Solution of Combined-Field Integral Equation Using Multilevel Fast Multipole Algorithm for Scattering by Homogeneous Bodies

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***Abstract*— In this paper, we present an accurate method of moments (MoM) solution of the combined field integral equation (CFIE) using the multilevel fast multipole algorithm (MLFMA) for scattering by large, three-dimensional (3-D), arbitrarily shaped, homogeneous objects. We first investigate several different MoM formulations of CFIE and propose a new formulation, which is both accurate and free of interior resonances. We then employ MLFMA to significantly reduce the memory requirement and computational complexity of the MoM solution. Numerical results are presented to demonstrate the accuracy and capability of the proposed method. The method can be extended in a straightforward manner to scatterers composed of different homogeneous dielectric and conducting objects.**

***Index Terms*—Electromagnetic scattering, moment methods.**

I. Introduction

THE calculation of electromagnetic scattering from arbitrarily shaped three-dimensional (3-D) homogeneous or layered homogeneous dielectric bodies has been of considerable current interest owing to the wide application of materials in a variety of radar targets. Analytical solutions are available for only very limited geometries such as a sphere and a spheroid. For dielectric objects having an arbitrary shape, one has to resort to some approximate numerical techniques based on either integral or differential equations. The integral equation approach is often preferred for homogeneous or layered homogeneous objects because it limits the discretization of the unknown quantity to the surface of the object and the discontinuous interfaces between different materials.

Given a homogeneous dielectric object, using either the equivalence principle or the vector Green's theorem, we can formulate a set of four integral equations to calculate the electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$ in terms of equivalent electric and magnetic currents $\mathbf{J}$ and $\mathbf{M}$ on the surface of the object [1]–[3]. The equation to calculate the electric field is known as the electric field integral equation (EFIE) and there are two such equations: one is for the field outside the object (EFIE-O) and the other is for the field inside the object (EFIE-I). The equation to calculate the magnetic field is known as the magnetic field integral equation (MFIE) and there are also two such equations: one is for the field outside the object (MFIE-O) and the other is for the field inside the object (MFIE-I). Since there are four equations and two unknowns, it is possible to develop a number of different formulations to solve for $\mathbf{J}$ and $\mathbf{M}$. For example, one can select one equation for the field outside the object (EFIE-O or MFIE-O) and another equation for the field inside the object (EFIE-I or MFIE-I) to form a system of equations for a solution of $\mathbf{J}$ and $\mathbf{M}$. Unfortunately, such a solution becomes inaccurate at frequencies that correspond to the resonant frequencies of a cavity formed by covering the surface of the object with a perfect conductor and filling its interior with the exterior medium. This problem is commonly referred to as the problem of interior resonance and it is particularly troublesome for large objects because of a large probability to hit a resonant frequency. One popular solution to this problem is to combine EFIE and MFIE linearly to form a combined field integral equation (CFIE) [4], [5].

Although the CFIE formulation has been used extensively for conducting and impedance bodies, few researchers have applied it to the analysis of scattering by 3-D dielectric bodies. Rao and Wilton claimed the first use of CFIE for this problem [5]. In their approach, $\mathbf{J}$ is expanded in terms of the Rao–Wilton–Glisson (RWG) [6] vector basis functions $(\mathbf{g}_i)$ and $\mathbf{M}$ is expanded in terms of another set of basis functions $(\hat{n}\times\mathbf{g}_i)$ that are orthogonal to the RWG functions, with $\hat{n}$ being the unit vector normal to the scatterer's surface. The resulting EFIE and MFIE are then converted into matrix equations using line-testing functions and a combined EFIE and MFIE is then solved for an approximate numerical solution of $\mathbf{J}$ and $\mathbf{M}$. Although Rao and Wilton argued that it is advantageous for a stable numerical procedure to use two sets of spatially orthogonal basis functions $\mathbf{g}_i$ and $\hat{n}\times\mathbf{g}_i$ to represent $\mathbf{J}$ and $\mathbf{M}$, representing $\mathbf{M}$ in terms of $\hat{n}\times\mathbf{g}_i$ actually violates the property of $\mathbf{M}$ at the edges of dielectric because $\hat{n}\times\mathbf{g}_i$ imposes the continuity of the $\mathbf{M}$ component tangential to the edge. Since $\mathbf{M}$ is related to the electric field by $\mathbf{M}=\mathbf{E}\times\hat{n}$; this, in turn, requires the surface tangential electric field normal to the edge to be continuous across the edge from one patch to another, which is not true if the two patches are not in the same plane.

Another formulation, which is widely used for scattering by 3-D dielectric bodies [7]–[9], is the so-called Poggio, Miller,

Manuscript received August 7, 1997; revised July 22, 1998. This work was supported in part by the Office of Naval Research under Grant N00014-95-1-0848, by a grant from the Air Force Office of Scientific Research via the MURI Program under Contract Number F49620-96-1-0025, and by the National Science Foundation under Grant NSF ECE 94-57735.

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Publisher Item Identifier S 0018-926X(98)08895-4.

Chang, Harrington, and Wu (PMCHW) [2] who originally developed the formulation [1], [10], [11]. In this formulation, the EFIE for the field outside the object is combined with the EFIE for the field inside the object to form a combined equation (EFIE-O + EFIE-I). Similarly, the MFIE for the field outside the object is combined with the MFIE for the field inside the object to form another combined equation (MFIE-O + MFIE-I). These two equations are then solved by the method of moments (MoM) for $\mathbf{J}$ and $\mathbf{M}$. This formulation is found to be free of interior resonances and yields accurate and stable solutions. However, by mixing the EFIE and MFIE for the fields inside and outside the object, the PMCHW formulation cannot produce a matrix equation that can be considered as a constraint for the field outside the object. As a result, the PMCHW formulation cannot be easily combined with other methods such as the finite-element method (FEM) for scattering by inhomogeneous objects [12].

In this paper, we first consider the MoM solution of CFIE for scattering by 3-D dielectric bodies. We use the RWG basis functions to expand both $\mathbf{J}$ and $\mathbf{M}$ and then use the RWG functions as the testing functions to convert EFIE and MFIE into matrix equations. By combining the resultant EFIE and MFIE in a traditional manner, we obtain four different formulations. We show that none of these formulations can yield accurate solutions and at the same time be immune to the problem of interior resonance. We then propose a new formulation of CFIE, which, like the PMCHW formulation, produces accurate MoM solutions and is free of interior resonance. We then apply the multilevel fast multipole algorithm (MLFMA) [13]–[18] to the new formulation to significantly reduce the memory requirement and computational complexity of the MoM solution.

II. Formulation and Analysis

In this section, we study a variety of discretization schemes using the RWG functions as both the expansion and testing functions for solving CFIE for scattering by a homogeneous body. Our goal is to identify an accurate formulation to be used for the implementation of MLFMA for a fast solution of the problem.

Consider the problem of electromagnetic wave scattering by an arbitrarily shaped and homogeneous body characterized by a permittivity ϵ_2 and a permeability μ_2 and immersed in an infinite and homogeneous medium having a permittivity ϵ_1 and a permeability μ_1. Introducing equivalent electric and magnetic currents $\mathbf{J}$ and $\mathbf{M}$ on the surface of the homogeneous body, which are related to the surface fields by $\mathbf{J} = \hat{n} \times \mathbf{H}$ and $\mathbf{M} = \mathbf{E} \times \hat{n}$, respectively, and applying the equivalence principle to the exterior fields, we obtain an electric field integral equation (EFIE)

$$[Z_1\mathbf{L}_1(\mathbf{J}) - \mathbf{K}_1(\mathbf{M}) = \mathbf{E}^i]_{\tan} \tag{1}$$

and a magnetic field integral equation (MFIE)

$$[Z_1\mathbf{K}_1(\mathbf{J}) + \mathbf{L}_1(\mathbf{M}) = Z_1\mathbf{H}^i]_{\tan} \tag{2}$$

TABLE I
Matrix Condition Numbers for the Three Scatterers in Fig. 1

Matrix	Sphere	Cylinder	Cube
$\langle \mathbf{g}_i, \mathbf{L}(\mathbf{g}_j)\rangle$	1.920×10^2	3.788×10^2	2.700×10^2
$\langle \mathbf{g}_i, \mathbf{K}(\mathbf{g}_j)\rangle$	2.182×10^9	7.206×10^8	3.974×10^6
$\langle \hat{n} \times \mathbf{g}_i, \mathbf{L}(\mathbf{g}_j)\rangle$	1.486×10^9	1.225×10^7	1.953×10^6
$\langle \hat{n} \times \mathbf{g}_i, \mathbf{K}(\mathbf{g}_j)\rangle$	1.520×10^1	2.566×10^1	1.755×10^1
$\langle \mathbf{g}_i, \mathbf{K}_s(\mathbf{g}_j)\rangle$	1.135×10^{13}	6.965×10^{11}	4.817×10^{10}
$\langle \mathbf{g}_i, G, \mathbf{g}_j\rangle$	2.182×10^2	3.467×10^2	2.613×10^2
$\langle \hat{n} \times \mathbf{g}_i, G, \mathbf{g}_j\rangle$	2.874×10^9	1.467×10^7	1.140×10^6
Unknowns	3,456	2,328	1,908

where $Z_1 = \sqrt{\mu_1/\epsilon_1}$, $(\mathbf{E}^i, \mathbf{H}^i)$ denote the incident fields, and the operators $\mathbf{L}_1$ and $\mathbf{K}_1$ are defined as

$$\mathbf{L}_1(\mathbf{X}) = jk_1 \int_S \left[\mathbf{X}(\mathbf{r}') + \frac{1}{k_1^2}\nabla\nabla' \cdot \mathbf{X}(\mathbf{r}')\right] G_1(\mathbf{r}, \mathbf{r}')\, dS' \tag{3}$$

$$\mathbf{K}_1(\mathbf{X}) = T_1\mathbf{Y}(\mathbf{r}) + \fint_S \mathbf{X}(\mathbf{r}') \times \nabla G_1(\mathbf{r}, \mathbf{r}')\, dS' \tag{4}$$

where $k_1 = \omega\sqrt{\mu_1\epsilon_1}$, S denotes the surface of the body, $\mathbf{Y}$ is related to $\mathbf{X}$ by $\mathbf{X} = \hat{n} \times \mathbf{Y}$, and

$$G_1(\mathbf{r}, \mathbf{r}') = \frac{e^{-jk_1R}}{4\pi R} \tag{5}$$

in which $R = |\mathbf{r} - \mathbf{r}'|$. The bar integral symbol is used to denote the principal value and the parameter T_1 is given by $T_1 = 1 - \Omega_1/4\pi$ where Ω_1 is the solid angle subtended by the observation point [1]. For a smooth surface, $\Omega_1 = 2\pi$ and $T_1 = 1/2$.

Equations (1) and (2) can be discretized by first expanding $\mathbf{J}$ and $\mathbf{M}$ as

$$\mathbf{J} = \sum_{i=1}^{N_S} \mathbf{g}_i J_i \tag{6}$$

$$\mathbf{M} = \sum_{i=1}^{N_S} \mathbf{g}_i M_i \tag{7}$$

where N_S denotes the total number of edges on S and $\mathbf{g}_i$ denotes the RWG vector basis functions [6]. Substituting (6) and (7) into (1) and using $\mathbf{g}_i$ as the testing function, we obtain the TE formulation (short for $\hat{t} \cdot \mathbf{E}$ where $\hat{t}$ denotes a unit vector tangential to S)

$$[P_1^{\mathrm{TE}}]\{M\} + [Q_1^{\mathrm{TE}}]\{J\} = \{b^{\mathrm{TE}}\} \tag{8}$$

where

$$P_{1ij}^{\mathrm{TE}} = -\int_S \mathbf{g}_i \cdot \mathbf{K}_1(\mathbf{g}_j)\, dS \tag{9}$$

$$Q_{1ij}^{\mathrm{TE}} = Z_1 \int_S \mathbf{g}_i \cdot \mathbf{L}_1(\mathbf{g}_j)\, dS \tag{10}$$

$$b_i^{\mathrm{TE}} = \int_S \mathbf{g}_i \cdot \mathbf{E}^i\, dS. \tag{11}$$

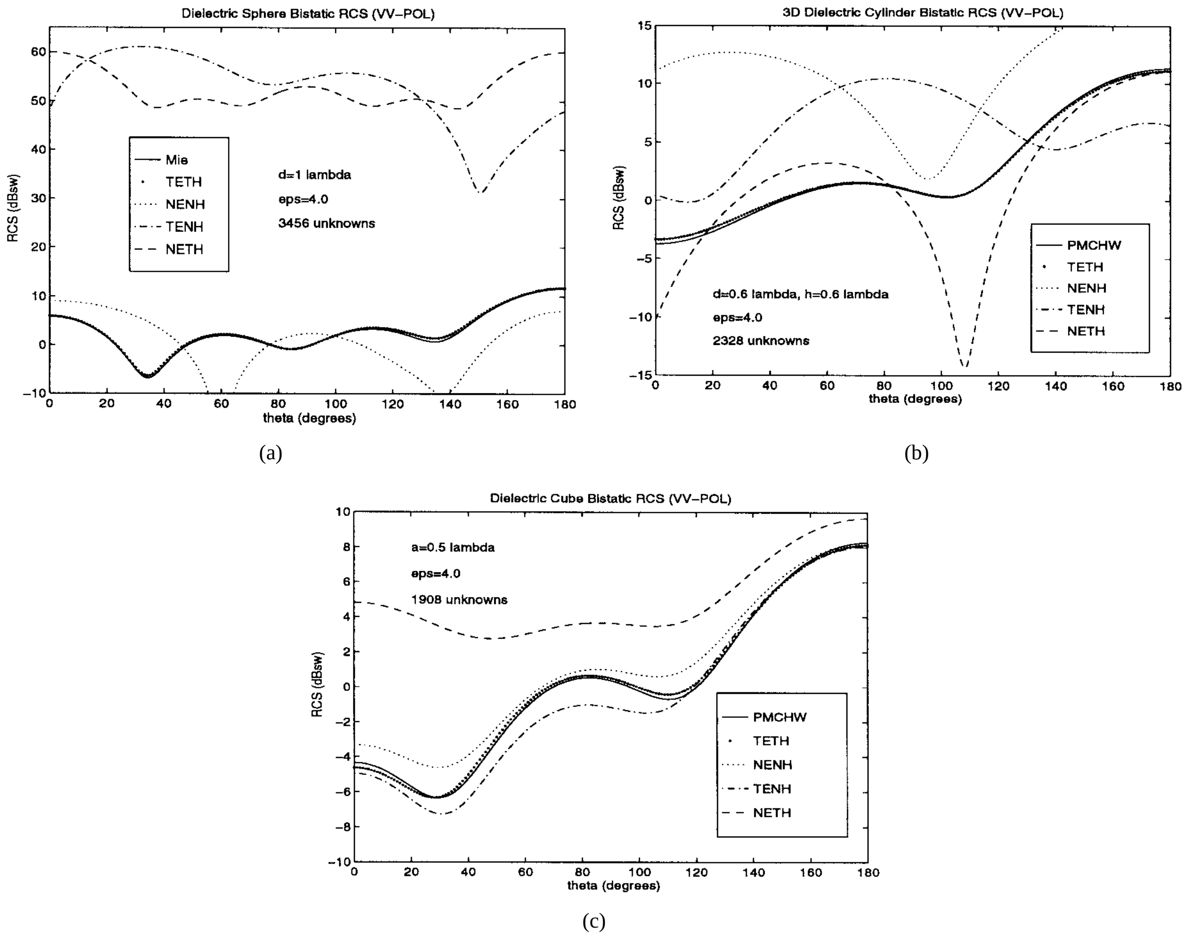


Fig. 1. Bistatic RCS. (a) Dielectric sphere. (b) Dielectric finite cylinder. (c) Dielectric cube.

Similarly, from (2) we obtain the TH formulation (short for $\hat{t} \cdot \mathbf{H}$)

$$[P_1^{\mathrm{TH}}]\{M\} + [Q_1^{\mathrm{TH}}]\{J\} = \{b^{\mathrm{TH}}\} \tag{12}$$

where

$$P_{1ij}^{\mathrm{TH}} = \int_S \mathbf{g}_i \cdot \mathbf{L}_1(\mathbf{g}_j)\, dS = Q_{1ij}^{\mathrm{TE}}/Z_1 \tag{13}$$

$$Q_{1ij}^{\mathrm{TH}} = Z_1 \int_S \mathbf{g}_i \cdot \mathbf{K}_1(\mathbf{g}_j)\, dS = -Z_1 P_{1ij}^{\mathrm{TE}} \tag{14}$$

$$b_i^{\mathrm{TH}} = Z_1 \int_S \mathbf{g}_i \cdot \mathbf{H}^i\, dS. \tag{15}$$

Alternatively, we may choose $\hat{n} \times \mathbf{g}_i$ as the testing function and obtain from (1) the NE formulation (short for $\hat{n} \times \mathbf{E}$)

$$[P_1^{\mathrm{NE}}]\{M\} + [Q_1^{\mathrm{NE}}]\{J\} = \{b^{\mathrm{NE}}\} \tag{16}$$

where

$$P_{1ij}^{\mathrm{NE}} = -\int_S \hat{n} \times \mathbf{g}_i \cdot \mathbf{K}_1(\mathbf{g}_j)\, dS \tag{17}$$

$$Q_{1ij}^{\mathrm{NE}} = Z_1 \int_S \hat{n} \times \mathbf{g}_i \cdot \mathbf{L}_1(\mathbf{g}_j)\, dS \tag{18}$$

$$b_i^{\mathrm{NE}} = \int_S \hat{n} \times \mathbf{g}_i \cdot \mathbf{E}^i\, dS \tag{19}$$

and from (2) the NH formulation (short for $\hat{n} \times \mathbf{H}$)

$$[P_1^{\mathrm{NH}}]\{M\} + [Q_1^{\mathrm{NH}}]\{J\} = \{b^{\mathrm{NH}}\} \tag{20}$$

where

$$P_{1ij}^{\mathrm{NH}} = \int_S \hat{n} \times \mathbf{g}_i \cdot \mathbf{L}_1(\mathbf{g}_j)\, dS = Q_{1ij}^{\mathrm{NE}}/Z_1 \tag{21}$$

$$Q_{1ij}^{\mathrm{NH}} = Z_1 \int_S \hat{n} \times \mathbf{g}_i \cdot \mathbf{K}_1(\mathbf{g}_j)\, dS = -Z_1 P_{1ij}^{\mathrm{NE}} \tag{22}$$

$$b_i^{\mathrm{NH}} = Z_1 \int_S \hat{n} \times \mathbf{g}_i \cdot \mathbf{H}^i\, dS. \tag{23}$$

Equations (16) and (20) can also be obtained by taking the cross product of $\hat{n}$ with (1) and (2) and then using $\mathbf{g}_i$ as the testing function. That is the reason we used the abbreviations NE and NH for the two equations.

Theoretically, any of (8), (12), (16), and (20) can be used to provide a matrix relation between $\{J\}$ and $\{M\}$, which can be written as

$$[P_1]\{M\} + [Q_1]\{J\} = \{b\}. \tag{24}$$

However, it is well known that each of them suffers from the problem of interior resonance and fails to produce accurate solution at and near frequencies corresponding to the resonant frequencies of the cavity formed by covering S with a perfect electric or magnetic conductor and filling it with the exterior medium. To eliminate this problem, one has to combine an equation from EFIE with another equation from MFIE to obtain a combined equation [5]. For example, one can combine (8) with (12) to obtain the TETH formulation or (8) with (20) to obtain the TENH formulation. One can also combine (16) with (12) to obtain the NETH formulation or (16) with (20) to obtain the NENH formulation. The combined matrix equation can still be written in the form of (24).

Equation (24) cannot be solved unless another relation between $\{J\}$ and $\{M\}$ is specified. Such a relation can be derived by applying the equivalence principle to the interior fields and doing so we obtain an EFIE

$$[Z_2\mathbf{L}_2(\mathbf{J}) - \mathbf{K}_2(\mathbf{M}) = 0]_{\tan} \quad (25)$$

and an MFIE

$$[Z_2\mathbf{K}_2(\mathbf{J}) + \mathbf{L}_2(\mathbf{M}) = 0]_{\tan} \quad (26)$$

where $Z_2 = \sqrt{\mu_2/\epsilon_2}$ and the operators $\mathbf{L}_2$ and $\mathbf{K}_2$ are defined similarly to $\mathbf{L}_1$ and $\mathbf{K}_1$, provided that all the subscripts are changed from "1" to "2." Equations (25) and (26) can be discretized in the same manner as for (1) and (2), resulting in the corresponding TETH, TENH, NETH, and NENH formulations, which provide a matrix relation

$$[P_2]\{M\} + [Q_2]\{J\} = \{0\}. \quad (27)$$

Equation (27) can be combined with (24) to form a complete set of linear algebraic equations for $\{J\}$ and $\{M\}$. In addition to the four formulations described above, another approach is to first combine (1) and (25) to form one equation

$$[Z_1\mathbf{L}_1(\mathbf{J}) + Z_2\mathbf{L}_2(\mathbf{J}) - \mathbf{K}_1(\mathbf{M}) - \mathbf{K}_2(\mathbf{M}) = \mathbf{E}^i]_{\tan} \quad (28)$$

and (2) and (26) to form another equation

$$[Z_1\mathbf{K}_1(\mathbf{J}) + Z_2\mathbf{K}_2(\mathbf{J}) + \mathbf{L}_1(\mathbf{M}) + \mathbf{L}_2(\mathbf{M}) = Z_1\mathbf{H}^i]_{\tan} \quad (29)$$

and then discretize these two equations to provide a complete set of linear algebraic equations for $\{J\}$ and $\{M\}$. This approach is commonly known as the PMCHW formulation [2], [7]–[11] and $\mathbf{g}_i$ instead of $\hat{n} \times \mathbf{g}_i$ is often used as the testing functions.

Among those described above, the PMCHW formulation is known to yield an accurate solution and is free of the interior resonance corruption. Although the CFIE for a homogeneous scatterer has been solved by a number of researchers using different expansion or testing functions, it has not been solved before using the RWG functions as both the expansion and testing functions as is done in this section. Therefore, it is not clear which of the four formulations (TETH, TENH, NETH, and NENH) can yield an accurate solution and is immune to the problem of interior resonance. For this, we consider the inner products $\langle \mathbf{g}_i, \mathbf{L}(\mathbf{g}_j)\rangle$ and $\langle \mathbf{g}_i, \mathbf{K}(\mathbf{g}_j)\rangle$, which arise in the TE and TH formulations. Apparently, both terms in $\mathbf{L}(\mathbf{g}_j)$ contribute to $\langle \mathbf{g}_i, \mathbf{L}(\mathbf{g}_j)\rangle$; however, when $i = j$, both terms in $\mathbf{K}(\mathbf{g}_j)$ do not contribute to $\langle \mathbf{g}_i, \mathbf{K}(\mathbf{g}_j)\rangle$ and moreover, when $\mathbf{g}_i$ and $\mathbf{g}_j$ are in the same plane the second term in $\mathbf{K}(\mathbf{g}_j)$ does not contribute $\langle \mathbf{g}_i, \mathbf{K}(\mathbf{g}_j)\rangle$ even if $i \neq j$. Therefore, the $\mathbf{K}(\mathbf{g}_j)$ is not well tested by $\mathbf{g}_i$. Next, we consider the inner products $\langle \hat{n}\times\mathbf{g}_i, \mathbf{L}(\mathbf{g}_j)\rangle$ and $\langle \hat{n}\times\mathbf{g}_i, \mathbf{K}(\mathbf{g}_j)\rangle$ encountered in the NE and NH formulations and, in this case, we find that when $i = j$ or when $\mathbf{g}_i$ and $\mathbf{g}_j$ are in the same plane the second term in $\mathbf{K}(\mathbf{g}_j)$ does not contribute to $\langle \hat{n}\times\mathbf{g}_i, \mathbf{K}(\mathbf{g}_j)\rangle$; however, the first term always has a dominant contribution. Furthermore, when $i = j$,

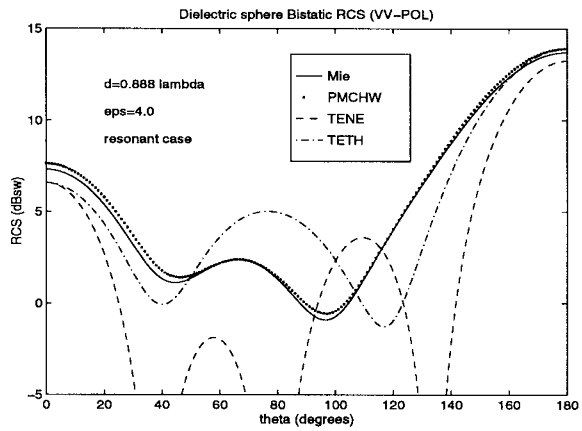


Fig. 2. Bistatic RCS of a dielectric sphere.

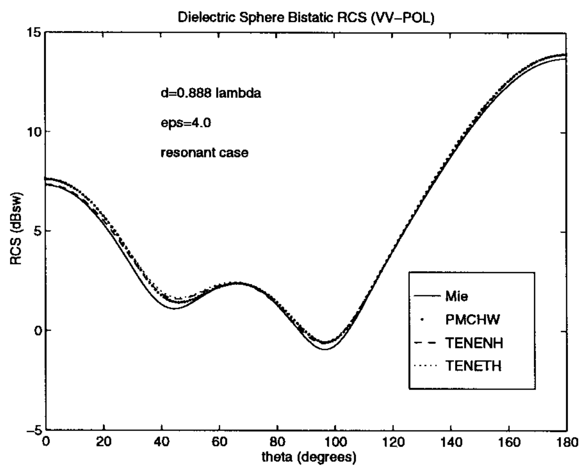


Fig. 3. Bistatic RCS of a dielectric sphere.

the first term in $\mathbf{L}(\mathbf{g}_j)$ has no contribution to $\langle \hat{n}\times\mathbf{g}_i, \mathbf{L}(\mathbf{g}_j)\rangle$. This suggests that $\mathbf{g}_i$ is not a good testing function for $\mathbf{K}(\mathbf{g}_j)$ and $\hat{n}\times\mathbf{g}_i$ is not a good testing function for $\mathbf{L}(\mathbf{g}_j)$. Therefore, in the TE formulation, $\mathbf{M}$ is not well tested, while in the TH formulation, $\mathbf{J}$ is not well tested. Similarly, $\mathbf{J}$ is not well tested in the NE formulation, while $\mathbf{M}$ is not well tested in the NH formulation. As a result, in the TENH formulation $\mathbf{M}$ is not well tested and in the THNE formulation $\mathbf{J}$ is not well tested. Therefore, both TENH and NETH would fail to produce the correct solution.

To verify the analysis above, we consider the problem of plane-wave scattering by a dielectric sphere, a finite dielectric cylinder, and a dielectric cube in a free-space. The sphere has a diameter of $1\lambda_0$, the cylinder has a diameter of $0.6\lambda_0$ and a length of $0.6\lambda_0$, the cube has a side length of $0.5\lambda_0$ and all have a relative permittivity $\epsilon_r = 4$. The first four rows in Table I give the condition number for the matrices involved and the result indicates that indeed $\mathbf{g}_i$ is a good testing function for $\mathbf{L}(\mathbf{g}_j)$ and a poor testing function for $\mathbf{K}(\mathbf{g}_j)$ and, on the other hand, $\hat{n} \times \mathbf{g}_i$ is a good testing function for $\mathbf{K}(\mathbf{g}_j)$ and a poor testing function for $\mathbf{L}(\mathbf{g}_j)$. Fig. 1 shows the bistatic radar cross section (RCS) of the three objects, obtained using the four formulations and the exact Mie series or the PMCHW formulation. It is observed that both TENH and NETH produce

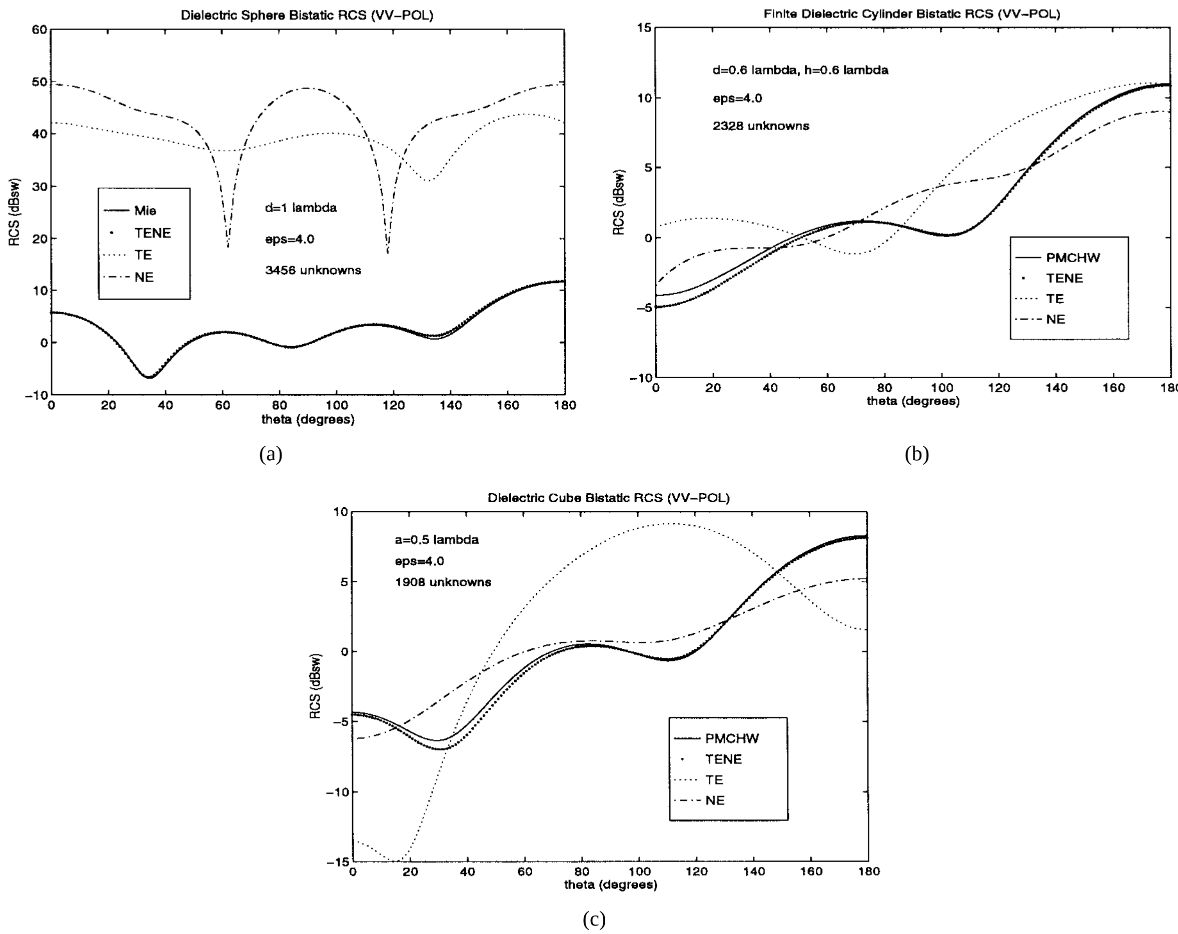


Fig. 4. Bistatic RCS. (a) Dielectric sphere. (b) Dielectric finite cylinder. (c) Dielectric cube.

completely incorrect results. This observation agrees well with our analysis. Among the two remaining formulations, TETH produces a very good result since both $\mathbf{J}$ and $\mathbf{M}$ are well tested, whereas NENH is rather inaccurate. However, our analysis above shows that $\hat{n} \times \mathbf{g}_i$ is a good testing function for $\mathbf{K}(\mathbf{g}_j)$ and the inaccuracy of NENH must be rooted in other causes. Further numerical experiment reveals that the matrix equation produced by NENH is a very ill-conditioned matrix. As a result, a small error introduced in the discretization can render the final solution meaningless.

It is apparent that among the four formulations, only TETH yields the most accurate solution. Unfortunately, because of its improper combination, TETH still suffers from the problem of interior resonance and this is demonstrated clearly in Fig. 2, where the bistatic RCS is given for a sphere having a diameter of $0.888\lambda_0$ and a relative permittivity $\epsilon_r = 4$. The size of the sphere corresponds to the first resonant frequency of the air-filled spherical cavity. It is well known that CFIE removes the interior resonance by combining EFIE and MFIE in such a manner that the resultant integral operators correspond to that for a cavity with a resistive wall. The proper combinations are TENH and NETH; however, neither of them produces accurate solution, as demonstrated earlier. Both TETH and NENH are the improper combinations in the sense that the combined integral operators do not correspond to those for a resistive cavity and, therefore, they still experience the interior resonance. However, because of the numerical discretization error, the singularity (resonance) of the TE (or NE) equation does not coincide exactly with that of the TH (or NH) equation. As a result, the bandwidth of the incorrect solution is extremely narrow (less than 1%), compared to those resulting from either EFIE or MFIE (about 10%).

To obtain a formulation that is both accurate and free of interior resonances, we should find a proper combination among TE, TH, NE, and NH, which tests both $\mathbf{J}$ and $\mathbf{M}$ well and contains a true CFIE. Based on the analysis described above, we find that any of the following combinations—TENENH, TENETH, THNHNE, and THNHTE—satisfies both requirements. Fig. 3 shows the RCS of a dielectric sphere obtained using TENENH and TENETH, along with the results obtained using PMCHW and Mie. Good results are obtained both away and at the frequency of interior resonance. Note that THNHNE and THNHTE are the dual formulations of TENENH and TENETH, respectively, and their validity is expected by duality, which has been verified numerically.

III. Further Analysis

In the preceding section, we provide a heuristic analysis to explain why NETH, TENH, and NENH fail while TETH succeeds to produce an accurate solution. In this section, we offer a different perspective to explain these issues since the understanding of these problems is vital to the development of

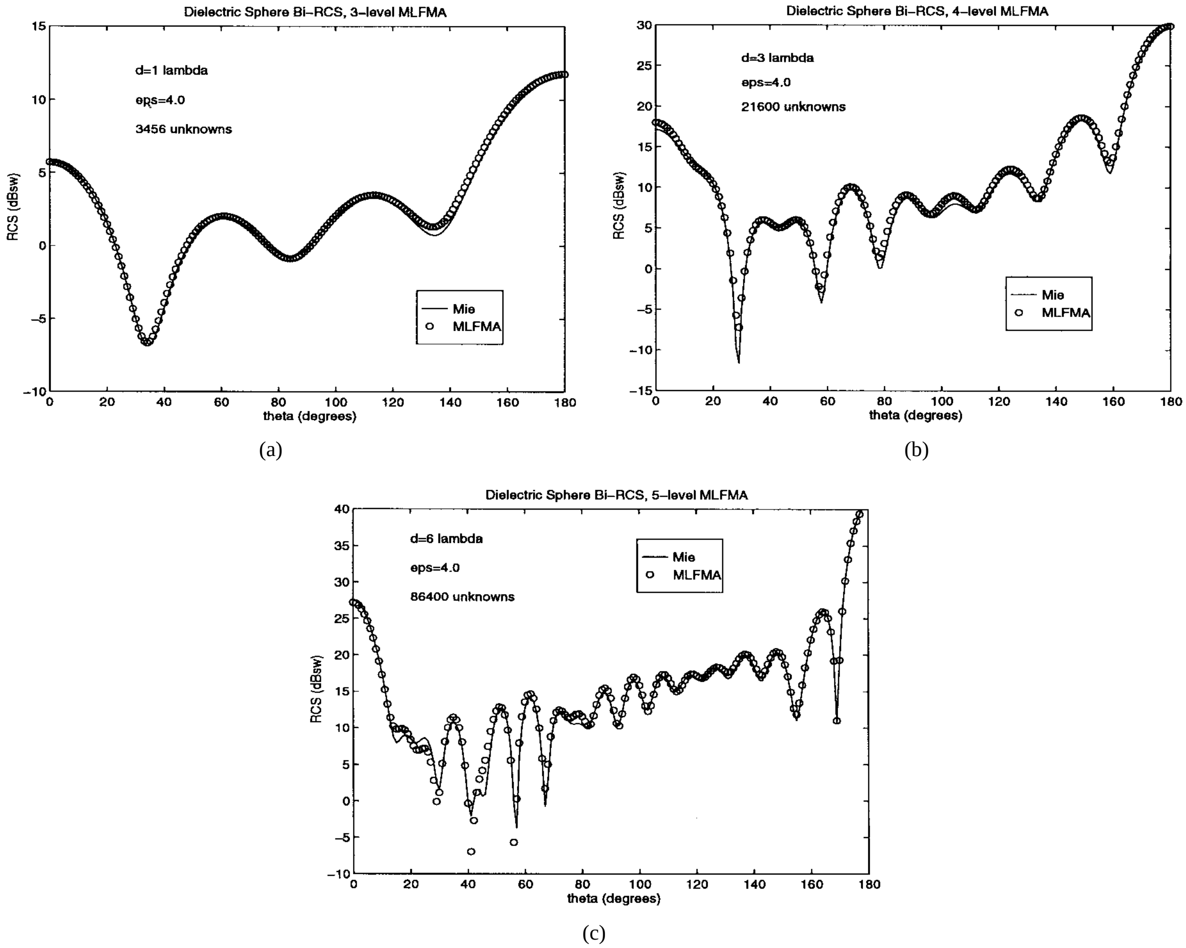


Fig. 5. Bistatic RCS of a dielectric sphere in the E plane. (a) $d = 1.0\lambda_0$. (b) $d = 3.0\lambda_0$. (c) $d = 6.0\lambda_0$.

an accurate numerical formulation. To understand the results of the numerical experiment, we note two very important facts.

1) If we define the space spanned by the set of the RWG functions $\{\mathbf{g}_i\}$ to be $S_{\rm RWG}$ and the space spanned by the set of $\{\hat{n} \times \mathbf{g}_i\}$ to be $\tilde{S}_{\rm RWG}$, then $S_{\rm RWG}$ is quite a different space from $\tilde{S}_{\rm RWG}$:
   a) if $\mathbf{f} \in S_{\rm RWG}$, then $\nabla \cdot \mathbf{f}$ is a piecewise constant function while $\nabla \times \mathbf{f}$ is a singular function that is not square integrable;
   b) if $\mathbf{f} \in \tilde{S}_{\rm RWG}$, then $\nabla \times \mathbf{f}$ is piecewise constant while $\nabla \cdot \mathbf{f}$ is singular and not square integrable; because $S_{\rm RWG}$ and $\tilde{S}_{\rm RWG}$ are not the same space, then the matrix $A_{ij} = \langle \mathbf{g}_i, \hat{n} \times \mathbf{g}_j \rangle$ is a singular matrix where the inner product is taken over the surface of the scatterer.
2) The $\mathbf{K}_1$ operator has the property that for a flat surface $\hat{n} \times \mathbf{K}_1(\mathbf{X}) = T_1\mathbf{X}$ and $\hat{t} \cdot \mathbf{K}_1(\mathbf{X}) = -T_1\hat{t} \cdot (\hat{n} \times \mathbf{X})$. In other words, the second term in (4) does not contribute to a tangential component for a flat surface. The above property implies that if we focus on the tangential component produced by $\mathbf{K}_1(\mathbf{X})$, then $\mathbf{K}_{1s}(\mathbf{X}) = -T_1\hat{n} \times \mathbf{X}$, where $\mathbf{K}_{1s}$ contains the tangential component of $\mathbf{K}_1$. Consequently, $\mathbf{K}_{1s}(\mathbf{g}_i) = -T_1\hat{n} \times \mathbf{g}_i$. Therefore, $\mathbf{K}_{1s}$, when operating on the domain of $S_{\rm RWG}$ space, maps to a range space spanned by $\tilde{S}_{\rm RWG}$. Consequently, the matrix $\langle \mathbf{g}_i, \mathbf{K}_{1s}(\mathbf{g}_j) \rangle$ is a very ill-conditioned matrix. This remains true for nonflat surfaces. For the three scatterers considered in Fig. 1, the condition number of this matrix is given in the fifth row of Table I, which clearly indicates the ill-conditioned nature of the matrix. Therefore, we can say that the space spanned by $\{\mathbf{g}_i\}$ is quasi-orthogonal to the space spanned by $\{\mathbf{K}_{1s}(\mathbf{g}_i)\}$ on the surface of the scatterer.

It is to be noted that when an electric field is produced by $\mathbf{E}_J = \mathbf{L}_1(\mathbf{J})$, the electric field is produced via two mechanisms: the first term in (3) corresponds to the induction term where the current first produces a magnetic field and then the time-varying magnetic field in turn produces an electric field. This is confirmed by the disappearance of the first term when $\omega \rightarrow 0$. The second term in (3) corresponds to the electric field produced by charge accumulation in the current. It remains finite in the limit $\omega \rightarrow 0$ as long as $\nabla \cdot \mathbf{J} \neq 0$.

On the other hand, the electric field produced by $\mathbf{E}_M = \mathbf{K}_1(\mathbf{M})$ is mainly produced by induction or nonzero curl in the $\mathbf{M}$. The induction field in $\mathbf{E}_J$ is quite different from the induction field in $\mathbf{E}_M$: They are quasi-orthogonal to each other. Numerical experiment shows that $\langle \hat{n} \times \mathbf{g}_i, G, \mathbf{g}_j \rangle$ and $\langle \mathbf{g}_i, \mathbf{K}(\mathbf{g}_j) \rangle$ are ill-conditioned while $\langle \mathbf{g}_i, G, \mathbf{g}_j \rangle$, and $\langle \hat{n} \times \mathbf{g}_i, \mathbf{K}(\mathbf{g}_j) \rangle$ are well conditioned (see Table I; G denotes scalar Green's function). But, the testing of the second term in (3) $\nabla \int_S G\nabla' \cdot \mathbf{X}\, dS'$ is always well conditioned because of its charge nature.

Therefore, if $\hat{n}\times\mathbf{g}_i$ alone is used to test (1), when the RWG function is used as an expansion function, both matrices for $\mathbf{K}_1$ and the charge term in $\mathbf{L}_1$ are well conditioned, but the induction term in $\mathbf{L}_1$ is poorly tested corresponding to an ill-conditioned matrix. However, the induction term is important physically and its negligence cannot yield the correct physics. Therefore, the NE formulation alone cannot work (as shown in Fig. 4).

If $\mathbf{g}_i$ alone is used to test (1) instead, the $\mathbf{K}_1$ term is poorly tested and, hence, the TE formulation alone cannot work. This is also shown in Fig. 4. As a remedy, we suggest using $\mathbf{t}_i=\mathbf{g}_i+\hat{n}\times\mathbf{g}_i$ as a testing function. When this is used, both the induction term and the charge term in $\mathbf{L}_1(\mathbf{J})$ are well-tested as well as $\mathbf{K}_1(\mathbf{M})$. We call this the TENE formulation. It is shown to work well for small scatterers (see Fig. 4) when the solution is not plagued by the interior resonance problem. At the interior resonance, it yields an incorrect solution, as demonstrated in Fig. 2.

As mentioned earlier, the interior resonance problem can be removed by using MFIE in addition to EFIE. Note that the TE formulation establishes little information on $\mathbf{K}_1(\mathbf{M})$ and hence $\mathbf{M}$, while a TH formulation would make the matrix from $\mathbf{L}_1(\mathbf{M})$ well-conditioned and hence provide information on $\mathbf{M}$. Hence, a TETH formulation will provide a stable formulation as validated by numerical results in Fig. 1.

However, the TETH formulation does not yield a lossy internal resonance problem where the internal resonance mode of the corresponding metallic cavity is damped. To alleviate this problem, we propose to use the TENENH or TENETH formulation. In these formulations, the TENE part provides a well-conditioned matrix for both $\mathbf{L}_1$ and $\mathbf{K}_1$ operators, while the NH or TH part generates a lossy interior resonance problem where the internal resonance modes are damped.

A final note is in order for the NENH formulation. This formulation generates a well-conditioned matrix for $\mathbf{K}_1(\mathbf{M})$ in (1) and it generates a well-conditioned matrix for $\mathbf{K}_1(\mathbf{J})$ in (2). However, $\mathbf{K}_1(\mathbf{M})$ and $\mathbf{K}_1(\mathbf{J})$ produce induction field which are generated primarily by $\nabla\times\mathbf{M}$ and $\nabla\times\mathbf{J}$, respectively. Since the curl of the RWG function is singular, they do not approximate the curl of a current well. Therefore, the error in the NENH formulation is in general higher than the TETH formulation while both are unable to induce a lossy internal resonance problem.

IV. Solution by MLFMA

It is clear now that both PMCHW and TENENH (and its variants) are accurate, reliable, and immune to the problem of interior resonance. However, their MoM solution requires the generation and solution of a fully populated matrix. The associated memory requirement is $O(N^2)$ and the computational complexity is $O(N^3)$ with N being the number of unknowns. This significantly limits the size of the scatterer to be handled. As a result, only very small dielectric objects have been considered in the past.

One solution to the problem discussed above is to employ an iterative method to solve the MoM matrix equation and within each iteration the required matrix–vector multiplication is performed by the fast multipole method (FMM) [13], [14].

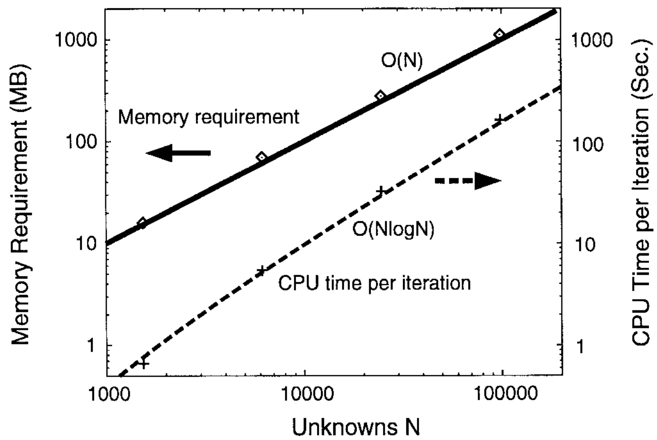


Fig. 6. Memory requirement and CPU time per iteration as functions of the number of unknowns for MLFMA solution of scattering from a dielectric sphere.

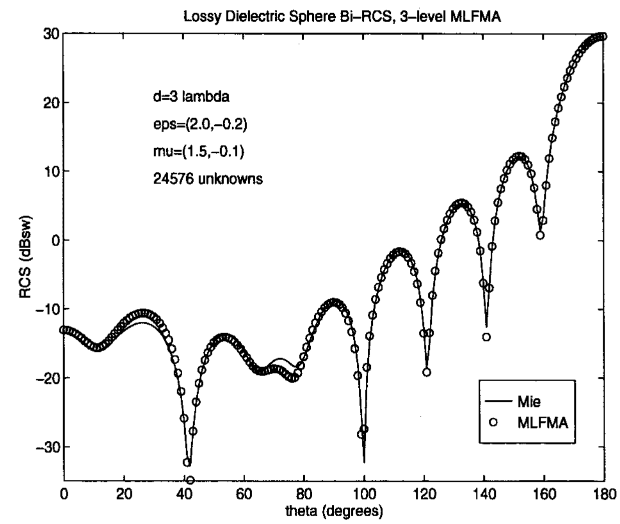


Fig. 7. Bistatic RCS of a lossy dielectric sphere in the E plane.

The basic idea of FMM is first to divide the subscatterers into groups. The addition theorem is then used to translate the scattered field of different scattering centers within a group into a single center, and this process is called aggregation. Doing this, the number of scattering centers is reduced significantly. Similarly, for each group, the field scattered by all the other group centers can be first received by the group center, and then redistributed to the subscatterers belonging to the group. This process is called disaggregation. It has been shown that the FMM can reduce the memory requirement and computational complexity to $O(N^{1.5})$.

The memory requirement and computational complexity can be further reduced to $O(N\log N)$ using the multilevel fast multipole algorithm (MLFMA) [15]–[18]. To implement MLFMA, the entire object is first enclosed in a large cube, which is divided into eight smaller cubes. Each subcube is then recursively subdivided into smaller cubes until the edge length of the finest cube is about half of a wavelength. For two points in the same or nearby finest cubes, their interaction is calculated in a direct manner, as is done in the direct MoM. However, when the two points reside in different nonnearby cubes, their interaction is calculated by the FMM, as described above. The level of cubes on which the FMM is applied depends on the distance between the two points. The detailed description of MLFMA is given in [16] and [18] and is not repeated here, although the equations to be treated are different.

The basic formulas, derived with the addition theorem to calculate the matrix elements for nonnearby groups are given by

$$P_{1ij}^{\mathrm{TE}} = \left(\frac{k_1}{4\pi}\right)^2 \oint \mathbf{V}_{1im}^{\mathrm{TEP}} T_{1mm'}(\hat{k}\cdot\hat{r}_{mm'})\cdot\mathbf{V}_{1jm'}d^2\hat{k} \quad (30)$$

$$Q_{1ij}^{\mathrm{TE}} = Z_1\left(\frac{k_1}{4\pi}\right)^2 \oint \mathbf{V}_{1im}^{\mathrm{TEQ}} T_{1mm'}(\hat{k}\cdot\hat{r}_{mm'})\cdot\mathbf{V}_{1jm'}d^2\hat{k} \quad (31)$$

$$P_{1ij}^{\mathrm{NE}} = \left(\frac{k_1}{4\pi}\right)^2 \oint \mathbf{V}_{1im}^{\mathrm{NEP}} T_{1mm'}(\hat{k}\cdot\hat{r}_{mm'})\cdot\mathbf{V}_{1jm'}d^2\hat{k} \quad (32)$$

$$Q_{1ij}^{\mathrm{NE}} = Z_1\left(\frac{k_1}{4\pi}\right)^2 \oint \mathbf{V}_{1im}^{\mathrm{NEQ}} T_{1mm'}(\hat{k}\cdot\hat{r}_{mm'})\cdot\mathbf{V}_{1jm'}d^2\hat{k} \quad (33)$$

where

$$\mathbf{V}_{1im}^{\mathrm{TEP}} = \int_S e^{-j\mathbf{k}_1\cdot\mathbf{r}_{im}}(\hat{k}\times\mathbf{g}_i)\,dS \quad (34)$$

$$\mathbf{V}_{1im}^{\mathrm{TEQ}} = \int_S e^{-j\mathbf{k}_1\cdot\mathbf{r}_{im}}(\bar{\mathbf{I}}-\hat{k}\hat{k})\cdot\mathbf{g}_i\,dS \quad (35)$$

$$\mathbf{V}_{1im}^{\mathrm{NEP}} = \int_S e^{-j\mathbf{k}_1\cdot\mathbf{r}_{im}}(\hat{k}\times\hat{n}\times\mathbf{g}_i)\,dS \quad (36)$$

$$\mathbf{V}_{1im}^{\mathrm{NEQ}} = \int_S e^{-j\mathbf{k}_1\cdot\mathbf{r}_{im}}(\bar{\mathbf{I}}-\hat{k}\hat{k})\cdot(\hat{n}\times\mathbf{g}_i)\,dS \quad (37)$$

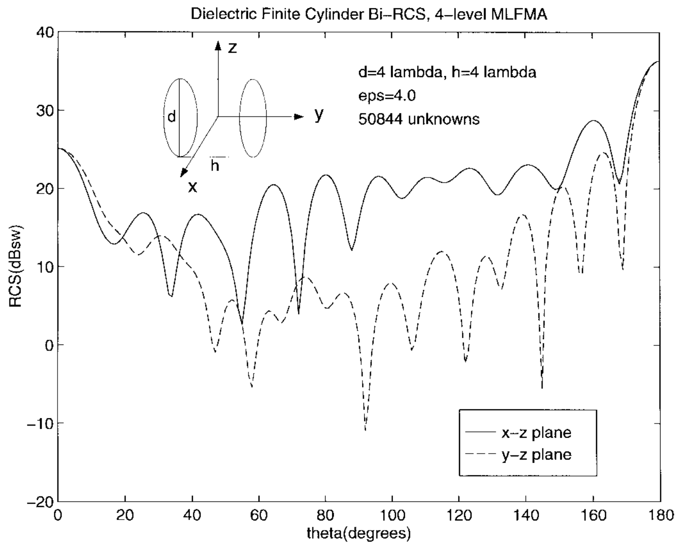
$$\mathbf{V}_{1jm'} = \int_S e^{j\mathbf{k}_1\cdot\mathbf{r}_{jm'}}\mathbf{g}_j\,dS \quad (38)$$

and

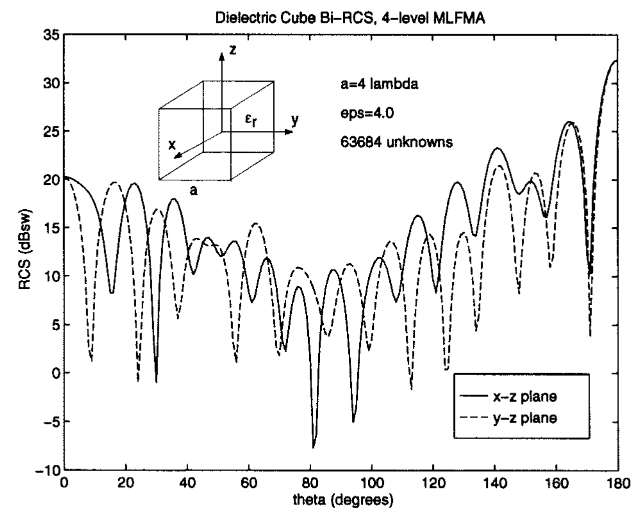
$$T_{1mm'}(\hat{k}\cdot\hat{r}_{mm'}) = \sum_{l=0}^{L}(-j)^l(2l+1)h_l^{(2)}(k_1 r_{mm'})P_l(\hat{r}_{mm'}\cdot\hat{k}). \quad (39)$$

In the above, the integrals in (30)–(33) are over the unit spherical surface, $\mathbf{g}_i$ resides in a group G_m centered at $\mathbf{r}_m$, $\mathbf{g}_j$ resides in a group $G_{m'}$ centered at $\mathbf{r}_{m'}$, $\mathbf{r}_{im} = \mathbf{r}_i - \mathbf{r}_m$, $\mathbf{r}_{jm'} = \mathbf{r}_j - \mathbf{r}_{m'}$, and $\mathbf{r}_{mm'} = \mathbf{r}_m - \mathbf{r}_{m'}$. Also in (39), $h_l^{(2)}$ denotes the spherical Hankel function of the second kind, P_l denotes the Legendre polynomial of degree l, and L denotes the number of multipole expansion terms, whose choice is discussed in [18]. Similar multipole expressions for other matrix elements can be obtained. The multipole expressions for the field inside the object can also be obtained provided that all the subscripts are changed from "1" to "2." Note that since k_2 is different from k_1, L in (39) must also be different for good accuracy.

As described earlier, MLFMA converts the direct interaction component P_{ij} or Q_{ij} between two "far-away" points i and j into three indirect components: the radiation component from the point j to the group center m', which is represented by $\mathbf{V}_{jm'}$; the translation component from the group center m' to another group center m, represented by $T_{mm'}$; and



(a)



(b)

Fig. 8. Bistatic RCS for a plane wave incident along the z axis from the top with the incident electric field in the xz plane. (a) A finite dielectric cylinder. (b) A dielectric cube.

the receiving component from the group center m to the point i, which is represented by $\mathbf{V}_{im}$. Among these three components, only the receiving component is different for different formulations, the other two components, the translation, and the radiation components, are the same. Therefore, even though the number of the terms in the TENENH and its variants is increased, the increase in computing time is insignificant provided that the different receiving components are merged into one receiving component before performing the matrix–vector multiplication. This advantage is, however, not shared by PMCHW since, as indicated in (28) and (29), there are two different operators $\mathbf{L}_1$ and $\mathbf{L}_2$ (or $\mathbf{K}_1$ and $\mathbf{K}_2$) on a single current $\mathbf{J}$ (or $\mathbf{M}$). Since the radiation, translation, and receiving components are all different for the two different operators, the matrix–vector multiplication has to be done twice on both $\mathbf{J}$ and $\mathbf{M}$ in each of (28) and (29).

The MLFMA described above is implemented for the solution of the TENENH formulation. Several representative results are given in Fig. 5 for a sphere whose diameter varies from $1\lambda_0$ to as large as $6\lambda_0$. The results are compared to those obtained using the Mie series and good agreement is observed. The memory requirement and the total central processing unit (CPU) time on one processor of an SGI Power Challenge (R8000) are given in Fig. 6. Fig. 7 shows the result for a lossy sphere having a diameter of $3\lambda_0$ and $\epsilon_r = 2.0 - j0.2$ and $\mu_r = 1.5 - j0.1$, demonstrating for the first time the applicability of MLFMA to lossy materials. Finally, Fig. 8 gives two additional examples: one for a finite dielectric cylinder having a diameter of $4\lambda_0$ and a length of $4\lambda_0$, and the other for a dielectric cube having a side length of $4\lambda_0$. The number of unknowns is 50 844 and 63 684, respectively.

V. Conclusion

In this paper, we have studied a variety of CFIE formulations for scattering by 3-D arbitrarily shaped homogeneous objects using the RWG functions as both the expansion and testing functions. We have shown that due to the deficiency of the RWG functions as the testing functions, among the four CFIE formulations (namely, TETH, TENH, NETH, and NENH), only TETH can test both the equivalent electric and magnetic currents well and thus can yield accurate solution. However, because of its improper combination, TETH suffers from the problem of interior resonance.

Based on the analysis, we then proposed new formulations (namely, TENENH, TENETH, THNHNE, and THNHTE) that have a good accuracy and are free of interior resonances. These formulations can be derived using two approaches. The first approach formulates the CFIE as a linear combination of the two tangential components of EFIE and MFIE (that is, $\text{CFIE} = \text{EFIE} + \hat{n} \times \text{EFIE} + \text{MFIE} + \hat{n} \times \text{MFIE}$) and then uses the RWG $\mathbf{g}_i$ as the testing functions. Since both $\mathbf{J}$ and $\mathbf{M}$ are already well tested in the first two terms ($\text{EFIE} + \hat{n} \times \text{EFIE}$), one can neglect either MFIE or $\hat{n} \times \text{MFIE}$ for the sake of efficiency. The other term, $\hat{n} \times$ MFIE or MFIE, is needed to remove the problem of interior resonance. The resulting formulation is TENENH or TENETH, depending on the neglected term. Similarly, since both $\mathbf{J}$ and $\mathbf{M}$ are also well tested in the last two terms ($\text{MFIE} + \hat{n} \times \text{MFIE}$), one can neglect either EFIE or $\hat{n} \times$ EFIE and the resulting formulation is THNHNE or THNHTE. The second approach formulates the CFIE as a linear combination of EFIE and MFIE ($\text{CFIE} = \text{EFIE} + \text{MFIE}$) and then uses $\mathbf{g}_i + \hat{n} \times \mathbf{g}_i$ as the testing functions. Again, one of the four terms can be dropped for the sake of efficiency, resulting in one of the four formulations described above.

Having identified the accurate and reliable CFIE formulations, we then applied MLFMA to significantly reduce the memory requirement and computational complexity of their MoM solutions. Numerical results were presented to demonstrate the accuracy and capability of the proposed method. The method can be extended in a straightforward manner to scatterers composed of different homogeneous dielectric and conducting objects.

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