}(\mathbf{x}',\mathbf{x}'')\psi(\mathbf{x}'') \end{aligned}\tag{30}$$

for x on S and

$$\begin{aligned}\frac{\partial\psi^{\rm out}}{\partial z}(\mathbf{x})+2\int_W ds'(\hat{\mathbf{n}}\cdot\nabla G(\mathbf{x},\mathbf{x}'))\frac{\partial\psi^{\rm out}}{\partial z}(\mathbf{x}') &= (\hat{\mathbf{n}}\cdot\nabla)\int_S ds'(\hat{\mathbf{n}}'\cdot\nabla' G(\mathbf{x},\mathbf{x}'))\psi(\mathbf{x}')\\ &+(\hat{\mathbf{n}}\cdot\nabla)\int_W ds'(\hat{\mathbf{n}}'\cdot\nabla' G(\mathbf{x},\mathbf{x}'))\psi(\mathbf{x}')\\ &+\frac{1}{2}\int_W ds'\tilde{H}(\mathbf{x},\mathbf{x}')\psi(\mathbf{x}')\\ &+\int_W ds'(\hat{\mathbf{n}}\cdot\nabla G(\mathbf{x},\mathbf{x}'))\int_W ds''\tilde{H}(\mathbf{x}',\mathbf{x}'')\psi(\mathbf{x}'') \end{aligned}\tag{31}$$

for x on W.

D. Discretization

While analytical solutions for waveguide modes are known for a few special cross sections, in general, modes must be computed numerically. Even when analytical solutions exist, it is more convenient (from a computational perspective) to use numerical solutions because then all interacting surfaces, whether physical or intangible (e.g. waveguide apertures), can be treated equivalently.

Assume the waveguide aperture has been discretized into a set of patches that support M basis functions $f_m(\mathbf{x})$. Following the procedure given in the Appendix, we can write approximate expressions for the N lowest waveguide modes in terms of basis functions defined on the aperture

$$u_n(\mathbf{x})=\sum_{m=1}^{M}A_{nm}f_m(\mathbf{x}).\tag{32}$$

In the usual method of moments fashion, we approximate the field ψ and its normal derivative σ on the aperture as linear combinations of the basis functions with unknowns coefficients S_m^W and I_m^W

$$\psi(\mathbf{x})\approx\sum_{m=1}^{M}S_m^W f_m(\mathbf{x})\tag{33}$$

$$\sigma(\mathbf{x})\approx\sum_{m=1}^{M}I_m^W f_m(\mathbf{x}).\tag{34}$$

We also approximate $H(\mathbf{x},\mathbf{x}')$ as a truncated sum over the N computed modes

$$H(\mathbf{x},\mathbf{x}')\approx\sum_{n=1}^{N}\frac{Z_n}{ik}u_n(\mathbf{x})u_n(\mathbf{x}').\tag{35}$$

Then, by substituting (32)–(35) into (12), and applying the testing operator $\int_W ds f_i(\mathbf{x})\cdot$ to both sides of the resultant equation, we arrive at the discretized form of (12)

$$2V^W=N^W S^W-X^W I^W\tag{36}$$

where

$$V_i^W=\int_W ds\psi^{\rm out}(\mathbf{x})f_i(\mathbf{x})\tag{37a}$$

$$N_{ij}^W=\int_W ds f_i(\mathbf{x})f_j(\mathbf{x})\tag{37b}$$

$$\begin{aligned}X_{ij}^W&=\int_W ds\int_W ds' f_i(\mathbf{x})H(\mathbf{x},\mathbf{x}')f_j(\mathbf{x}')\\ &=[(AN^W)^T\Lambda(AN^W)]_{ij}\end{aligned}\tag{37c}$$

and

$$\Lambda_{mn}=\frac{Z_n}{ik}\delta_{mn}.\tag{38}$$

A similar procedure produces the discretized form of (18), namely

$$2\tilde{V}^W=N^W I^W-\tilde{X}^W S^W\tag{39}$$

where

$$\tilde{V}_i^W=-\int_W ds\frac{\partial\psi^{\rm out}}{\partial z}(\mathbf{x})f_i(\mathbf{x})\tag{40a}$$

$$\begin{aligned}\tilde{X}_{ij}^W&=\int_W ds\int_W ds' f_i(\mathbf{x})\tilde{H}(\mathbf{x},\mathbf{x}')f_j(\mathbf{x}')\\ &=[(AN^W)^T\tilde{\Lambda}(AN^W)]_{ij}\end{aligned}\tag{40b}$$

and

$$\tilde{\Lambda}_{mn}=\frac{ik}{Z_n}\delta_{mn}=(\Lambda^{-1})_{mn}.\tag{41}$$

Equations (12) and (18) and their discretized equivalents (36) and (39) may be viewed as nonlocal inhomogeneous boundary conditions that must be obeyed on the waveguide aperture. They are nonlocal because the "surface impedance" terms X^W and $\tilde{X}^W$ relate the field at one point on the aperture to its derivative not just at the same point, but everywhere on the aperture, and vice versa. The equations are inhomogeneous if excitations V^W and $\tilde{V}^W$ are nonzero.

The discretized forms of the coupled integral equations for Dirichlet boundary conditions on S are obtained by first approximating the source on S in terms of basis functions as

$$\sigma(\mathbf{x})\approx\sum_{m=1}^{M}I_m^S f_m(\mathbf{x})\tag{42}$$

then substituting this approximation and the approximate expressions for $\psi(\mathbf{x})$, $\sigma(\mathbf{x})$, and $H(\mathbf{x}_\perp,\mathbf{x}'_\perp)$ on W into (22) and (23) and finally applying the testing function operator $\int_{S\ominus W} ds f_i(\mathbf{x})\cdot$ to both sides. The result in block matrix form is

$$\begin{bmatrix}-2Y^{SW}(N^W)^{-1}V^W\\ V^W\end{bmatrix}=\begin{bmatrix}Z^{SS} & Z^{SW}+Y^{SW}(N^W)^{-1}X^W\\ Z^{WS} & Z^{WW}-\frac{1}{2}X^W\end{bmatrix}\begin{bmatrix}I^S\\ I^W\end{bmatrix}\tag{43}$$

where

$$Y_{ij}^{SW} = \int_S ds \int_W ds' f_i(\mathbf{x})(\hat{\mathbf{n}}' \cdot \nabla' G(\mathbf{x}, \mathbf{x}'))f_j(\mathbf{x}') \tag{44}$$

$$Z_{ij}^{\alpha\beta} = \int_\alpha ds \int_\beta ds' f_i(\mathbf{x})G(\mathbf{x}, \mathbf{x}')f_j(\mathbf{x}') \tag{45}$$

with S or W replacing α and β.

An analogous result is obtained for the case of Neumann boundary conditions on S. We approximate the source on S as

$$\psi(\mathbf{x}) \approx \sum_{m=1}^{M} S_m^S f_m(\mathbf{x}) \tag{46}$$

substitute this expression and the approximate expressions for $\psi(\mathbf{x})$, $\sigma(\mathbf{x})$, and $\tilde{H}(\mathbf{x}_\perp, \mathbf{x}'_\perp)$ on W into (28) and (29) and then apply the testing operator. The result is

$$\begin{bmatrix} 2Y^{SW}(N^W)^{-1}\tilde{V}^W \\ -\tilde{V}^W \end{bmatrix} = \begin{bmatrix} \tilde{Z}^{SS} & \tilde{Z}^{SW} - \tilde{Y}^{SW}(N^W)^{-1}\tilde{X}^W \\ \tilde{Z}^{WS} & \tilde{Z}^{WW} + \frac{1}{2}\tilde{X}^W \end{bmatrix} \begin{bmatrix} S^S \\ S^W \end{bmatrix} \tag{47}$$

where

$$\tilde{Y}_{ij}^{SW} = \int_S ds \int_W ds' f_i(\mathbf{x})(\hat{\mathbf{n}} \cdot \nabla G(\mathbf{x}, \mathbf{x}'))f_j(\mathbf{x}') \tag{48}$$

$$\tilde{Z}_{ij}^{\alpha\beta} = \int_\alpha ds \int_\beta ds' [f_i(\mathbf{x})[\hat{\mathbf{n}}(\mathbf{x}) \times \nabla G(\mathbf{x}, \mathbf{x}')] \cdot [\hat{\mathbf{n}}(\mathbf{x}') \times \nabla' f_j(\mathbf{x}')] - k^2(\hat{\mathbf{n}}(\mathbf{x}) \cdot \hat{\mathbf{n}}(\mathbf{x}'))f_i(\mathbf{x})G(\mathbf{x}, \mathbf{x}')f_j(\mathbf{x}')] \tag{49}$$

with S or W replacing α and β.

E. Modal Decomposition

In preparation for computing the power flowing across the waveguide aperture in either direction, it is useful to write ψ and $\partial\psi/\partial z$ in terms of modes propagating in either direction.

By employing the completeness relation for the modes we can decompose the field on W into a sum over modes as

$$\psi(\mathbf{x}) = \sum_n \eta_n u_n(\mathbf{x}) \tag{50}$$

where

$$\eta_n = \int_W ds u_n(\mathbf{x})\psi(\mathbf{x}) \tag{51}$$

is the amplitude of the nth mode contained in $\psi(\mathbf{x})$. It is useful to further decompose $\psi(\mathbf{x})$ into its incoming and outgoing components

$$\psi(\mathbf{x}) = \psi^{\text{in}}(\mathbf{x}) + \psi^{\text{out}}(\mathbf{x}). \tag{52}$$

Since the discretized representation of $\psi^{\text{out}}(\mathbf{x})$ is given by V^W, we may write the discretized form of η_n^{out} as

$$\eta_n^{\text{out}} = \sum_m A_{nm} V_m^W. \tag{53}$$

Using (12) to eliminate $\psi(\mathbf{x})$, we arrive at the discretized form of η_n^{in}

$$\eta_n^{\text{in}} = \sum_m A_{nm}(V^W + X^W I^W)_m. \tag{54}$$

Similarly, we may decompose the longitudinal derivative of the field as

$$\frac{\partial\psi(\mathbf{x})}{\partial z} = \sum_n \tilde{\eta}_n u_n(\mathbf{x}) \tag{55}$$

where

$$\tilde{\eta}_n = \int_W ds u_n(\mathbf{x}) \frac{\partial\psi(\mathbf{x})}{\partial z}. \tag{56}$$

Then, using

$$\frac{\partial\psi(\mathbf{x})}{\partial z} = \frac{\partial\psi^{\text{in}}(\mathbf{x})}{\partial z} + \frac{\partial\psi^{\text{out}}(\mathbf{x})}{\partial z} \tag{57}$$

and (18), we can write $\tilde{\eta}_n^{\text{out}}$ and $\tilde{\eta}_n^{\text{in}}$ in discretized form as

$$\tilde{\eta}_n^{\text{out}} = -\sum_m A_{nm}\tilde{V}_m^W \tag{58}$$

and

$$\eta_n^{\text{in}} = -\sum_m A_{nm}(\tilde{V}^W + \tilde{X}^W S^W)_m. \tag{59}$$

F. Power

The time-averaged power-flow density vector (the scalar equivalent to the Poynting vector) is [5]

$$\langle \mathbf{S}(\mathbf{x}) \rangle = \frac{1}{2} \operatorname{Re}[ic\omega\psi(\mathbf{x})\nabla\psi(\mathbf{x})^*] \tag{60}$$

where c is a constant.

The total power flowing across the waveguide aperture in the $\hat{\mathbf{z}}$ direction is made up of an incoming part associated with the incoming parts of ψ and $\partial\psi/\partial z$ and an outgoing part associated with the outgoing parts of ψ and $\partial\psi/\partial z$. The total power exiting (entering) the waveguide aperture is given by

$$P^\alpha = \int_W ds \langle \mathbf{S}^\alpha(\mathbf{x}) \cdot \hat{\mathbf{z}} \rangle = \frac{1}{2}\int_W ds \operatorname{Re}\left[ic\omega\psi^\alpha(\mathbf{x})\frac{\partial\psi^\alpha(\mathbf{x})^*}{\partial z}\right] \tag{61}$$

for $\alpha = \text{out (in)}$. This integral is most conveniently evaluated by decomposing ψ^α and $\partial\psi^\alpha/\partial z$ into their modal components. The reason is that since the modes are orthogonal, the power in the sum over modes is equal to the sum of the powers in each mode.

The amplitude of the nth outgoing (incoming) mode contained in $\psi(\mathbf{x})$ is η_n^{out} (η_n^{in}). Therefore, the time-averaged power exiting (entering) the waveguide aperture is

$$P^\alpha = c\omega k \sum_n^{n_{\max}} \frac{|\eta_n^\alpha|^2}{2Z_n} \tag{62}$$

for $\alpha = \text{out (in)}$, where $n_{\max}$ is the largest value of n for which β_n is real. We exclude modes with imaginary propagation constants since such modes do not transport any power into or out of the guide on average.

The amplitude of the nth outgoing (incoming) mode contained in $\partial\psi/\partial z$ is $\tilde{\eta}_n^{\rm out}$ ($\tilde{\eta}_n^{\rm in}$). Therefore, the time-averaged power exiting (entering) the waveguide aperture is

$$P^{\alpha} = c\frac{\omega}{k}\sum_{n}^{n_{\max}} \frac{Z_n\left|\tilde{\eta}_n^{\alpha}\right|^2}{2} \tag{63}$$

for $\alpha = $ out (in).

1) Acoustic Waves: If ψ is the velocity potential, i.e., $\mathbf{v} = \nabla\psi$, and ρ is the mass density, then the constant c in (60) is given by

$$c = \rho \tag{64}$$

Furthermore, the acoustic impedance [5] is related to our modal impedance by

$$Z_n^{\rm acoustic} = \frac{\omega}{k}\rho Z_n. \tag{65}$$

2) Electromagnetic Waves in Two Dimensions: Suppose a waveguide whose axis is parallel to $\hat{\mathbf{z}}$ is also translationally invariant in the $\hat{\mathbf{y}}$ direction, i.e., the waveguide consists of a pair of half-infinite plates parallel to the yz plane. When a geometry is translationally invariant in one direction, the electromagnetic scattering problem can be decoupled into two independent problems, each of which is isomorphic to a 2-D scalar scattering problem with a different boundary condition. If the 3-D surfaces are perfectly conducting, the boundary conditions for the corresponding scalar fields on the corresponding 2-D surfaces become either Dirichlet or Neumann.

Solutions to the scalar waveguide problem with Dirichlet boundary conditions inside the waveguide correspond to solutions to the electromagnetic waveguide problem with exclusively TE modes inside the waveguide according to

$$\mathbf{E}(\mathbf{x}) = \psi(x)\hat{\mathbf{x}}, \quad \mathbf{H}(x) = \frac{\sigma(x)}{i\omega\mu}\hat{\mathbf{x}}\times\hat{\mathbf{z}} \quad \text{Dirichlet/TE} \tag{66}$$

and solutions to the scalar waveguide problem with Neumann boundary conditions inside the waveguide correspond to solutions to the electromagnetic waveguide problem with exclusively TM modes inside the waveguide according to

$$\mathbf{H}(\mathbf{x}) = \psi(x)\hat{\mathbf{x}}, \quad \mathbf{E}(x) = \frac{\sigma(x)}{i\omega\epsilon}\hat{\mathbf{z}}\times\hat{\mathbf{x}} \quad \text{Neumann/TM}. \tag{67}$$

Note how the correspondence between TM or TE polarization and Dirichlet or Neumann boundary conditions in the waveguide mode case differs from the correspondence between TM or TE polarization and Dirichlet or Neumann boundary conditions in the case of scattering from perfect conductors. On a perfect conductor we associate TM-polarized electromagnetic scattering with solutions to the scalar scattering problem with Dirichlet boundary conditions according to

$$\mathbf{E}(\mathbf{x}) = \psi(x)\hat{\mathbf{y}}, \quad \mathbf{H}(x) = \frac{\sigma(x)}{i\omega\mu}\hat{\mathbf{y}}\times\hat{\mathbf{n}} \quad \text{Dirichlet/TM} \tag{68}$$

and we associate TE-polarized electromagnetic scattering with solutions to the scalar scattering problem with Neumann boundary conditions according to

$$\mathbf{H}(\mathbf{x}) = \psi(x)\hat{\mathbf{y}}, \quad \mathbf{E}(x) = \frac{\sigma(x)}{i\omega\epsilon}\hat{\mathbf{n}}\times\hat{\mathbf{y}} \quad \text{Neumann/TE} \tag{69}$$

where $\hat{\mathbf{y}}$ is the direction of translational invariance and $\hat{\mathbf{n}}$ is the outward surface normal. Therefore, the waveguide-excited electromagnetic scattering problem with TM (TE) polarization in which all the scattering surfaces are perfect conductors, is equivalent to the waveguide-excited scalar problem, in which Neumann (Dirichlet) boundary conditions hold on the inner walls of the waveguide and Dirichlet (Neumann) boundary conditions hold on all the surfaces of all the scatterers.

For electromagnetic waves in two dimensions, the constant c in (60) is given by

$$c = \begin{cases} \frac{1}{\mu\omega^2} & \text{Dirichlet/TE} \\ \frac{1}{\epsilon\omega^2} & \text{Neumann/TM} \end{cases} \tag{70}$$

where μ and ϵ are appropriate to the material inside the guide.

III. Electromagnetic Waveguide Equations

A. Modes

The electric and magnetic fields inside a waveguide with perfectly conducting walls can be decomposed into modal components just as the field and its normal derivative were in the scalar case. The essential difference is that now there are three distinct categories of modal fields, namely TM, TE, and transverse electromagnetic (TEM); each is a vector function rather than scalar function. For our purposes, it is sufficient to consider only the transverse components of the electric and magnetic fields. Assuming the guide is uniformly filled with a nondissipative medium having dielectric constant ϵ and magnetic permeability μ, we may write[4] [6]

$$\mathbf{E}_\perp(\mathbf{x}_\perp, z) = \sum_n (a_n e^{i\beta_n z} + b_n e^{-i\beta_n z})\mathbf{u}_n(\mathbf{x}_\perp) \tag{71}$$

$$\mathbf{H}_\perp(\mathbf{x}_\perp, z) = \sum_n (a_n e^{i\beta_n z} - b_n e^{-i\beta_n z})\frac{1}{Z_n}\hat{\mathbf{z}}\times\mathbf{u}_n(\mathbf{x}_\perp) \tag{72}$$

where the modal impedance Z_n is given by

$$Z_n = \sqrt{\frac{\mu}{\epsilon}} \times \begin{cases} \frac{\beta_n}{k}, & \text{for } n \in \text{TM modes} \\ 1, & \text{for } n \in \text{TEM modes} \\ \frac{k}{\beta_n}, & \text{for } n \in \text{TE modes.} \end{cases} \tag{73}$$

The modes are the eigensolutions to the transverse Helmholtz equation

$$\left(\nabla_\perp^2 + k^2 - \beta_n^2\right)\mathbf{u}_n(\mathbf{x}_\perp) = 0 \tag{74}$$

for $\mathbf{x}_\perp$ inside the waveguide aperture W and $\mathbf{u}_n(\mathbf{x}_\perp)$ constrained by the perfect electrical conductor boundary condition on ∂W. With proper normalization, the modes form a complete

[4] As in the scalar case, cutoff modes are neglected.

[5] $\overleftrightarrow{\delta}(\mathbf{x}-\mathbf{x}')$ is a tensor distribution, which, for any vector-valued surface functions $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$ on W obeys

$$\int_W ds'\,\mathbf{f}(\mathbf{x})\cdot\overleftrightarrow{\delta}(\mathbf{x}-\mathbf{x}')\cdot\mathbf{g}(\mathbf{x}') = \mathbf{f}(\mathbf{x})\cdot\mathbf{g}(\mathbf{x}).$$

and orthonormal set of functions over W, i.e.,

$$\sum_n \mathbf{u}_n(\mathbf{x}_\perp)\mathbf{u}_n(\mathbf{x}'_\perp) = \overleftrightarrow{\delta}(\mathbf{x}_\perp - \mathbf{x}'_\perp)$$
$$\sum_n (\hat{\mathbf{z}} \times \mathbf{u}_n(\mathbf{x}_\perp))(\hat{\mathbf{z}} \times \mathbf{u}_n(\mathbf{x}'_\perp)) = \overleftrightarrow{\delta}(\mathbf{x}_\perp - \mathbf{x}'_\perp) \quad \text{Completeness}^5 \quad (75)$$

and

$$\int_W d\mathbf{x}_\perp \mathbf{u}_m(\mathbf{x}_\perp) \cdot \mathbf{u}_n(\mathbf{x}_\perp) = \delta_{mn} \quad \text{Orthonormality.} \quad (76)$$

B. Computation of Vector Modes from Scalar Functions

The TM and TE modes can be deduced from the solutions to the scalar Helmholtz equation on W with Dirichlet and Neumann boundary conditions, respectively, on ∂W [6]. The TM mode corresponding to the nth scalar waveguide mode $\varphi_n(\mathbf{x}_\perp)$ obeying Dirichlet boundary conditions on ∂W is

$$\mathbf{u}_n(\mathbf{x}_\perp) = \frac{\nabla_\perp \varphi_n(\mathbf{x}_\perp)}{\sqrt{k^2 - \beta_n^2}} \quad (77)$$

and the TE mode corresponding to the nth scalar waveguide mode $\psi_n(\mathbf{x}_\perp)$ obeying Neumann boundary conditions on ∂W is

$$\mathbf{u}_n(\mathbf{x}_\perp) = \frac{\hat{\mathbf{z}} \times \nabla_\perp \psi_n(\mathbf{x}_\perp)}{\sqrt{k^2 - \beta_n^2}}. \quad (78)$$

TEM modes are possible if and only if W is multiply connected, in which case they are related to solutions to the electrostatic potential problem on W. The TEM mode corresponding to the solution $\zeta_n(\mathbf{x})$ to the electrostatic potential problem on W with all except the nth boundary at zero potential is given by

$$\mathbf{u}_n(\mathbf{x}_\perp) \propto \nabla_\perp \zeta_n(\mathbf{x}_\perp). \quad (79)$$

The scale factor should be chosen to enforce orthonormality for the TEM modes. This amounts to assigning a particular value to the otherwise arbitrary potential on the nth boundary. For all TEM modes, $\beta_n = k$.

C. Waveguide Integral Equation

Let $\mathbf{E}_\perp^{\text{out}}(\mathbf{x}_\perp, z)$ be the transverse component of electric field for a specified outgoing wave. Using the modal expansions and the first completeness relation for the modes, we can write the following expression for $\mathbf{E}_\perp^{\text{out}}(\mathbf{x}_\perp, 0)$ in terms of the transverse components of the electric and magnetic fields on W:

$$\begin{aligned}\mathbf{E}_\perp^{\text{out}}(\mathbf{x}_\perp, 0) &= \sum_n a_n \mathbf{u}_n(\mathbf{x}_\perp) \\ &= \frac{1}{2}\sum_n (a_n + b_n)\mathbf{u}_n(\mathbf{x}_\perp) \\ &\quad + \frac{1}{2}\sum_n Z_n(a_n - b_n)\frac{1}{Z_n}\mathbf{u}_n(\mathbf{x}_\perp) \\ &= \frac{1}{2}\mathbf{E}_\perp(\mathbf{x}_\perp, 0) - \frac{1}{2}\int_W d\mathbf{x}'_\perp \overleftrightarrow{\mathbf{H}}(\mathbf{x}_\perp, \mathbf{x}'_\perp) \\ &\quad \cdot (\hat{\mathbf{z}} \times \mathbf{H}_\perp(\mathbf{x}'_\perp, 0))\end{aligned} \quad (80)$$

where the dyad

$$\overleftrightarrow{\mathbf{H}}(\mathbf{x}_\perp, \mathbf{x}'_\perp) = \sum_n Z_n \mathbf{u}_n(\mathbf{x}_\perp)\mathbf{u}_n(\mathbf{x}'_\perp) \quad (81)$$

is the analogue of the scalar function $H(\mathbf{x}_\perp, \mathbf{x}'_\perp)$. Dropping the spatial coordinate z, we get the following expression for the waveguide integral equation on W, which relates the transverse components of the electric field, the magnetic field, and the specified electric field waveguide excitation on W:

$$\begin{aligned}2\mathbf{E}_\perp^{\text{out}}(\mathbf{x}_\perp) = \mathbf{E}_\perp(\mathbf{x}_\perp) - \int_W d\mathbf{x}'_\perp \overleftrightarrow{\mathbf{H}}(\mathbf{x}_\perp, \mathbf{x}'_\perp) \\ \cdot (\hat{\mathbf{z}} \times \mathbf{H}_\perp(\mathbf{x}'_\perp)).\end{aligned} \quad (82)$$

Defining equivalent electric and magnetic currents on W by

$$\mathbf{J}(\mathbf{x}_\perp) = \hat{\mathbf{z}} \times \mathbf{H}_\perp(\mathbf{x}_\perp) \quad (83)$$
$$\mathbf{M}(\mathbf{x}_\perp) = -\hat{\mathbf{z}} \times \mathbf{E}_\perp(\mathbf{x}_\perp) \quad (84)$$

allows us to write the waveguide integral equation in terms of equivalent currents as

$$2\mathbf{E}_\perp^{\text{out}}(\mathbf{x}_\perp) = \hat{\mathbf{z}} \times \mathbf{M}(\mathbf{x}_\perp) - \int_W d\mathbf{x}'_\perp \overleftrightarrow{\mathbf{H}}(\mathbf{x}_\perp, \mathbf{x}'_\perp) \cdot \mathbf{J}(\mathbf{x}'_\perp). \quad (85)$$

If $\mathbf{H}_\perp^{\text{out}}(\mathbf{x}_\perp, z)$ is specified instead of $\mathbf{E}_\perp^{\text{out}}(\mathbf{x}_\perp, z)$, we may write

$$\begin{aligned}\mathbf{H}_\perp^{\text{out}}(\mathbf{x}_\perp, 0) &= \sum_n a_n \frac{1}{Z_n}\hat{\mathbf{z}} \times \mathbf{u}_n(\mathbf{x}_\perp) \\ &= \frac{1}{2}\sum_n (a_n + b_n)\frac{1}{Z_n}\hat{\mathbf{z}} \times \mathbf{u}_n(\mathbf{x}_\perp) \\ &\quad + \frac{1}{2}\sum_n \frac{1}{Z_n}(a_n - b_n)\hat{\mathbf{z}} \times \mathbf{u}_n(\mathbf{x}_\perp) \\ &= \frac{1}{2}\mathbf{H}_\perp(\mathbf{x}_\perp, 0) + \frac{1}{2}\int_W d\mathbf{x}'_\perp \overleftrightarrow{\tilde{\mathbf{H}}}(\mathbf{x}_\perp, \mathbf{x}'_\perp) \\ &\quad \cdot (\hat{\mathbf{z}} \times \mathbf{E}_\perp(\mathbf{x}'_\perp, 0))\end{aligned} \quad (86)$$

where the dyad

$$\overleftrightarrow{\tilde{\mathbf{H}}}(\mathbf{x}_\perp, \mathbf{x}'_\perp) = \sum_n \frac{1}{Z_n}(\hat{\mathbf{z}} \times \mathbf{u}_n(\mathbf{x}_\perp))(\hat{\mathbf{z}} \times \mathbf{u}_n(\mathbf{x}'_\perp)) \quad (87)$$

is the analogue of the scalar distribution $\tilde{H}(\mathbf{x}_\perp, \mathbf{x}'_\perp)$. Dropping the spatial coordinate z, we get an alternative form of the waveguide integral equation

$$\begin{aligned}2\mathbf{H}_\perp^{\text{out}}(\mathbf{x}_\perp) = \mathbf{H}_\perp(\mathbf{x}_\perp) + \int_W d\mathbf{x}'_\perp \overleftrightarrow{\tilde{\mathbf{H}}}(\mathbf{x}_\perp, \mathbf{x}'_\perp) \\ \cdot (\hat{\mathbf{z}} \times \mathbf{E}_\perp(\mathbf{x}'_\perp))\end{aligned} \quad (88)$$

or in terms of equivalent currents

$$\begin{aligned}2\mathbf{H}_\perp^{\text{out}}(\mathbf{x}_\perp) = -\hat{\mathbf{z}} \times \mathbf{J}_\perp(\mathbf{x}_\perp) - \int_W d\mathbf{x}'_\perp \overleftrightarrow{\tilde{\mathbf{H}}}(\mathbf{x}_\perp, \mathbf{x}'_\perp) \\ \cdot \mathbf{M}_\perp(\mathbf{x}'_\perp).\end{aligned} \quad (89)$$

Equations (85) and (89) are the electromagnetic counterparts of the scalar waveguide integral equations given in (12) and (18).

D. Discretization

As stated above, the TM and TE vector modes on W are derivable from the scalar modes on W with Dirichlet and Neumann boundary conditions on ∂W, respectively, and the TEM vector modes (if any) are derivable from the solutions to the electrostatic potential problem on W. One can compute approximate solutions for the scalar modes and the electrostatic potential by putting scalar basis functions on W and following the procedure given in the Appendix. Once this has been accomplished, one has to choose between keeping the representation of the modes in terms of the scalar discretization or converting it to an equivalent vector discretization. If the scalar discretization is kept on the aperture, specialized code must be written to handle interactions with the waveguide aperture. On the other hand, if the waveguide modes are converted to a vector discretization early on, then the interactions between the various scattering surfaces, whether physical or waveguide aperture, can be handled in a consistent fashion, i.e., entirely in terms of vector basis functions. For computations involving more than just the waveguide alone, we find the later choice to be the simplest and cleanest to implement.

If we discretize the electric current $\mathbf{J}(\mathbf{x}_\perp)$ and magnetic current $\mathbf{M}(\mathbf{x}_\perp)$ on W in terms of M vector basis functions $\mathbf{f}_m(\mathbf{x}_\perp)$ using

$$\mathbf{J}(\mathbf{x}_\perp) \approx \sum_{m=1}^{M} I_m^W \mathbf{f}_m(\mathbf{x}_\perp) \tag{90}$$

$$\mathbf{M}(\mathbf{x}_\perp) \approx \sum_{m=1}^{M} S_m^W (\mathbf{f}_m(\mathbf{x}_\perp) \times \hat{\mathbf{z}}) \tag{91}$$

we may write the first waveguide integral equation (85) in its discretized form as

$$2V^W = N^W S^W - X^W I^W \tag{92}$$

where

$$V_i^W = \int_W d\mathbf{x}_\perp \mathbf{E}_\perp^{\text{out}}(\mathbf{x}_\perp) \cdot \mathbf{f}_i(\mathbf{x}_\perp) \tag{93a}$$

$$N_{ij}^W = \int_W d\mathbf{x}_\perp \mathbf{f}_i(\mathbf{x}_\perp) \cdot \mathbf{f}_j(\mathbf{x}_\perp) \tag{93b}$$

$$\begin{aligned} X_{ij}^W &= \int_W d\mathbf{x}_\perp \int_W d\mathbf{x}'_\perp \mathbf{f}_i(\mathbf{x}_\perp) \cdot \overleftrightarrow{\mathbf{H}}(\mathbf{x}_\perp, \mathbf{x}'_\perp) \cdot \mathbf{f}_j(\mathbf{x}'_\perp) \\ &= [(BN^W)^T \Lambda (BN^W)]_{ij} \end{aligned} \tag{93c}$$

and

$$\mathbf{u}_m(\mathbf{x}) = \sum_n B_{mn} \mathbf{f}_n(\mathbf{x}) \tag{94}$$

$$\Lambda_{mn} = Z_n \delta_{mn}. \tag{95}$$

We get the elements of B_{mn} by computing inner products of the vector basis functions with gradients of the scalar basis functions. For example, if $\mathbf{u}_m$ corresponds to a TM mode, it is clear from (32), (77), and (94) and the definition of N^W that the entries in the mth row of B_{mn} are given by

$$\begin{aligned} B_{mn} = \frac{1}{\sqrt{k^2 - \beta_m^2}} \sum_{jk} A_{mj} \int_W d\mathbf{x}_\perp \nabla_\perp f_j(\mathbf{x}_\perp) \\ \cdot \mathbf{f}_k(\mathbf{x}_\perp)((N^W)^{-1})_{kn}. \end{aligned} \tag{96}$$

Similarly, the discretized form of the second waveguide integral equation (89) becomes

$$2\tilde{V}^W = N^W I^W - \tilde{X}^W S^W \tag{97}$$

where

$$\tilde{V}_i^W = \int_W d\mathbf{x}_\perp \mathbf{H}_\perp^{\text{out}}(\mathbf{x}_\perp) \cdot (\mathbf{f}_i(\mathbf{x}_\perp) \times \hat{\mathbf{z}}) \tag{98a}$$

$$\begin{aligned} \tilde{X}_{ij}^W &= \int_W d\mathbf{x}_\perp \int_W d\mathbf{x}'_\perp (\mathbf{f}_i(\mathbf{x}_\perp) \times \hat{\mathbf{z}}) \cdot \overleftrightarrow{\tilde{\mathbf{H}}}(\mathbf{x}_\perp, \mathbf{x}'_\perp) \\ &\quad \cdot (\mathbf{f}_j(\mathbf{x}'_\perp) \times \hat{\mathbf{z}}) \\ &= [(BN^W)^T \tilde{\Lambda} (BN^W)]_{ij} \end{aligned} \tag{98b}$$

and

$$\tilde{\Lambda}_{mn} = \frac{1}{Z_n} \delta_{mn} = (\Lambda^{-1})_{mn}. \tag{99}$$

E. Coupled Integral Equations in the Perfect Conductor Case

Suppose the waveguide W is the primary source of radiation for a general antenna problem in which all other scattering surfaces S may be treated as perfect conductors. If there are no other sources, the electric field integral equation (EFIE) for $\mathbf{x}$ on $S \oplus W$ is [7]

$$\begin{aligned} 0 = -\frac{1}{2}\hat{\mathbf{n}}(\mathbf{x}) \times \mathbf{M}(\mathbf{x}) + \oint_{S \ominus W} ds' \Bigg[i\omega\mu \left(\overleftrightarrow{\mathbf{1}} + \frac{1}{k^2} \nabla' \nabla' \right) \\ \times G(\mathbf{x}, \mathbf{x}') \cdot \mathbf{J}(\mathbf{x}') + \nabla' G(\mathbf{x}, \mathbf{x}') \times \mathbf{M}(\mathbf{x}') \Bigg]_{\tan}. \end{aligned} \tag{100}$$

The tangential component of the electric field vanishes on a perfect conductor; hence, $\mathbf{M} = 0$ on S. At this point, we could rewrite the above equation in the separate forms appropriate to $\mathbf{x}$ on S and $\mathbf{x}$ on W and eliminate $\mathbf{M}$ on W by means of (85), thereby obtaining a set of coupled integral equations for the fields on S and W, just as we did in the scalar case. Then we could convert them to discretized form. Alternatively, we could discretize (100) as it stands, eliminate the unknown equivalent magnetic current amplitudes on W using (92) and achieve the discretized form directly. For brevity, we follow the latter approach.

A discretized version of (100) in block matrix form is

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} Z^{SS} & Z^{SW} & Y^{SW} \\ Z^{WS} & Z^{WW} & -\frac{1}{2}N^W \end{bmatrix} \begin{bmatrix} I^S \\ I^W \\ S^W \end{bmatrix} \tag{101}$$

where

$$\begin{aligned} Z_{ij}^{\alpha\beta} = i\omega\mu \int_\alpha ds \int_\beta ds' \mathbf{f}_i^\alpha(\mathbf{x}) \cdot \left(\overleftrightarrow{\mathbf{1}} + \frac{1}{k^2} \nabla' \nabla' \right) G(\mathbf{x}, \mathbf{x}') \\ \cdot \mathbf{f}_j^\beta(\mathbf{x}') \end{aligned} \tag{102}$$

$$Y_{ij}^{\alpha\beta} = \int_\alpha ds \int_\beta ds' \mathbf{f}_i^\alpha(\mathbf{x}) \cdot \left(\nabla' G(\mathbf{x}, \mathbf{x}') \times \left(\mathbf{f}_j^\beta(\mathbf{x}') \times \hat{\mathbf{n}}' \right) \right) \tag{103}$$

with S or W replacing α and β and I^S representing the block of unknown current amplitudes on S, which is related to the electric current $\mathbf{J}$ on S by

$$\mathbf{J}(\mathbf{x}) \approx \sum_m I_m^S \mathbf{f}_m(\mathbf{x}). \tag{104}$$

Rewriting (92) as

$$S^W = 2(N^W)^{-1} V^W + (N^W)^{-1} X^W I^W \tag{105}$$

we can eliminate the block of unknowns S^W in favor of I^W to obtain the discretized version of (100) in its simplest block form

$$\begin{bmatrix} -2Y^{SW}(N^W)^{-1}V^W \\ V^W \end{bmatrix} = \begin{bmatrix} Z^{SS} & Z^{SW}+Y^{SW}(N^W)^{-1} \\ Z^{WS} & Z^{WW}-\frac{1}{2}X^W \end{bmatrix} \begin{bmatrix} I^S \\ I^W \end{bmatrix}. \tag{106}$$

F. Modal Decomposition

By employing the first completeness relation for the modes, we can decompose the transverse part of the electric field into a sum over modes as

$$\mathbf{E}_\perp(\mathbf{x}) = \sum_n \eta_n \mathbf{u}_n(\mathbf{x}) \tag{107}$$

where

$$\eta_n = \int_W ds \mathbf{u}_n(\mathbf{x}) \cdot \mathbf{E}_\perp(\mathbf{x}) \tag{108}$$

is the amplitude of the nth mode contained in $\mathbf{E}_\perp(\mathbf{x})$. It is useful to further decompose $\mathbf{E}_\perp(\mathbf{x})$ into its incoming and outgoing components

$$\mathbf{E}_\perp(\mathbf{x}) = \mathbf{E}_\perp^{\text{in}}(\mathbf{x}) + \mathbf{E}_\perp^{\text{out}}(\mathbf{x}). \tag{109}$$

Since the discretization of $\mathbf{E}_\perp^{\text{out}}(\mathbf{x})$ is given by V^W, we may write the discretized form of η_n^{out} as

$$\eta_n^{\text{out}} = \sum_m A_{nm} V_m^W. \tag{110}$$

Using (85) to eliminate $\mathbf{E}_\perp(\mathbf{x})$, we arrive at the discretized form of η_n^{in}

$$\eta_n^{\text{in}} = \sum_m A_{nm}(V^W + X^W I^W)_m. \tag{111}$$

Similarly, by employing the second completeness relation for the modes, we may decompose the transverse part of the magnetic field as

$$\mathbf{H}_\perp(\mathbf{x}) = \sum_n \tilde{\eta}_n(\hat{\mathbf{z}} \times \mathbf{u}_n(\mathbf{x})) \tag{112}$$

where

$$\tilde{\eta}_n = \int_W ds(\hat{\mathbf{z}} \times \mathbf{u}_n(\mathbf{x})) \cdot \mathbf{H}_\perp(\mathbf{x}). \tag{113}$$

Then, using

$$\mathbf{H}_\perp(\mathbf{x}) = \mathbf{H}_\perp^{\text{in}}(\mathbf{x}) + \mathbf{H}_\perp^{\text{out}}(\mathbf{x}) \tag{114}$$

and (89), we can write $\tilde{\eta}_n^{\text{out}}$ and $\tilde{\eta}_n^{\text{in}}$ in discretized form as

$$\tilde{\eta}_n^{\text{out}} = -\sum_m A_{nm} \tilde{V}_m^W \tag{115}$$

and

$$\eta_n^{\text{in}} = -\sum_m A_{nm}(\tilde{V}^W + \tilde{X}^W S^W)_m. \tag{116}$$

G. Power

The time-averaged power-flow-density vector (Poynting vector) is [6]

$$\langle \mathbf{S}(\mathbf{x}) \rangle = \frac{1}{2} \operatorname{Re}[\mathbf{E}_\perp(\mathbf{x}) \times \mathbf{H}_\perp(\mathbf{x})^*]. \tag{117}$$

The total power flowing across the waveguide aperture in the $\hat{\mathbf{z}}$ direction is made up of an incoming part associated with the incoming parts of $\mathbf{E}_\perp$ and $\mathbf{H}_\perp$ and an outgoing part associated with the outgoing parts of $\mathbf{E}_\perp$ and $\mathbf{H}_\perp$. The total power exiting (entering) the waveguide aperture is given by

$$\begin{aligned} P^\alpha &= \int_W ds \langle \mathbf{S}^\alpha(\mathbf{x}) \cdot \hat{\mathbf{z}} \rangle \\ &= \frac{1}{2} \int_W ds \operatorname{Re}[\mathbf{E}_\perp^\alpha(\mathbf{x}) \times \mathbf{H}_\perp^\alpha(\mathbf{x})^*] \end{aligned} \tag{118}$$

for $\alpha = \text{out (in)}$. This integral is most conveniently evaluated by decomposing $\mathbf{E}_\perp^\alpha$ and $\mathbf{H}_\perp^\alpha$ into their modal components, since the modes are orthogonal and the power in the sum over modes is equal to the sum of the powers in each mode.

The amplitude of the nth outgoing (incoming) mode contained in $\mathbf{E}_\perp(\mathbf{x})$ is $\eta_n^{\text{out}}(\eta_n^{\text{in}})$. Therefore, the time-averaged power exiting (entering) the waveguide aperture is

$$P^\alpha = \sum_n^{n_{\max}} \frac{|\eta_n^\alpha|^2}{2Z_n} \tag{119}$$

for $\alpha = \text{out (in)}$ where $n_{\max}$ is the largest value of n for which β_n is real. We exclude modes with imaginary propagation constants since such modes do not transport any power into or out of the guide on average.

The amplitude of the nth outgoing (incoming) mode contained in $\mathbf{H}_\perp$ is $\tilde{\eta}_n^{\text{out}}$ $(\tilde{\eta}_n^{\text{in}})$. Therefore, the time-averaged power exiting (entering) the waveguide aperture is

$$P^\alpha = \sum_n^{n_{\max}} \frac{Z_n |\tilde{\eta}_n^\alpha|^2}{2} \tag{120}$$

for $\alpha = \text{out (in)}$.

IV. Extensions

Up to this point, we have assumed that all energy coupled into incoming traveling modes is completely absorbed. It is possible (at the cost of some extra complication) to relax this assumption, as we now demonstrate for scalar scattering.

Suppose a uniform waveguide is terminated after length L by a wall (oriented perpendicular to the axis of the guide) whose reflectivity for the mth waveguide mode is r_m. For the time being, assume no independent sources are located inside the guide. Every mode that enters with amplitude b_n, exits with amplitude $a_n = r_n e^{i\beta_n 2L} b_n$, i.e., if $\psi^{\text{in}}(\mathbf{x}) = \sum_n b_n u_n(\mathbf{x})$ comes in, then $\psi^{\text{out}}(\mathbf{x}) = \sum_n r_n e^{i\beta_n 2L} b_n u_n(\mathbf{x})$ goes out. This expression for $\psi^{\text{out}}(\mathbf{x})$ can be rewritten as

$$\psi^{\text{out}}(\mathbf{x}) = \int_W ds' R(\mathbf{x}, \mathbf{x}') \psi^{\text{in}}(\mathbf{x}') \tag{121}$$

where

$$R(\mathbf{x}, \mathbf{x}') = \sum_n r_n e^{i\beta_n 2L} u_n(\mathbf{x}) u_n(\mathbf{x}'). \tag{122}$$

After discretization, (121) becomes

$$N^W S^{\rm out} = (AN^W)^T R (AN^W) S^{\rm in} \tag{123}$$

where R is a diagonal reflectivity matrix whose elements are

$$R_{ij} = r_i e^{i\beta_i 2L} \delta_{ij}. \tag{124}$$

A boundary condition relating ψ and σ on W can be obtained by applying the operator $\int_W ds'(\delta(\mathbf{x},\mathbf{x}') + R(\mathbf{x},\mathbf{x}'))\cdot$ to both sides of (12) and using (52). The result is

$$\int_W ds'(\delta(\mathbf{x}-\mathbf{x}') + R(\mathbf{x},\mathbf{x}')) \int_W ds'' H(\mathbf{x}',\mathbf{x}'')\sigma(\mathbf{x}'') = \int_W ds'(\delta(\mathbf{x},\mathbf{x}') - R(\mathbf{x},\mathbf{x}'))\psi(\mathbf{x}') \tag{125}$$

or in discretized form

$$(N^W + R)(N^W)^{-1} X^W I^W = (N^W - R) S^W. \tag{126}$$

The discretized relation takes a particularly simple and appealing form if: 1) the basis functions used on W are orthonormal in which case $N^W = 1$ and 2) if as many modes are computed as there are basis functions on W in which case $A^T A = 1$. Then (126) is equivalent to

$$TX^W I^W = S^W \tag{127}$$

where

$$T_{ij} = [A^T t A]_{ij} \tag{128}$$

and

$$t_{mn} = \frac{1 + r_m}{1 - r_m} \delta_{mn} \tag{129}$$

is the diagonal transmission matrix giving the amplitude transmission of each mode at the waveguide aperture.

It is easy to modify these relations to allow for a specified outgoing wave. Suppose the field $\psi^{\rm spec}(\mathbf{x})$ is specified as being emitted from the aperture in addition to the reflected wave, i.e., $\psi^{\rm out}(\mathbf{x}) = \psi^{\rm spec}(\mathbf{x}) + \psi^{\rm refl}(\mathbf{x})$. We use $\psi^{\rm refl}(\mathbf{x})$ here to refer to the quantity on the left side of (121). The result is

$$\int_W ds'(\delta(\mathbf{x}-\mathbf{x}') + R(\mathbf{x},\mathbf{x}')) \int_W ds'' H(\mathbf{x}',\mathbf{x}'')\sigma(\mathbf{x}'') = \int_W ds'(\delta(\mathbf{x},\mathbf{x}') - R(\mathbf{x},\mathbf{x}'))\psi(\mathbf{x}') - 2\psi^{\rm spec}(\mathbf{x}). \tag{130}$$

Its discretized form

$$2V^{\rm spec} = (N^W - R)S^W - (N^W + R)(N^W)^{-1} X^W I^W \tag{131}$$

is the obvious analog to (36) and reduces to it for $R \to 0$.

Even more generally, one can imagine the situation in which each incoming mode can be scattered into one or more outgoing modes. Any number of practical effects (such as nonuniformities in the cross section or imperfect termination) could cause this to happen. In such a case, the reflectivity matrix R contains the amplitude for every mode to scatter into every other mode and is no longer diagonal.

Analogous results obtain for the alternative form of the scalar waveguide boundary condition and for the vector cases.

V. Summary

As the previous discussion illustrates, the equations that describe scattering interactions with waveguides can be put into simple forms that are common to scalar scattering and vector scattering. For example, the boundary condition on a waveguide aperture may be written in both cases as

$$2V^W = N^W S^W - X^W I^W \tag{132}$$

or

$$2\tilde{V}^W = N^W I^W - \tilde{X}^W S^W. \tag{133}$$

In the scalar case, the unknown amplitudes I^W and S^W are related to the field ψ and its longitudinal derivative σ according to (33) and (34); the matrices N^W, X^W, and $\tilde{X}^W$ and the vectors V^W and $\tilde{V}^W$ are given by (37) and (40). In the vector case, the unknown amplitudes I^W and S^W are related to the equivalent electric and magnetic currents $\mathbf{J}$ and $\mathbf{M}$, according to (90) and (91); the matrices N^W, X^W, and $\tilde{X}^W$ and the vectors V^W and $\tilde{V}^W$ are given by (93) and (98). The discretized equations for scalar scattering when W obeys the waveguide boundary condition and S obeys Dirichlet boundary conditions [see (43)] are also identical to the equations for vector scattering when W obeys the waveguide boundary condition and S is perfectly conducting [see (106)]. The commonality extends to the expressions for power transport into and out of the waveguide as well.

Appendix

Construction of the X and $\tilde{X}$ matrices that appear in the discretized expressions for the waveguide boundary condition requires an approximate representation of the eigenmodes in terms of basis functions on patches covering the waveguide aperture as well as the eigenvalues associated with these eigenmodes. For a few geometries such as rectangular waveguide and coaxial waveguide, complete analytical solutions for the eigenmodes are known. In such cases, it is a simple matter to calculate the projection of a given eigenmode onto the set of basis functions. In the general case, an eigenvalue equation must be constructed for computing the modes.

In this Appendix we describe a means for computing the modes of cylindrical waveguides of arbitrary cross section. There are three subsections. The first and second subsections describe methods for numerically solving the scalar Helmholtz equation for the waveguide modes when the waveguide walls obey either Dirichlet or Neumann boundary conditions, respectively. The third subsection describes a method for numerically solving the scalar Laplace equation for the electrostatic potential of a multiply-connected cylindrical waveguide, all but one of whose surfaces is held at zero potential.

The Helmholtz modes are directly applicable to scalar problems such as acoustic radiation and scattering. The Helmholtz and Laplace modes are applicable to electromagnetic radiation and scattering problems in that the TM and TE modes can be deduced from the scalar Helmholtz modes with Dirichlet and Neumann boundary conditions, respectively, and the TEM modes are derivable from the scalar Laplace modes. The correspondence is described further in Section III-B of the main text.

We will assume the availability of scalar basis functions that are continuous across patch boundaries. A simple example of such a basis function is a function that spans two triangular patches sharing a common edge and whose value goes linearly from unity on the common edge to zero at the opposing vertices. The extension of continuous scalar basis functions to higher order polynomials in the surface parameterization results in three types of basis functions that may be classified according to whether they span two patches that share a common edge, span multiple patches that share a common vertex, or have single patch support. Basis functions of the first variety go to zero at the opposing vertices and are nonzero on the common edge; basis functions of the second variety go to zero on all edges not touching the central vertex (where they are nonzero); basis functions of the third variety are zero on the boundary of a patch and nonzero in its interior.

A. Scalar Helmholtz Modes

1) Dirichlet Boundary Conditions on ∂W: Operating on both sides of (5) by $\int_W d\mathbf{x}_\perp f_m(\mathbf{x}_\perp)$ turns it into an integral equation, which may be written as

$$-\int_W d\mathbf{x}_\perp f_m(\mathbf{x}_\perp)(\nabla_\perp \cdot \nabla_\perp u_n(\mathbf{x}_\perp)) = \left(k^2-\beta_n^2\right)\int_W d\mathbf{x}_\perp f_m(\mathbf{x}_\perp)u_n(\mathbf{x}_\perp). \quad (134)$$

Integrating the left-hand side by parts and applying Gauss' theorem to convert one of the resulting surface integrals into a boundary integral, we get

$$\int_W d\mathbf{x}_\perp \nabla_\perp f_m(\mathbf{x}_\perp)\cdot\nabla_\perp u_n(\mathbf{x}_\perp) - \oint_{\partial W} dl f_m(\mathbf{x}_\perp)(\hat{\mathbf{e}}_\perp(\mathbf{x}_\perp)\cdot\nabla_\perp u_n(\mathbf{x}_\perp)) = \left(k^2-\beta_n^2\right)\int_W d\mathbf{x}_\perp f_m(\mathbf{x}_\perp)u_n(\mathbf{x}_\perp) \quad (135)$$

where $\hat{\mathbf{e}}_\perp(\mathbf{x}_\perp)$ is the unit edge normal to ∂W at $\mathbf{x}_\perp$. The unit edge normal is in the plane of W and points into the waveguide wall.

The Dirichlet boundary condition demands that $u_n(\mathbf{x}_\perp \in \partial W) = 0$. If we expand the modes u_n in a set of basis functions f_m that are continuous and vanish on the boundary of W, i.e.,

$$u_n(\mathbf{x}_\perp) = \sum_m A_{nm} f_m(\mathbf{x}_\perp) \quad (136)$$

then the boundary integral term vanishes and (135) becomes a generalized eigenvalue equation for the mode coefficients

$$\sum_{m'} M_{mm'}A_{nm'} = \left(k^2-\beta_n^2\right)\sum_{m'} N_{mm'}A_{nm'} \quad (137)$$

where

$$N_{mm'} \equiv \int_W d\mathbf{x}_\perp f_m(\mathbf{x}_\perp) f_{m'}(\mathbf{x}_\perp) \quad (138)$$

$$M_{mm'} \equiv \int_W dx_\perp \nabla_\perp f_m(\mathbf{x}_\perp)\cdot\nabla_\perp f_{m'}(\mathbf{x}_\perp). \quad (139)$$

2) Neumann Boundary Conditions on ∂W: The Neumann boundary condition demands that $(\hat{\mathbf{e}}_\perp \cdot \nabla_\perp)u_n(\mathbf{x}_\perp \in \partial W) = 0$. If we had basis functions whose values were nonzero on the boundary but whose edge derivatives vanished on the boundary, we could construct the modes directly from them, just as we did in the Dirichlet case. Since we do not, we need to augment our usual set of basis functions on the interior of W with extra basis functions associated with the boundary of W. Edge-based basis functions supported on the patch pairs (one each from S and W) that share a common edge on ∂W comprise this set.

The generalized eigenvalue equation again derives from (135) and (136). In this case, however, the unknown coefficients A_{nm} also need to obey the added constraint that the edge derivative of each eigenmode must vanish on the boundary. We may write this constraint in integral form as

$$\oint_{\partial W} dl\hat{\mathbf{e}}_\perp(\mathbf{x}_\perp)\cdot\nabla_\perp u_n(\mathbf{x}_\perp) = 0 \quad (140)$$

which, after substituting the discretized approximation for u_n, becomes

$$\sum_m C_m A_{nm} = 0 \quad (141)$$

where

$$C_m \equiv \int_{\partial W} dl\hat{\mathbf{e}}_\perp(\mathbf{x}_\perp)\cdot\nabla_\perp f_m(\mathbf{x}_\perp). \quad (142)$$

Thus, we seek solutions to the eigenvalue equation

$$\sum_{m'}(M_{mm'}-L_{mm'})A_{nm'} = \left(k^2-\beta_n^2\right)\sum_{m'} N_{mm'}A_{nm'} \quad (143)$$

where

$$L_{mm'} \equiv \oint_{\partial W} dl f_m(\mathbf{x}_\perp)(\hat{\mathbf{e}}_\perp(\mathbf{x}_\perp)\cdot\nabla_\perp f_{m'}(\mathbf{x}_\perp)) \quad (144)$$

and the matrices M and N are defined as in the Dirichlet case, subject to the constraint given by (141).

We can subsume the constraint information directly into the eigenvalue equation by use of the projection operator P defined by

$$P \equiv 1 - C^T(CC^T)^{-1}C \quad (145)$$

where C is given above and 1 represents the identity matrix of the proper dimensionality. P has the property that it reproduces vectors x that obey $Cx = 0$ and it annihilates vectors that do not. P also has the property that the vectors x that simultaneously obey the eigenvalue equation $Qx = \lambda x$ and the constraint equation $Cx = 0$, are the same vectors that obey the eigenvalue equation

$$PQPx = \lambda x. \quad (146)$$

Applying this to (143), we obtain the following the generalized eigenvalue equation for Neumann boundary conditions:

$$\sum_{m'}\left[PN^{-1}(M-L)P\right]_{mm'}A_{nm'} = \left(k^2-\beta_n^2\right)A_{nm}. \quad (147)$$

Rows of A (i.e., eigenvectors) corresponding to eigenmodes that do not obey the constraint will vanish (to numerical precision) when left multiplied by P. All such eigenmodes and eigenvectors should be discarded.

B. Scalar Laplace Modes

We seek solutions $u_n(\mathbf{x}_\perp)$ that obey the Laplace equation

$$\nabla_\perp^2 u_n(\mathbf{x}_\perp) = 0 \tag{148}$$

inside W and vanish on all boundaries of W except one (call it ∂W_n), where we may arbitrarily set it to unity. Since our basis functions vanish on the boundary, we need to construct a special function $\psi_n(\mathbf{x}_\perp)$ that is continuous and evaluates to unity on ∂W_n. For example, given triangular patches parameterized by the three (nonindependent) triangle coordinates u_1, u_2, and u_3, we could take $\psi_n = 0$ on all patches that are not in contact with the boundary, $\psi_n = u_i$ on all patches that have the vertex $u_i = 1$ on the boundary, and $\psi_n = 1 - u_i$ on all patches that have edge $u_i = 0$ on the boundary. Then we want to approximately solve

$$\nabla_\perp^2 \left(\psi_n(\mathbf{x}_\perp) + \sum_{m'} A_{nm'} f_{m'}(\mathbf{x}_\perp) \right) = 0. \tag{149}$$

Applying the operator $\int_W d\mathbf{x}_\perp f_m(\mathbf{x}_\perp)$ to both sides and integrating the resulting equation by parts produces the following linear equation for the basis function coefficients $A_{nm'}$ for the potential function associated with the nth boundary:

$$\sum_{m'} M_{mm'} A_{nm'} = \int_W d\mathbf{x}_\perp \nabla_\perp f_m(x_\perp) \cdot \nabla_\perp \psi_n(x_\perp) \tag{150}$$

where M is as defined in (139).

To make normalized TEM modes out of these Laplace modes, we need them to obey

$$\begin{aligned} 1 &= \int_W d\mathbf{x}_\perp \mathbf{u}_n(x_\perp) \cdot \mathbf{u}_n(x_\perp) \\ &= \int_W d\mathbf{x}_\perp \nabla_\perp u_n(x_\perp) \cdot \nabla_\perp u_n(x_\perp) \\ &= \int_W d\mathbf{x}_\perp \nabla_\perp \cdot (u_n(x_\perp) \nabla_\perp u_n(x_\perp)) \\ &\quad - \int_W d\mathbf{x}_\perp \nabla_\perp^2 u_n(x_\perp) \\ &= \oint_{\partial W} dl u_n(\mathbf{x}_\perp)(\hat{\mathbf{e}}_\perp(\mathbf{x}_\perp) \cdot \nabla_\perp u_n(\mathbf{x}_\perp)) \\ &= \int_{\partial W_n} dl (\hat{\mathbf{e}}_\perp(\mathbf{x}_\perp) \cdot \nabla_\perp u_n(\mathbf{x}_\perp)) \end{aligned} \tag{151}$$

which means the coefficients of the discretized representation of u_n must be scaled to make

$$\begin{aligned} 1 &= \int_{\partial W_n} dl \hat{\mathbf{e}}_\perp(\mathbf{x}_\perp) \cdot \nabla_\perp \left(\sum_{m'} A_{nm'} f_{m'}(\mathbf{x}_\perp) \right) \\ &= \sum_{m'} A_{nm'} \int_{\partial W_n} dl \hat{\mathbf{e}}_\perp(\mathbf{x}_\perp) \cdot \nabla_\perp f_{m'}(\mathbf{x}_\perp). \end{aligned} \tag{152}$$

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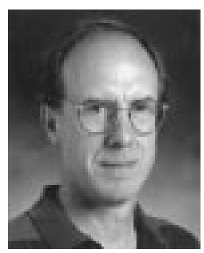

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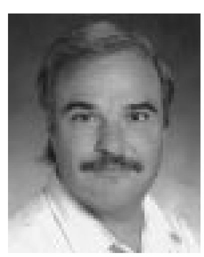

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