Radiation and Scattering from Isorefractive Bodies of Revolution

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***Abstract*—The radiation from an electric or magnetic dipole located on the symmetry axis of an isorefractive body of revolution (BOR) and axially oriented is considered. The boundary-value problem is solved exactly for three BOR's: the isorefractive prolate and oblate spheroids and the isorefractive paraboloid. Furthermore, for the isorefractive circular cone, the radiation from an arbitrarily located and radially oriented dipole, and the scattering from an obliquely incident plane wave are determined exactly.**

***Index Terms*—Electromagnetic radiation, electromagnetic scattering.**

I. Introduction

An isorefractive body is made of a linear, homogeneous, and isotropic material whose refractive index is equal to the refractive index of the medium surrounding the body, although the intrinsic impedances of the two media have different values. If the surface that separates the body from the surrounding medium is a coordinate surface in a system of orthogonal curvilinear coordinates for which the wave equation is separable, then the boundary conditions at the interface between the two media can be satisfied by mode matching and an exact canonical solution to the boundary-value problem is obtained. This is possible because the eigenfunction expansions of the electromagnetic field on either side of the interface contain the wavenumber as a parameter (and this has the same value in both media), but not the intrinsic impedance. If the body's material has electric permittivity ε_2 and magnetic permeability μ_2, and the body is immersed in a medium of permittivity ε_1 and permeability μ_1, then the isorefractive condition is

$$\varepsilon_1\mu_1 = \varepsilon_2\mu_2. \tag{1}$$

Condition (1) has been used to obtain exact solutions for scattering from elliptic and parabolic cylinders in [1] and from a paraboloid of revolution in [2]. The scattering from a wedge has been solved in [3] via a transform technique. The fact that lateral waves on either side of a planar isorefractive boundary propagate with the same phase velocity has been utilized in obtaining an exact solution to plane wave scattering from a class of wedge structures [4].

In this paper, we solve exactly the boundary-value problems for primary sources that are either electric or magnetic dipoles located on the axis of symmetry of an isorefractive body of revolution (BOR) and axially oriented for three different types of BOR's: a prolate spheroid (Section II), an oblate spheroid (Section III), and a paraboloid (Section IV). For the isorefractive circular cone, we solve exactly the problems of radiation from an arbitrarily located and radially oriented electric or magnetic dipole and of scattering by an obliquely incident plane wave (Section V). The analysis is conducted in the frequency domain with $\exp(j\omega t)$ as the time-dependent factor.

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II. The Isorefractive Prolate Spheroid

The prolate spheroidal coordinates (ξ, η, φ) are related to the rectangular coordinates (x, y, z) by

$$\begin{cases} x = \frac{d}{2}\sqrt{(\xi^2-1)(1-\eta^2)}\cos\varphi, \\ y = \frac{d}{2}\sqrt{(\xi^2-1)(1-\eta^2)}\sin\varphi, \\ z = \frac{d}{2}\xi\eta \end{cases} \tag{2}$$

where $1 \le \xi < \infty$, $-1 \le \eta \le 1$, $0 \le \varphi < 2\pi$ and z is the axis of symmetry. The surfaces $\xi =$ constant, $\eta =$ constant, and $\varphi =$ constant are, respectively, confocal prolate spheroids of interfocal distance d, major axis ξd, and minor axis $\sqrt{\xi^2-1}d$, confocal semi-hyperboloids of revolution with interfocal distance d, and semi-planes originating in the z axis.

The prolate spheroid with surface $\xi = \xi_1$ is made of a material with permittivity ε_2 and permeability μ_2 and is immersed in a medium with constitutive parameters ε_1 and μ_1; the two media are isorefractive, i.e., condition (1) applies.

For an electric dipole located at $\xi = \xi_0 \ge \xi_1$, $\eta = 1$ on the positive z axis and corresponding to a primary electric Hertz vector

$$\underline{\pi}^i = \hat{z}\exp(-jkR)/(kR) \tag{3}$$

where R is the distance between the dipole and the observations point and

$$k = \omega\sqrt{\varepsilon_1\mu_1} = \omega\sqrt{\varepsilon_2\mu_2} \tag{4}$$

is the wavenumber, the primary magnetic field is $\underline{H}^i = \hat{\varphi}H^i_\varphi$, with

$$\begin{aligned} H^i_\varphi &= -k^2cY_1[(kR)^{-2} - j(kR)^{-3}]\sqrt{(\xi^2-1)(1-\eta^2)}e^{-jkR} \\ &= \frac{2k^2Y_1}{\sqrt{\xi_o^2-1}}\sum_{n=0}^{\infty}\frac{j^{n-1}}{\rho_{1n}N_{1n}}R^{(1)}_{1n}(c,\xi_<)R^{(4)}_{1n}(c,\xi_>)S_{1n}(c,\eta) \end{aligned} \tag{5}$$

where $c = kd/2$, the notation for the prolate spheroidal wave functions is as in [5] and [6] and $\xi_<(\xi_>)$ is the smaller (larger) between ξ and ξ_o. The scattered magnetic field in the region $\xi \geq \xi_1$ and the total magnetic field inside the spheroid $(1 \leq \xi \leq \xi_1)$ are also φ polarized, and are given, respectively, by

$$H_\varphi^s = \frac{2k^2 Y_1}{\sqrt{\xi_o^2 - 1}} \sum_{n=0}^{\infty} \frac{j^{n-1}}{\rho_{1n} N_{1n}} a_{en} \times R_{1n}^{(4)}(c,\xi) R_{1n}^{(4)}(c,\xi_o) S_{1n}(c,\eta) \tag{6}$$

$$H_{2\varphi} = \frac{2k^2 Y_2}{\sqrt{\xi_o^2 - 1}} \sum_{n=0}^{\infty} \frac{j^{n-1}}{\rho_{1n} N_{1n}} c_{en} R_{1n}^{(1)}(c,\xi) \times R_{1n}^{(4)}(c,\xi_o) S_{1n}(c,\eta) \tag{7}$$

where a_{en} and c_{en} are modal coefficients determined by the boundary conditions, i.e., by imposing the continuity of H_φ and E_η across the interface $\xi = \xi_1$. The quantities $Y_1 = Z_1^{-1} = \sqrt{\varepsilon_1/\mu_1}$ and $Y_2 = Z_2^{-1} = \sqrt{\varepsilon_2/\mu_2}$ are the intrinsic admittances of the surrounding medium and of the spheroid.

The component E_η is given by

$$E_{\ell\eta} = \frac{-j}{c} Z_\ell \sqrt{(\xi^2-1)/(\xi^2-\eta^2)} \left(\frac{\partial}{\partial \xi} + \frac{\xi}{\xi^2 - 1} \right) H_{\ell\varphi} \qquad \ell = 1,2 \tag{8}$$

where $H_{1\varphi} = H_\varphi^i + H_\varphi^s$ and $E_{1\eta} = E_\eta^i + E_\eta^s$. Hence

$$c_{en} = 1 + a_{en} A_n \tag{9}$$

$$a_{en} = \frac{(Y_2 - Y_1) R_{1n}^{(1)}(c,\xi_1)}{Y_1 R_{1n}^{(4)}(c,\xi_1) - Y_2 A_n R_{1n}^{(1)}(c,\xi_1)} \tag{10}$$

where

$$A_n = \frac{R_{1n}^{(4)'}(c,\xi_1) + \frac{\xi_1}{\xi_1^2 - 1} R_{1n}^{(4)}(c,\xi_1)}{R_{1n}^{(1)'}(c,\xi_1) + \frac{\xi_1}{\xi_1^2 - 1} R_{1n}^{(1)}(c,\xi_1)} \tag{11}$$

and the prime means the derivative with respect to ξ_1. Note that $a_{en} = 0$ and $c_{en} = 1$ if $Y_1 = Y_2$, as expected.

For a magnetic dipole located at $\xi = \xi_o \geq \xi_1$, $\eta = 1$ on the positive z axis and axially oriented, corresponding to a primary magnetic Hertz vector given by (3), the electric field is everywhere oriented in the $\hat{\varphi}$ direction. The primary electric field E_φ^i, the scattered electric field E_φ^s in the region $\xi \geq \xi_1$ outside the spheroid, and the total electric field $E_{2\varphi}$ inside the spheroid $(1 \leq \xi \leq \xi_1)$ are given by

$$E_\varphi^i = k^2 c Z_1 [(kR)^{-2} - j(kR)^{-3}] \sqrt{(\xi^2-1)(1-\eta^2)} e^{-jkR} = \frac{-2k^2 Z_1}{\sqrt{\xi_o^2 - 1}} \sum_{n=0}^{\infty} \frac{j^{n-1}}{\rho_{1n} N_{1n}} R_{1n}^{(1)}(c,\xi_<) R_{1n}^{(4)}(c,\xi_>) S_{1n}(c,\eta) \tag{12}$$

$$E_\varphi^s = \frac{-2k^2 Z_1}{\sqrt{\xi_o^2 - 1}} \sum_{n=0}^{\infty} \frac{j^{n-1}}{\rho_{1n} N_{1n}} a_{hn} \times R_{1n}^{(4)}(c,\xi) R_{1n}^{(4)}(c,\xi_o) S_{1n}(c,\eta) \tag{13}$$

$$E_{2\varphi} = \frac{-2k^2 Z_2}{\sqrt{\xi_o^2 - 1}} \sum_{n=0}^{\infty} \frac{j^{n-1}}{\rho_{1n} N_{1n}} c_{hn} R_{1n}^{(1)}(c,\xi) \times R_{1n}^{(4)}(c,\xi_o) S_{1n}(c,\eta) \tag{14}$$

where the modal constants a_{hn} and c_{hn} are determined by imposing the boundary conditions at $\xi = \xi_1$, yielding

$$c_{hn} = 1 + a_{hn} A_n \tag{15}$$

$$a_{hn} = \frac{(Z_2 - Z_1) R_{1n}^{(1)}(c,\xi_1)}{Z_1 R_{1n}^{(4)}(c,\xi_1) - Z_2 A_n R_{1n}^{(1)}(c,\xi_1)} \tag{16}$$

where A_n is given by (11). As expected, $a_{hn} = 0$ and $c_{hn} = 1$ if $Z_1 = Z_2$. Note that a_{hn} is obtained from a_{en} by duality, i.e., by replacing Y_ℓ with Z_ℓ, $(\ell = 1, 2)$.

Results for an arbitrarily located and oriented dipole are unavailable, even for the simpler case of a metallic spheroid.

III. The Isorefractive Oblate Spheroid

The oblate spheroidal coordinates (ξ, η, φ) are related to the rectangular coordinates (x, y, z) by

$$\begin{cases} x &= \frac{d}{2}\sqrt{(\xi^2+1)(1-\eta^2)} \cos\varphi, \\ y &= \frac{d}{2}\sqrt{(\xi^2+1)(1-\eta^2)} \sin\varphi, \\ z &= \frac{d}{2}\xi\eta \end{cases} \tag{17}$$

where $0 \leq \xi < \infty$, $-1 \leq \eta \leq 1$, $0 \leq \varphi < 2\pi$, and z is the symmetry axis. The surfaces $\xi =$ constant, $\eta =$ constant, and $\varphi =$ constant are, respectively, confocal oblate spheroids of interfocal distance d, minor axis ξd and major axis $\sqrt{\xi^2 + 1} d$; confocal semi-hyperboloids of revolution with interfocal distance d; and semi-planes originating in the z axis.

The oblate spheroid with surface $\xi = \xi_1$ is made of a material with permittivity ε_2 and permeability μ_2 and is immersed in a medium with constitutive parameters ε_1 and μ_1. The isorefractive condition (1) applies.

For an electric dipole located at $\xi = \xi_o \geq \xi_1$, $\eta = 1$ on the positive z axis and axially oriented, corresponding to the primary electric Hertz vector (3), the magnetic field is everywhere oriented in the $\hat{\varphi}$ direction. The primary magnetic field H_φ^i, the scattered magnetic field H_φ^s in the region $\xi \geq \xi_1$ outside the spheroid, and the total magnetic field $H_{2\varphi}$ inside the spheroid $(0 \leq \xi \leq \xi_1)$ are given by

$$H_\varphi^i = -k^2 c Y_1 [(kR)^{-2} - j(kR)^{-3}] \sqrt{(\xi^2+1)(1-\eta^2)} e^{-jkR} = \frac{2k^2 Y_1}{\sqrt{\xi_o^2 + 1}} \sum_{n=0}^{\infty} \frac{j^n}{\tilde{\rho}_{1n} \tilde{N}_{1n}} R_{1n}^{(1)}(jc, -j\xi_<) \times R_{1n}^{(4)}(jc, -j\xi_>) S_{1n}(jc,\eta) \tag{18}$$

$$H_\varphi^s = \frac{2k^2 Y_1}{\sqrt{\xi_o^2 + 1}} \sum_{n=0}^{\infty} \frac{j^n}{\tilde{\rho}_{1n} \tilde{N}_{1n}} \tilde{a}_{en} R_{1n}^{(4)}(jc, -j\xi) \times R_{1n}^{(4)}(jc, -j\xi_o) S_{1n}(jc,\eta) \tag{19}$$

$$H_{2\varphi} = \frac{2k^2 Y_2}{\sqrt{\xi_o^2 + 1}} \sum_{n=0}^{\infty} \frac{j^n}{\tilde{\rho}_{1n} \tilde{N}_{1n}} \tilde{c}_{en} R_{1n}^{(1)}(jc, -j\xi) \times R_{1n}^{(4)}(jc, -j\xi_o) S_{1n}(jc,\eta) \tag{20}$$

where $c = kd/2$, the notation for the oblate spheroidal wave functions is that adopted in [5] and [6] and $\xi_<(\xi_>)$ is the smaller (larger) between ξ and ξ_o. The modal coefficients $\tilde{a}_{en}$ and $\tilde{c}_{en}$ are found by imposing the boundary conditions at $\xi = \xi_1$

$$\tilde{c}_{en} = 1 + \tilde{a}_{en} \tilde{A}_n \tag{21}$$

$$\tilde{a}_{en} = \frac{(Y_2 - Y_1)R_{1n}^{(1)}(jc, -j\xi_1)}{Y_1 R_{1n}^{(4)}(jc, -j\xi_1) - Y_2 \tilde{A}_n R_{1n}^{(1)}(jc, -j\xi_1)} \tag{22}$$

where

$$\tilde{A}_n = \frac{R_{1n}^{(4)'}(jc, -j\xi_1) + \frac{\xi_1}{\xi_1^2+1} R_{1n}^{(4)}(jc, -j\xi_1)}{R_{1n}^{(1)'}(jc, -j\xi_1) + \frac{\xi_1}{\xi_1^2+1} R_{1n}^{(1)}(jc, -j\xi_1)} \tag{23}$$

and the prime means the derivative with respect to ξ_1. Note that $\tilde{c}_{en} = 1$ and $\tilde{a}_{en} = 0$ if $Y_1 = Y_2$.

For a magnetic dipole located at $\xi = \xi_o \geq \xi_1$, $\eta = 1$ on the positive z axis and axially oriented, corresponding to a primary Hertz vector given by (3), the electric field is everywhere oriented in the $\hat{\varphi}$ direction. The primary electric field E_φ^i, the scattered electric field E_φ^s in the region $\xi \geq \xi_1$ outside the spheroid, and the total electric field $E_{2\varphi}$ inside the spheroid $(0 \leq \xi \leq \xi_1)$ are given by

$$\begin{aligned} E_\varphi^i &= k^2 c Z_1[(kR)^{-2} - j(kR)^{-3}]\sqrt{(\xi^2+1)(1-\eta^2)}e^{-jkR} \\ &= \frac{-2k^2 Z_1}{\sqrt{\xi_o^2+1}} \sum_{n=0}^{\infty} \frac{j^n}{\tilde{\rho}_{1n}\tilde{N}_{1n}} R_{1n}^{(1)}(jc, -j\xi_<) \\ &\quad \times R_{1n}^{(4)}(jc, -j\xi_>) S_{1n}(jc, \eta) \end{aligned} \tag{24}$$

$$\begin{aligned} E_\varphi^s &= \frac{-2k^2 Z_1}{\sqrt{\xi_o^2+1}} \sum_{n=0}^{\infty} \frac{j^n}{\tilde{\rho}_{1n}\tilde{N}_{1n}} \tilde{a}_{hn} R_{1n}^{(4)}(jc, -j\xi) \\ &\quad \times R_{1n}^{(4)}(jc, -j\xi_o) S_{1n}(jc, \eta) \end{aligned} \tag{25}$$

$$\begin{aligned} E_{2\varphi} &= \frac{-2k^2 Z_2}{\sqrt{\xi_o^2+1}} \sum_{n=0}^{\infty} \frac{j^n}{\tilde{\rho}_{1n}\tilde{N}_{1n}} \tilde{c}_{hn} R_{1n}^{(1)}(jc, -j\xi) \\ &\quad \times R_{1n}^{(4)}(jc, -j\xi_o) S_{1n}(jc, \eta). \end{aligned} \tag{26}$$

The modal coefficients $\tilde{a}_{hn}$ and $\tilde{c}_{hn}$ are found by imposing the boundary conditions at $\xi = \xi_1$

$$\tilde{c}_{hn} = 1 + \tilde{a}_{hn}\tilde{A}_n \tag{27}$$

$$\tilde{a}_{hn} = \frac{(Z_2 - Z_1)R_{1n}^{(1)}(jc, -j\xi_1)}{Z_1 R_{1n}^{(4)}(jc, -j\xi_1) - Z_2 \tilde{A}_n R_{1n}^{(1)}(jc, -j\xi_1)} \tag{28}$$

where $\tilde{A}_n$ is given by (23). As expected, $\tilde{c}_{hn} = 1$ and $\tilde{a}_{hn} = 0$ when $Z_1 = Z_2$. Observe that $\tilde{a}_{hn}$ is obtained from $\tilde{a}_{en}$ by duality, i.e., by replacing Y_ℓ with Z_ℓ, $(\ell = 1, 2)$.

IV. The Isorefractive Paraboloid

The parabolic coordinates (ξ, η, φ) are related to the rectangular coordinates (x, y, z) by the transformation

$$\begin{cases} x &= 2\sqrt{\xi\eta}\cos\varphi \\ y &= 2\sqrt{\xi\eta}\sin\varphi \\ z &= \xi - \eta \end{cases} \tag{29}$$

where $0 \leq \xi < \infty$, $0 \leq \eta < \infty$, and $0 \leq \varphi < 2\pi$. The z axis is the axis of symmetry. The surfaces $\xi =$ constant and $\eta =$ constant are paraboloids of revolution with foci at the origin $x = y = z = 0$ and the surfaces $\varphi =$ constant are semi-planes originating in the z axis.

The paraboloid $\eta = \eta_1$ separates two regions of space; the volume $\eta_1 \leq \eta < \infty$ is filled with a linear, homogeneous, and isotropic medium of permittivity ε_1 and permeability μ_1; the volume $0 \leq \eta \leq \eta_1$ is filled with a linear, homogeneous, and isotropic medium of permittivity ε_2 and permeability μ_2. The isorefractive condition (1) applies. The source is an electric or magnetic dipole located at $\xi_o = 0$, $\eta_o \geq \eta_1$ on the negative z axis and axially oriented.

For an electric dipole with a primary Hertz vector given by (3), the magnetic field is everywhere directed along $\hat{\varphi}$. The incident magnetic field is

$$\begin{aligned} H_\varphi^i = \frac{-Y_1}{2\eta_o\sqrt{\xi\eta}} \int_{-\gamma-j\infty}^{-\gamma+j\infty} d\tau \frac{\tau}{\sin\pi\tau} W_{-\tau,\frac{1}{2}}(j2k\xi) \\ \times M_{\tau,\frac{1}{2}}(j2k\eta_<) W_{\tau,\frac{1}{2}}(j2k\eta_>) \end{aligned} \tag{30}$$

where $|\gamma| < 1$ and $\eta_<(\eta_>)$ is the smaller (larger) between η and η_o. The functions M and W are Whittaker functions; their relations to other eigenfunctions introduced by Pinney, Buchholz, and Fock may be found in [6, ch. 16]. The scattered magnetic field H_φ^s in $\eta_1 \leq \eta < \infty$ on the convex side of the paraboloidal interface $\eta = \eta_1$ and the total magnetic field $H_{2\varphi}$ in $0 \leq \eta \leq \eta_1$ on the concave side of the interface are given by

$$\begin{aligned} H_\varphi^s = \frac{-Y_1}{2\eta_o\sqrt{\xi\eta}} \int_{-\gamma-j\infty}^{-\gamma+j\infty} d\tau \frac{\tau}{\sin\pi\tau} b_\tau^s W_{-\tau,\frac{1}{2}}(j2k\xi) \\ \times W_{\tau,\frac{1}{2}}(j2k\eta_o) W_{\tau,\frac{1}{2}}(j2k\eta) \end{aligned} \tag{31}$$

$$\begin{aligned} H_{2\varphi} = \frac{-Y_2}{2\eta_o\sqrt{\xi\eta}} \int_{-\gamma-j\infty}^{-\gamma+j\infty} d\tau \frac{\tau}{\sin\pi\tau} b_{2\tau} W_{-\tau,\frac{1}{2}}(j2k\xi) \\ \times W_{\tau,\frac{1}{2}}(j2k\eta_o) M_{\tau,\frac{1}{2}}(j2k\eta) \end{aligned} \tag{32}$$

where b_τ^s and $b_{2\tau}$ are functions of τ to be determined by imposing the boundary conditions at the interface $\eta = \eta_1$. The above expressions are a generalization of the results obtained by Buchholz in 1948 for the perfectly conducting paraboloid, as reported in [6].

The total tangential electric field at the interface $\eta = \eta_1$ is

$$\begin{aligned} (E_\xi^i + E_\xi^s)_{\eta=\eta_1+} = \frac{-jZ_1}{k}\sqrt{\frac{\eta_1}{\xi+\eta_1}}\left(\frac{1}{2\eta_1} + \frac{\partial}{\partial\eta_1}\right) \\ \cdot (H_\varphi^i + H_\varphi^s)_{\eta=\eta_1} \end{aligned} \tag{33}$$

$$(E_{2\xi})_{\eta=\eta_1-} = \frac{-jZ_2}{k}\sqrt{\frac{\eta_1}{\xi+\eta_1}}\left(\frac{1}{2\eta_1} + \frac{\partial}{\partial\eta_1}\right)(H_{2\varphi})_{\eta=\eta_1}. \tag{34}$$

Continuity of E_ξ and H_φ across the interface $\eta = \eta_1$ yields

$$b_{2\tau} = 1 + b_\tau^s \frac{W'_{\tau,\frac{1}{2}}(j2k\eta_1)}{M'_{\tau,\frac{1}{2}}(j2k\eta_1)} \tag{35}$$

$$b_\tau^s = \frac{Y_2 - Y_1}{Y_1 \frac{W_{\tau,\frac{1}{2}}(j2k\eta_1)}{M_{\tau,\frac{1}{2}}(j2k\eta_1)} - Y_2 \frac{W'_{\tau,\frac{1}{2}}(j2k\eta_1)}{M'_{\tau,\frac{1}{2}}(j2k\eta_1)}} \tag{36}$$

where the prime means derivative with respect to η_1. Observe that $b_\tau^s = 0$ and $b_{2\tau} = 1$ for $Y_1 = Y_2$, as expected.

For a magnetic dipole with a primary Hertz vector given by (3), the electric field is everywhere directed along $\hat{\varphi}$. The incident and scattered electric fields E_φ^i and E_φ^s in $\infty > \eta \geq \eta_1$ and the total electric field $E_{2\varphi}$ in $0 \leq \eta \leq \eta_1$ are obtained from H_φ^i, H_φ^s and $H_{2\varphi}$ of (30)–(32), respectively, by replacing

$Y_{1,2}$ with $-Z_{1,2}$, b_τ^s with d_τ^s, and $b_{2\tau}$ with $d_{2\tau}$. The boundary conditions at $\eta = \eta_1$ yield

$$d_{2\tau} = 1 + d_\tau^s \frac{W'_{\tau,\frac{1}{2}}(j2k\eta_1)}{M'_{\tau,\frac{1}{2}}(j2k\eta_1)} \tag{37}$$

$$d_\tau^s = \frac{Z_2 - Z_1}{Z_1 \frac{W_{\tau,\frac{1}{2}}(j2k\eta_1)}{M_{\tau,\frac{1}{2}}(j2k\eta_1)} - Z_2 \frac{W'_{\tau,\frac{1}{2}}(j2k\eta_1)}{M'_{\tau,\frac{1}{2}}(j2k\eta_1)}}. \tag{38}$$

For $Z_1 = Z_2$, we have that $d_\tau^s = 0$ and $d_{2\tau} = 1$. Also, d_τ^s is obtained from b_τ^s by replacing $Y_{1,2}$ with $Z_{1,2}$.

The exact scattering of an axially incident plane wave by an isorefractive paraboloid has been obtained previously [2]; the remarkable fact about this exact solution is that it coincides with geometrical optics.

V. The Isorefractive Circular Cone

With reference to spherical polar coordinates (r, θ, ϕ), consider the conical surface $\theta = \theta_1$, which separates the portion of space $0 \leq \theta \leq \theta_1$ filled with a medium of constitutive parameters ε_1, μ_1 from the region of space $\theta_1 \leq \theta \leq \pi$ filled with a medium of constitutive parameters ε_2, μ_2. The semi-aperture angle θ_1 of the cone may take any value in the range $\frac{\pi}{2} \leq \theta_1 < \pi$ and the isorefractive condition (1) applies.

A radially oriented electric dipole located at $\underline{r}_o \equiv (r_o, \theta_o, \varphi_o)$, $0 \leq \theta_o \leq \theta_1$, with moment $\hat{r}_o 4\pi\varepsilon_1/k$ corresponding to an incident electric Hertz vector

$$\underline{\pi}^i = \hat{r}_o \exp(-jkR)/(kR) \tag{39}$$

leads to a total (incident $\underline{E}^i$ plus scattered $\underline{E}^s$) electric field in $0 \leq \theta \leq \theta_1$

$$\begin{aligned} \underline{E}_1(\underline{r}) &= \underline{E}^i(\underline{r}) + \underline{E}^s(\underline{r}) \\ &= \left\{ \hat{r}\left(\frac{\partial^2}{\partial r^2} + k^2\right) + \frac{\hat{\theta}}{r}\frac{\partial^2}{\partial r \partial\theta} + \frac{\hat{\varphi}}{r\sin\theta}\frac{\partial^2}{\partial r\partial\varphi} \right\} \\ &\quad \cdot \left(\frac{\partial^2}{\partial r_o^2} + k^2\right) r r_o u_{1e} \end{aligned} \tag{40}$$

and to a total electric field $\underline{E}_2(\underline{r})$ in $\theta_1 \leq \theta \leq \pi$ that is given by (40) with u_{1e} replaced by u_{2e}. The corresponding magnetic fields are

$$\underline{H}_\ell(\underline{r}) = jkY_\ell \left(\frac{\hat{\theta}}{\sin\theta}\frac{\partial}{\partial\varphi} - \hat{\varphi}\frac{\partial}{\partial\theta}\right)\left(\frac{\partial^2}{\partial r_o^2} + k^2\right) r_o u_{\ell e}, \quad \ell = 1, 2. \tag{41}$$

The functions $u_{\ell e}$ are given by

$$\begin{aligned} u_{\ell e} &= \sum_{m=0}^{\infty} \frac{\varepsilon_m}{2}\cos m(\varphi - \varphi_o) \\ &\quad \times \int_c d\nu \frac{(2\nu+1)\Gamma(\nu+m+1)}{\nu(\nu+1)\Gamma(\nu-m+1)\sin(\nu-m)\pi} \\ &\quad \times j_\nu(kr_<) h_\nu^{(2)}(kr_>) f_{\ell e} \end{aligned} \tag{42}$$

where $\varepsilon_0 = 1$, $\varepsilon_m = 2$ for all positive m,

$$f_{1e} = P_\nu^{-m}(\cos\theta_<)\left[P_\nu^{-m}(-\cos\theta_>) + a_{\mathrm{em}} P_\nu^{-m}(\cos\theta_>)\right] \tag{43}$$

$$f_{2e} = c_{\mathrm{em}} P_\nu^{-m}(\cos\theta_o) P_\nu^{-m}(-\cos\theta) \tag{44}$$

and $\theta_<(\theta_>)$ is the smaller (larger) between) θ and θ_o. The integration contour C encloses the positive $\mathrm{Re}\, v$ axis in the clockwise direction and j_ν and $h_\nu^{(2)}$ are the spherical Bessel function and the spherical Hankel function of the second kind, respectively:

$$j_\nu(\xi) = \sqrt{\frac{\pi}{2\xi}} J_{\nu+\frac{1}{2}}(\xi), \quad h_\nu^{(2)}(\xi) = \sqrt{\frac{\pi}{2\xi}} H_{\nu+\frac{1}{2}}^{(2)}(\xi). \tag{45}$$

The above solution is akin to the solution obtained by Felsen in 1957 for the perfectly conducting cone (see chapter 18 in [6]). It is useful to remember that for $\mathrm{Re}\, q > 0$, the solutions of the associated Legendre equation behave as follows (see, e.g., [7, p. 315]): $P_\nu^{-q}(\cos\theta)$ is bounded at $\theta = 0$, but not at $\theta = \pi$; $P_\nu^{-q}(-\cos\theta)$ is bounded at $\theta = \pi$, but not at $\theta = 0$.

The modal coefficients a_{em} and c_{em} are found by imposing the boundary conditions at $\theta = \theta_1$

$$c_{em} = 1 + a_{em}\frac{P_\nu^{-m}(\cos\theta_1)}{P_\nu^{-m}(-\cos\theta_1)} \tag{46}$$

$$a_{em} = \frac{(Y_2 - Y_1)\frac{\partial}{\partial\theta_1}P_\nu^{-m}(-\cos\theta_1)}{Y_1\frac{\partial}{\partial\theta_1}P_\nu^{-m}(\cos\theta_1) - Y_2\frac{P_\nu^{-m}(\cos\theta_1)}{P_\nu^{-m}(-\cos\theta_1)}\frac{\partial}{\partial\theta_1}P_\nu^{-m}(-\cos\theta_1)}. \tag{47}$$

Observe that if $Y_1 = Y_2$, then $a_{em} = 0$ and $c_{em} = 1$, as expected.

The particular case $\theta_1 = \pi/2$, when the cone becomes a planar interface separating two isorefractive materials of semi-infinite extent, deserves special attention. From the formulas (see, e.g., [8, p. 1009])

$$P_\nu^{-m}(0) = \frac{2^{-m}\pi^{1/2}}{\Gamma\left(\frac{\nu+m}{2}+1\right)\Gamma\left(\frac{-\nu+m+1}{2}\right)} \tag{48}$$

$$\left[\frac{\partial}{\partial\theta_1}P_\nu^{-m}(\pm\cos\theta_1)\right]_{\theta_1=\pi/2} = \frac{\mp 2^{1-m}\sin\left(\frac{\nu-m}{2}\pi\right)\Gamma\left(\frac{\nu-m}{2}+1\right)}{\pi^{1/2}\Gamma\left(\frac{\nu+m+1}{2}\right)} \tag{49}$$

it follows that

$$(a_{em})_{\theta_1=\pi/2} = \frac{Y_1 - Y_2}{Y_1 + Y_2} \quad (c_{em})_{\theta_1=\pi/2} = \frac{2Y_1}{Y_1 + Y_2} \tag{50}$$

which are the reflection and transmission coefficients for the electric field at the interface, respectively.

A radially oriented magnetic dipole located at $\underline{r}_o \equiv (r_o, \theta_o, \varphi_o)$, $0 \leq \theta_o \leq \theta_1$, with moment $\hat{r}_o 4\pi/k$ corresponding to the incident magnetic Hertz vector (39), yields a total (incident plus scattered) magnetic field in $0 \leq \theta \leq \theta_1$

$$\underline{H}_1(\underline{r}) = \underline{H}^i(\underline{r}) + \underline{H}^s(\underline{r}) \tag{51}$$

which is given by the right-hand side of (40) with u_{1e} replaced by u_{1h} and yields a total magnetic field $\underline{H}_2(\underline{r})$ in $\theta_1 \leq \theta \leq \pi$

that is given by the right-hand side of (40) with u_{1e} replaced by u_{2h}. The corresponding electric fields are, respectively

$$\underline{E}_\ell(\underline{r}) = -jkZ_\ell\left(\frac{\hat{\theta}}{\sin\theta}\frac{\partial}{\partial\varphi} - \hat{\varphi}\frac{\partial}{\partial\theta}\right)\left(\frac{\partial^2}{\partial r_o^2}+k^2\right)r_o u_{\ell h}, \quad \ell = 1,2. \tag{52}$$

The functions $u_{\ell h}$ are obtained from $u_{\ell e}$ of (42) by replacing $f_{\ell e}$ with $f_{\ell h}$, where, in turn, $f_{\ell h}$ is obtained from $f_{\ell e}$ of (43) and (44) by replacing a_{em} with a_{hm} and c_{em} with c_{hm}. Imposition of the boundary conditions yields

$$c_{hm} = 1 + a_{hm}\frac{P_\nu^{-m}(\cos\theta_1)}{P_\nu^{-m}(-\cos\theta_1)} \tag{53}$$

$$a_{hm} = \frac{(Z_2 - Z_1)\frac{\partial}{\partial\theta_1}P_\nu^{-m}(-\cos\theta_1)}{Z_1\frac{\partial}{\partial\theta_1}P_\nu^{-m}(\cos\theta_1) - Z_2\frac{P_\nu^{-m}(\cos\theta_1)}{P_\nu^{-m}(-\cos\theta_1)}\frac{\partial}{\partial\theta_1}P_\nu^{-m}(-\cos\theta_1)}. \tag{54}$$

For $Z_1 = Z_2$, $a_{hm} = 0$ and $c_{hm} = 1$, as expected. The particular case of a planar interface ($\theta_1 = \pi/2$) yield

$$(a_{hm})_{\theta_1=\pi/2} = \frac{Z_1 - Z_2}{Z_1 + Z_2}, \quad (c_{hm})_{\theta_1=\frac{\pi}{2}} = \frac{2Z_1}{Z_1+Z_2} \tag{55}$$

which are the reflection and transmission coefficients for the magnetic field at the interface, respectively.

Let us now consider an incident plane wave

$$\begin{cases}\underline{E}^i &= (\hat{\theta}_o\sin\beta + \hat{\varphi}_o\cos\beta)\exp(-jk\hat{k}^i\cdot\underline{r})\\ \underline{H}^i &= Y_1(\hat{\theta}_o\cos\beta - \hat{\varphi}_o\sin\beta)\exp(-jk\hat{k}^i\cdot\underline{r})\end{cases} \tag{56}$$

where $\hat{k}^i = -\hat{r}_o = \hat{\varphi}_o \times \hat{\theta}_o, \quad 0 \le \theta_o \le \theta_1$, and

$$\begin{aligned}\hat{\theta}_o &= \hat{x}\cos\theta_o\cos\varphi_o + \hat{y}\cos\theta_o\sin\varphi_o - \hat{z}\sin\theta_o\\ \hat{\varphi}_o &= -\hat{x}\sin\varphi_o + \hat{y}\cos\varphi_o.\end{aligned} \tag{57}$$

The total electric $\underline{E}_1$ and magnetic $\underline{H}_1$ fields in the region $0 \le \theta \le \theta_1$ are

$$\begin{aligned}\underline{E}_1 &= \underline{E}^i + \underline{E}^s\\ &= -jkZ_1\left(\frac{\hat{\theta}}{\sin\theta}\frac{\partial}{\partial\varphi} - \hat{\varphi}\frac{\partial}{\partial\theta}\right)v_1\\ &\quad + \left[\hat{r}\left(\frac{\partial^2}{\partial r^2}+k^2\right) + \frac{\hat{\theta}}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\hat{\varphi}}{r\sin\theta}\frac{\partial^2}{\partial r\partial\varphi}\right](ru_1)\end{aligned} \tag{58}$$

$$\begin{aligned}\underline{H}_1 &= \underline{H}^i + \underline{H}^s\\ &= jkY_1\left(\frac{\hat{\theta}}{\sin\theta}\frac{\partial}{\partial\varphi} - \hat{\varphi}\frac{\partial}{\partial\theta}\right)u_1\\ &\quad + \left[\hat{r}\left(\frac{\partial^2}{\partial r^2}+k^2\right) + \frac{\hat{\theta}}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\hat{\varphi}}{r\sin\theta}\frac{\partial^2}{\partial r\partial\varphi}\right](rv_1)\end{aligned} \tag{59}$$

whereas in the region $\theta_1 \le \theta \le \pi$, the total fields $\underline{E}_2$ and $\underline{H}_2$ are obtained from $\underline{E}_1$ and $\underline{H}_1$, respectively, by replacing Z_1 with Z_2, u_1 with u_2, and v_1 with v_2.

The Debye potentials u_ℓ and $v_\ell (\ell = 1,2)$ are given by

$$\begin{aligned}u_\ell = \frac{1}{2k}\sum_{m=0}^{\infty}\varepsilon_m\int_c d\nu\frac{(2\nu+1)\Gamma(\nu+m+1)e^{j\frac{\pi}{2}\nu}}{\nu(\nu+1)\Gamma(\nu-m+1)\sin(\nu-m)\pi}\\ \times j_\nu(kr)\cdot\left[m\sin m(\varphi-\varphi_o)\cos\beta\frac{f_{1\ell}}{\sin\theta_o}\right.\\ \left.+\cos m(\varphi-\varphi_o)\sin\beta\frac{\partial f_{1\ell}}{\partial\theta_o}\right]\end{aligned} \tag{60}$$

$$\begin{aligned}v_\ell = \frac{Y_\ell}{2k}\sum_{m=0}^{\infty}\varepsilon_m\int_c d\nu\frac{(2\nu+1)\Gamma(\nu+m+1)e^{j\frac{\pi}{2}\nu}}{\nu(\nu+1)\Gamma(\nu-m+1)\sin(\nu-m)\pi}\\ \times j_\nu(kr)\cdot\left[\cos m(\varphi-\varphi_o)\cos\beta\frac{\partial f_{2\ell}}{\partial\theta_o}\right.\\ \left.-m\sin m(\varphi-\varphi_o)\sin\beta\frac{f_{2\ell}}{\sin\theta_o}\right]\end{aligned} \tag{61}$$

where f_{11} is given by f_{1e} of (43) with a_{em} replaced by a new modal coefficient a_m, f_{21} is given by f_{1e} of (43) with a_{em} replaced by b_m, f_{12} is given by f_{2e} of (44) with c_{em} replaced by c_m, and f_{22} is given by f_{2e} of (44) with c_{em} replaced by d_m. The solution (57)–(60) is similar to that given by Felsen in 1957–1958 for a metallic cone, as reported in [6].

By imposing the boundary conditions, it is found that the modal coefficients a_m, b_m, c_m, and d_m are given by

$$c_m = 1 + a_m\frac{P_\nu^{-m}(\cos\theta_1)}{P_\nu^{-m}(-\cos\theta_1)} \tag{62}$$

$$d_m = 1 + b_m\frac{\frac{\partial}{\partial\theta_1}P_\nu^{-m}(\cos\theta_1)}{\frac{\partial}{\partial\theta_1}P_\nu^{-m}(-\cos\theta_1)} \tag{63}$$

a_m is equal to a_{em} of (47) [consequently, c_m equals c_{em} of (46)] and b_m is equal to a_{hm} of (54).

It should be noted that the case of metallic cone ($Z_2 = 0$) cannot be expected to follow as a particular case of the above result for the isorefractive cone because of condition (1). Nevertheless, in the limit $Z_2 \to 0$, the above isorefractive solution yields the correct result for the metallic cone.

The contour integrals in (42), (60), and (61) may be evaluated as a series of pole contributions, using Cauchy's residue theorem. The results would be akin to those obtained by Bailin and Silver [9] for the metallic cone. This procedure was followed in [3] for the isorefractive wedge.

VI. Conclusion

In this paper, we have solved several new canonical problems involving radiation or scattering by penetrable bodies of revolution that are isorefractive to the surrounding medium. These new, exact solutions are important not only because they enrich the catalog of exact solutions for penetrable bodies, but also for two additional reasons. First, they provide limiting cases to test the correctness of analytical solutions that may be developed in the future for bodies with the same shapes treated here, but made of more general materials. Second, they provide test cases for the validation of numerical codes developed for penetrable bodies whose boundary surfaces have varying radii of curvature and/or singularities.

To our knowledge, and with the exception of the result reported in [2], the results given herein are the only existing exact solutions for three-dimensional penetrable bodies whose boundary surface is not a sphere.

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