

Decomposition of Electromagnetic Boundary Conditions at Planar Interfaces with Applications to TE and TM Field Solutions

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Abstract—Electromagnetic fields in homogeneous source-free regions can be decomposed into fields that are TE and TM with respect to a particular reference direction (e.g., the z direction). If transverse sources exist, both TE and TM fields may be excited simultaneously. This paper considers the case of two infinite regions having a common planar interface and prescribed sources (surface currents) on the interface. The source currents are decomposed in a manner consistent with the decomposition of the fields. Accordingly, a procedure is established for describing the boundary conditions at the interface in terms of the longitudinal field components E_z , H_z , and the surface currents \bar{J}_s . The development is unique in that the continuity of the transverse field components at the boundary are not explicitly considered but interpreted in terms of z -directed fields. This boundary condition approach is shown to give results consistent with those obtained by matching the tangential fields at the interface using vector transforms. A simple example illustrating the procedure using a ring of current in free-space is presented.

Index Terms— Boundary value problems, electromagnetic fields.

I. INTRODUCTION

THE problem of determining solutions for electromagnetic fields in two different source-free homogeneous regions having sources on a common planar interface is of great interest to researchers. It has important applications to such problems as the analysis of microstrip antennas. One method of analysis decomposes the electromagnetic fields in each region into fields that are TE and TM with respect to an axis normal to the boundary (the z axis) [1] while meeting the boundary conditions at the interface. Such a geometry and decomposition is illustrated in Fig. 1.

The longitudinal components of the fields E_z and H_z may both be excited by transverse sources [2]; the corresponding transverse components of the fields can be derived from the z components. In Section II of this paper, a brief review of the relationship between the transverse and longitudinal field components is presented. The boundary conditions at the planar interface are decomposed into equivalent conditions which specify the two-dimensional (2-D) divergence and curl of the transverse field components in planes tangent to the

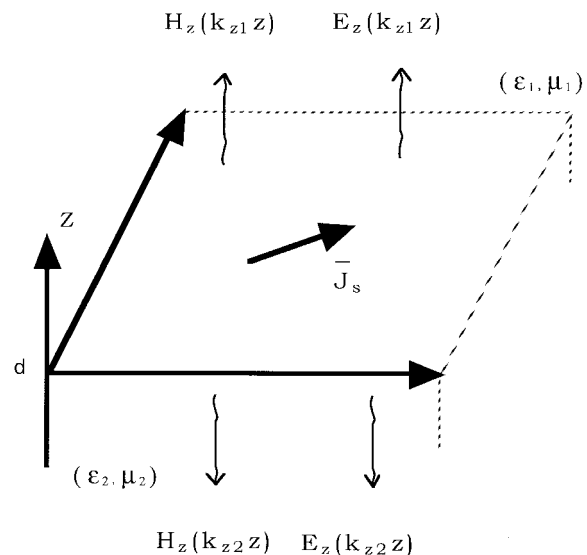


Fig. 1. The z components of an electromagnetic field excited by a prescribed current source.

boundary. This decomposition is then reinterpreted in terms of z -directed field components. Accordingly, the boundary condition decomposition conforms to the E_z , H_z , field decomposition in the source-free homogeneous regions.

Two-dimensional integral transforms can be used to obtain exact integral solutions for fields satisfying the boundary conditions at the interface. Assuming a general 2-D transform, spectral representations of the boundary conditions are presented in Section III. These representations, which require a z -directed coordinate axis, are demonstrated to be consistent with the vector transform method of meeting boundary conditions proposed by Chew *et al.* [3]–[4]. As an illustration, the cylindrical coordinate system is specified and the representation of the boundary conditions using the approach of this paper is demonstrated to be identical to that found using a Vector Hankel transform. In Section IV, the fields generated by a ring of current in free-space are determined illustrating the convenience of the paper's results.

II. DECOMPOSITION OF ELECTROMAGNETIC FIELDS AND THEIR BOUNDARY CONDITIONS

In a homogeneous source-free region the transverse electromagnetic fields may be determined from the z -directed field

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components \bar{E}_z and \bar{H}_z . A brief overview of the procedure is presented here. Maxwell's source-free curl equations

$$\nabla \times \bar{E} = -j\omega\mu\bar{H} \quad (1)$$

$$\nabla \times \bar{H} = +j\omega\epsilon\bar{E} \quad (2)$$

may be expanded into the following form:

$$\left(\bar{z}_0 \frac{\partial}{\partial z} + \nabla_t\right) \times (\bar{z}_0 E_z + \bar{E}_t) = -j\omega\mu(\bar{z}_0 H_z + \bar{H}_t) \quad (3)$$

$$\left(\bar{z}_0 \frac{\partial}{\partial z} + \nabla_t\right) \times (\bar{z}_0 H_z + \bar{H}_t) = +j\omega\epsilon(\bar{z}_0 E_z + \bar{E}_t) \quad (4)$$

where ∇_t denotes transverse derivatives and \bar{E}_t and \bar{H}_t are the transverse components of the electric and magnetic fields.

Equations (3) and (4) may be solved for \bar{E}_t and \bar{H}_t in terms of \bar{E}_z and \bar{H}_z

$$\left[\frac{\partial^2}{\partial z^2} + k^2\right] \bar{E}_t = \left[\nabla_t \frac{\partial}{\partial z} E_z + j\omega\mu\bar{z}_0 \times (\nabla_t H_z)\right] \quad (5)$$

$$\left[\frac{\partial^2}{\partial z^2} + k^2\right] \bar{H}_t = \left[\nabla_t \frac{\partial}{\partial z} H_z - j\omega\epsilon\bar{z}_0 \times (\nabla_t E_z)\right] \quad (6)$$

E_z and H_z are required to satisfy the scalar Helmholtz equation.

Our development pertains to Fig. 1, which is unbounded in the transverse directions, although this does not necessarily need to be the case. For example, modal field solutions (fixed k_z) for a bounded structure such as a waveguide can be developed using this methodology. This paper uses the term "modal" in the broadest sense where the guiding structure may be thought of as having walls that recede to infinity allowing the "modes" to coalesce into a continuous spectrum. Assuming a *modal* field solution for which the method of separation of variables applies, the wavenumber will satisfy the separation equation

$$k^2 = k_z^2 + k_t^2, \quad k^2 = \epsilon_r \mu_r k_0^2, \quad k_0^2 = \omega^2 \epsilon_0 \mu_0. \quad (7)$$

In (7), the component k_t^2 contains wavenumbers associated with the transverse geometry of the problem (e.g., $k_t^2 = k_x^2 + k_y^2$ in Cartesian coordinates and $k_t^2 = k_\rho^2$ in cylindrical coordinates).

For the same separable solution, second-order derivatives with respect to "z" may be obtained by the replacement

$$\frac{\partial^2}{\partial z^2} = -k_z^2. \quad (8)$$

The equations for the transverse modal fields become

$$k_t^2 \bar{E}_t = \left[\nabla_t \frac{\partial}{\partial z} E_z + j\omega\mu\bar{z}_0 \times (\nabla_t H_z)\right] \quad (9)$$

$$k_t^2 \bar{H}_t = \left[\nabla_t \frac{\partial}{\partial z} H_z - j\omega\epsilon\bar{z}_0 \times (\nabla_t E_z)\right]. \quad (10)$$

Therefore, the transverse field components are completely determined from the z components. The fields given by (9) and (10) exclude contributions of waves TEM with respect to z^1 .

¹ Formally, such waves may be included by setting $E_z = k_t^2 F$.

The boundary conditions that must be enforced for the tangential components of an electromagnetic field across a planar interface are given in (11) and (12)

$$\bar{z}_0 \times [\bar{H}_t(d^+) - \bar{H}_t(d^-)] = \bar{J}_s \quad (11)$$

$$\bar{z}_0 \times [\bar{E}_t(d^+) - \bar{E}_t(d^-)] = -\bar{M}_s = 0. \quad (12)$$

Our development explicitly employs the simple electric surface current \bar{J}_s of (11). The surface current \bar{J}_s may be generalized to include axial magnetic currents as shown by Michalski [5, eq. (10b)]. The alternate solution for transverse magnetic currents and axial electric currents is easily obtained using duality [1].

Equations (11) and (12) do not explicitly indicate the manner in which the z components of the fields are related to prescribed transverse excitation(s) at the boundary. Since we wish to work in terms of z components and derive the transverse components from the z -component solution, it is desirable to develop an equivalent set of boundary condition equations which *are* interpreted directly in terms of the z components.

Consider, for example, the boundary condition specified by (11). This equation defines the limiting behavior of vector fields (vector point functions) in the transverse plane. According to Helmholtz's theorem (in 2-D form), these transverse vectors are completely determined to within an additive constant if their transverse divergence and curl are specified everywhere in the plane. The additive constant, corresponding to a static field unrelated to the prescribed excitation, may be set to zero without altering the results of the problem. Therefore, the 2-D divergence and curl equivalent to (11) are

$$\nabla_t \cdot [\bar{z}_0 \times [\bar{H}_t(d^+) - \bar{H}_t(d^-)]] = \nabla_t \cdot \bar{J}_s \quad (13)$$

$$\nabla_t \times [\bar{z}_0 \times [\bar{H}_t(d^+) - \bar{H}_t(d^-)]] = \nabla_t \times \bar{J}_s. \quad (14)$$

Using (A.3) and (A.4) from Appendix A, (13) and (14) may be written as

$$-\bar{z}_0 \cdot [\nabla_t \times [\bar{H}_t(d^+) - \bar{H}_t(d^-)]] = \nabla_t \cdot \bar{J}_s \quad (15)$$

$$\bar{z}_0 [\nabla_t \cdot [\bar{H}_t(d^+) - \bar{H}_t(d^-)]] = \nabla_t \times \bar{J}_s. \quad (16)$$

Equations (15) and (16) can be interpreted in terms of the z components of the electric and magnetic fields. On inspecting (2) or (4) it is evident that

$$\nabla_t \times \bar{H}_t = j\omega\epsilon\bar{E}_z \quad (17)$$

the transverse curl of the tangential magnetic field is interpreted in terms of the z component of the electric field. Substituting (17) into (15) gives

$$-j\omega[\epsilon_1 E_z(d^+) - \epsilon_2 E_z(d^-)] = \nabla_t \cdot \bar{J}_s. \quad (18)$$

In homogeneous source-free media

$$\nabla \cdot \bar{H} = 0. \quad (19)$$

When the z and transverse components are separated

$$\nabla_t \cdot \bar{H}_t = -\frac{\partial}{\partial z} H_z. \quad (20)$$

Consequently, the transverse divergence of the magnetic field at the interface is interpreted in terms of the derivative of the

z component of the magnetic field. Substituting (20) into (16) gives

$$\left[\frac{\partial}{\partial z} H_z(d^+) - \frac{\partial}{\partial z} H_z(d^-) \right] = -\bar{z}_0 \cdot (\nabla_t \times \bar{J}_s) = \nabla_t \cdot (\bar{z}_0 \times \bar{J}_s). \quad (21)$$

In (21), (A.3) has been used to express the transverse curl of the surface current in terms of a transverse divergence. This form is useful for analytical purposes. Equations (18) and (21) are, therefore, equivalent to the boundary condition of (11), our basic result.

An alternate set of equations relating z components equivalent to (12) must be developed. This is easily accomplished by applying the duality concept to (18) and (21) and then by setting the magnetic surface current \bar{M}_s equal to zero. The boundary conditions corresponding to the tangential electric fields become

$$[\mu_1 H_z(d^+) - \mu_2 H_z(d^-)] = 0 \quad (22)$$

$$\left[\frac{\partial}{\partial z} E_z(d^+) - \frac{\partial}{\partial z} E_z(d^-) \right] = 0. \quad (23)$$

Equations (18) and (21)–(23) constitute four independent equations which, presumably, contain four unknown coefficients. Given the fields on one side of the boundary (e.g., at $z = d^-$), they may be solved for the unknown fields at $z = d^+$. An example will be given in Section IV.

III. REPRESENTATION USING 2-D TRANSFORMS

Two-dimensional transforms (operating on the transverse plane) have been found of value in obtaining spectral solutions for \bar{E}_z and \bar{H}_z at the boundary of two regions. Here, a general approach for such a solution is presented which can later be specialized to any separable z -directed coordinate system of interest. Let L represent a 2-D transform from the spatial to the spectral domain (i.e., integrations are with respect to spatial transverse coordinates). Multiplication k_t^2 by and differentiation with respect to “ z ” may be interchanged with L . The transformed vectors are denoted with a “ \sim .” Let L^{-1} designate the inverse transform. Differential operations with respect to transverse spatial coordinates may be exchanged with L^{-1} . Using these transforms, (18) and (21) may be written in matrix form as

$$L^{-1} \begin{bmatrix} \frac{\partial}{\partial z} \tilde{H}_z(d^+) - \frac{\partial}{\partial z} \tilde{H}_z(d^-) \\ j\omega[\varepsilon_1 \tilde{E}_z(d^+) - \varepsilon_2 \tilde{E}_z(d^-)] \end{bmatrix} = L^{-1} \begin{bmatrix} L \nabla_t \cdot (\bar{z}_0 \times \bar{J}_s) \\ -L \nabla_t \cdot \bar{J}_s \end{bmatrix}. \quad (24)$$

This equation, representing the boundary condition approach advanced by this paper, can be written more explicitly upon once a coordinate system and the transforms L and L^{-1} are specified. Using cylindrical coordinates, let the surface current be prescribed as

$$\bar{J}_s = [(\mathcal{J}' \bar{\rho}_0 + \mathcal{J}^{\phi'} \bar{\phi}_0) \delta(\rho - \rho') \delta(\phi - \phi') / \rho] \quad (\text{A/m}). \quad (25)$$

where $\mathcal{J}' = \mathcal{J}^0 \cos(\alpha)$ and $\mathcal{J}^{\phi'} = \mathcal{J}^0 \sin(\alpha)$ with “ α ” being the angle describing the relationship between the $\bar{\rho}_0$ and $\bar{\phi}_0$ components of the current. The current moment \mathcal{J}^0 has dimensions of one (A · m).

Transforms for the cylindrical coordinate system (Fourier-Bessel Transforms) may now be used. Such a pair is given by

$$\begin{aligned} L\psi(\rho, \phi, z) &= \tilde{\psi}_n(k_\rho, z) \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty J_n(k_\rho \rho) \psi(\rho, \phi, z) e^{-jn\phi} \rho d\rho d\phi \end{aligned} \quad (26)$$

$$\begin{aligned} L^{-1}\tilde{\psi}_n(k_\rho, z) &= \psi(\rho, \phi, z) \\ &= \sum_{n=-\infty}^{n=+\infty} e^{jn\phi} \int_0^\infty J_n(k_\rho \rho) \tilde{\psi}_n(k_\rho, z) k_\rho dk_\rho. \end{aligned} \quad (27)$$

Substituting (26) and (27) into (24) and using the fact that

$$\int_{-\infty}^{+\infty} \delta'(x) f(x) dx = - \int_{-\infty}^{+\infty} \delta(x) f'(x) dx \quad (28)$$

we find

$$\begin{aligned} &\sum_{n=-\infty}^{n=+\infty} e^{jn\phi} \int_0^\infty k_\rho J_n(k_\rho \rho) \left[j\omega[\varepsilon_1 \tilde{E}_z(d^+) - \varepsilon_2 \tilde{E}_z(d^-)] \right] dk_\rho \\ &= \frac{1}{2\pi} \sum_{n=-\infty}^{n=+\infty} e^{jn(\phi - \phi')} \int_0^\infty k_\rho^2 J_n(k_\rho \rho) \bar{J}_n(k_\rho \rho') \left[\frac{\mathcal{J}^{\phi'}}{\mathcal{J}^{\rho'}} \right] dk_\rho. \end{aligned} \quad (29)$$

In (29), we have followed the work of Chew and Habashy [4] where the kernel of their Vector Hankel transform is defined with

$$\bar{J}_n(k_\rho \rho) = \begin{bmatrix} +J'_n(k_\rho \rho) & +\frac{jn}{k_\rho \rho} J_n(k_\rho \rho) \\ -\frac{jn}{k_\rho \rho} J_n(k_\rho \rho) & +J'_n(k_\rho \rho) \end{bmatrix}. \quad (30)$$

Note that $J'_n(k_\rho \rho)$ denotes the derivative with respect to the argument of the Bessel function.

From (29) it is evident that we must have

$$\left[\frac{\partial}{\partial z} [\tilde{H}_z(d^+) - \tilde{H}_z(d^-)] \right] = \frac{1}{2\pi} k_\rho e^{-jn\phi'} \bar{J}_n(k_\rho \rho') \left[\frac{\mathcal{J}^{\phi'}}{\mathcal{J}^{\rho'}} \right]. \quad (31)$$

Not surprisingly, exactly the same result (31) may be found using the tangential components of the magnetic field at the boundary. The process is briefly outlined here. Equation (10) may be written as

$$L^{-1} L k_t^2 \bar{H}_t = \nabla_t L^{-1} L \frac{\partial}{\partial z} H_z - j\omega \varepsilon \bar{z}_0 \times (\nabla_t L^{-1} L E_z) \quad (32)$$

or (after exchanging orders of operation)

$$L^{-1} k_t^2 \tilde{\bar{H}}_t = L^{-1} \nabla_t \frac{\partial}{\partial z} \tilde{H}_z - L^{-1} j\omega \varepsilon \bar{z}_0 \times (\nabla_t \tilde{E}_z). \quad (33)$$

In (32) and (33), it is understood that *the transverse derivatives operate on the spatial coordinates* of the L^{-1} transform. Equation (33) may be solved for the transverse components of the magnetic field

$$\bar{H}_t = L^{-1} \tilde{\bar{H}}_t = L^{-1} \left[\frac{1}{k_t^2} \nabla_t \frac{\partial}{\partial z} \tilde{H}_z - j\omega \varepsilon \frac{1}{k_t^2} \bar{z}_0 \times (\nabla_t \tilde{E}_z) \right]. \quad (34)$$

The individual transverse components of the magnetic field at the boundary may now be written

$$\begin{aligned} & \begin{bmatrix} H_\rho(d^+) - H_\rho(d^-) \\ -(H_\phi(d^+) - H_\phi(d^-)) \end{bmatrix} \\ &= L^{-1} \frac{1}{k_t^2} \begin{bmatrix} +\bar{\rho}_0 \\ -\bar{\phi}_0 \end{bmatrix} \cdot [\nabla_t - \bar{z}_0 \times \nabla_t] \\ & \quad \times \begin{bmatrix} \frac{\partial}{\partial z} [\tilde{H}_z(d^+) - \tilde{H}_z(d^-)] \\ j\omega[\varepsilon_1 \tilde{E}_z(d^+) - \varepsilon_2 \tilde{E}_z(d^-)] \end{bmatrix} \\ &= L^{-1} \frac{1}{k_t^2} \begin{bmatrix} +\bar{\rho}_0 \\ -\bar{\phi}_0 \end{bmatrix} \cdot [\nabla_t - \bar{z}_0 \times \nabla_t] L \nabla_t \cdot \begin{bmatrix} \bar{z}_0 \times \bar{J}_s \\ -\bar{J}_s \end{bmatrix} \quad (35) \end{aligned}$$

where (35) expresses the boundary conditions met by the tangential magnetic fields at the interface.

Substituting (25)–(27) into (35), the following may be obtained:

$$\begin{aligned} & \sum_{n=-\infty}^{n=+\infty} e^{jn\phi} \int_0^\infty \bar{J}_n(k_\rho \rho) \begin{bmatrix} \frac{\partial}{\partial z} [\tilde{H}_z(d^+) - \tilde{H}_z(d^-)] \\ j\omega[\varepsilon_1 \tilde{E}_z(d^+) - \varepsilon_2 \tilde{E}_z(d^-)] \end{bmatrix} dk_\rho \\ &= \frac{1}{2\pi} \sum_{n=-\infty}^{n=+\infty} e^{jn(\phi-\phi')} \int_0^\infty k_\rho \bar{J}_n(k_\rho \rho) \bar{J}_n(k_\rho \rho') \begin{bmatrix} \mathcal{J}^{\phi'} \\ \mathcal{J}^{\rho'} \end{bmatrix} dk_\rho. \quad (36) \end{aligned}$$

Equation (36) may now be used to obtain a solution for using the boundary conditions of the tangential magnetic fields. Inspection of (36) reveals that it leads to the same requirement of (29) (i.e., (31)) as was expected. Consequently, the two techniques are equivalent.

IV. SAMPLE PROBLEM

The methodology developed so far may now be applied to a specific problem. The simple problem of the ring source with a radius of “ a ” will be examined. The regions above and below the ring are free-space and the ring is located at $z = 0$ on the x - y plane. The problem will be solved by determining the unknown spectral functions due to point source excitations. Once these are known, spectral representations of the Green’s functions are readily determined. The Green’s functions are then multiplied by the surface current distribution of a ring source and integrated over the primed (source) coordinates determining E_z and H_z .

To account for the (transformed) boundary conditions of (29), we assume general modal (transformed) solutions having undetermined (spectral) coefficients. Anticipating outgoing waves, such solutions for the electric and magnetic fields will be of the form

$$\tilde{E}_z = A_n(k_\rho) e^{-jk_z z} \quad z \geq 0 \quad (37)$$

$$\tilde{E}_z = B_n(k_\rho) e^{+jk_z z} \quad z \leq 0 \quad (38)$$

$$\tilde{H}_z = C_n(k_\rho) e^{-jk_z z} \quad z \geq 0 \quad (39)$$

$$\tilde{H}_z = D_n(k_\rho) e^{+jk_z z} \quad z \leq 0. \quad (40)$$

The boundary conditions specified in (22) and (23) will hold in the spectral domain. Consequently

$$C_n(k_\rho) = D_n(k_\rho) \quad (41)$$

$$A_n(k_\rho) = -B_n(k_\rho) \quad (42)$$

and the problem is reduced to determining two unknowns. Accordingly, the left-hand side of (24) becomes

$$\begin{bmatrix} \frac{\partial}{\partial z} [\tilde{H}_z(d^+) - \tilde{H}_z(d^-)] \\ j\omega[\varepsilon_1 \tilde{E}_z(d^+) - \varepsilon_2 \tilde{E}_z(d^-)] \end{bmatrix} = \begin{bmatrix} -j2k_z C_n(k_\rho) \\ j2\omega\varepsilon_0 A_n(k_\rho) \end{bmatrix}. \quad (43)$$

Using (31) which was developed using electric current point source excitation, the spectral functions $C_n(k_\rho)$ and $A_n(k_\rho)$ in (43) are explicitly determined

$$\begin{aligned} & \begin{bmatrix} C_n(k_\rho) \\ A_n(k_\rho) \end{bmatrix} \\ &= \frac{1}{4\pi} k_\rho e^{-jn\phi'} \begin{bmatrix} +\frac{j}{k_z} J'_n(k_\rho \rho') & -\frac{1}{k_z} \frac{n}{k_\rho \rho'} J_n(k_\rho \rho') \\ -\frac{1}{\omega\varepsilon_0} \frac{n}{k_\rho \rho'} J_n(k_\rho \rho') & -\frac{j}{\omega\varepsilon_0} J'_n(k_\rho \rho') \end{bmatrix} \\ & \quad \times \begin{bmatrix} \mathcal{J}^{\phi'} \\ \mathcal{J}^{\rho'} \end{bmatrix}. \quad (44) \end{aligned}$$

Since the spectral coefficients are found from point sources, the solutions for the z -directed electric and magnetic fields give the Green’s functions directly

$$\begin{aligned} & \begin{bmatrix} G_H^z(\bar{r}, \bar{r}') \\ G_E^z(\bar{r}, \bar{r}') \end{bmatrix} = \sum_{n=-\infty}^{n=+\infty} e^{jn(\phi-\phi')} \int_0^\infty k_\rho J_n(k_\rho \rho) \begin{bmatrix} C_n(k_\rho) \\ A_n(k_\rho) \end{bmatrix} \\ & \quad \times e^{-jk_z z} dk_\rho \quad z \geq 0 \quad \begin{bmatrix} \text{A/m} \\ \text{V/m} \end{bmatrix}. \quad (45) \end{aligned}$$

Substituting (44) into (45) gives the explicit spectral representation of the Green’s functions

$$\begin{aligned} & \begin{bmatrix} G_H^z(\bar{r}, \bar{r}') \\ G_E^z(\bar{r}, \bar{r}') \end{bmatrix} = \frac{1}{4\pi} \sum_{n=-\infty}^{n=+\infty} e^{jn(\phi-\phi')} \int_0^\infty k_\rho^2 e^{-jk_z z} J_n(k_\rho \rho) \\ & \quad \cdot \begin{bmatrix} +\frac{j}{k_z} J'_n(k_\rho \rho') & -\frac{1}{k_z} \frac{n}{k_\rho \rho'} J_n(k_\rho \rho') \\ -\frac{1}{\omega\varepsilon_0} \frac{n}{k_\rho \rho'} J_n(k_\rho \rho') & -\frac{j}{\omega\varepsilon_0} J'_n(k_\rho \rho') \end{bmatrix} \\ & \quad \times \begin{bmatrix} \mathcal{J}^{\phi'} \\ \mathcal{J}^{\rho'} \end{bmatrix} dk_\rho \quad \begin{bmatrix} \text{A/m} \\ \text{V/m} \end{bmatrix}. \quad (46) \end{aligned}$$

For the problem of the ring source $\alpha = \pi/2$ in (25). Assuming no variation over ϕ' , the surface currents are written as

$$\begin{bmatrix} J_s^{\phi'}(\rho') \\ J_s^{\rho'}(\rho') \end{bmatrix} = \begin{bmatrix} \mathcal{J}^0 \delta(\rho'-a) \\ 0 \end{bmatrix}. \quad (47)$$

The z -components of the fields are found after multiplying by the surface current distributions and integrating

$$\begin{aligned} & \begin{bmatrix} H_z(\bar{r}) \\ E_z(\bar{r}) \end{bmatrix} \\ &= \frac{1}{4\pi} \int_0^{2\pi} \int_0^\infty \sum_{n=-\infty}^{n=+\infty} e^{jn(\phi-\phi')} \int_0^\infty k_\rho^2 e^{-jk_z z} J_n(k_\rho \rho) \\ & \quad \cdot \begin{bmatrix} +\frac{j}{k_z} J'_n(k_\rho \rho') & -\frac{1}{k_z} \frac{n}{k_\rho \rho'} J_n(k_\rho \rho') \\ -\frac{1}{\omega\varepsilon_0} \frac{n}{k_\rho \rho'} J_n(k_\rho \rho') & -\frac{j}{\omega\varepsilon_0} J'_n(k_\rho \rho') \end{bmatrix} \\ & \quad \times \begin{bmatrix} J_s^{\phi'}(\rho') \\ J_s^{\rho'}(\rho') \end{bmatrix} dk_\rho \rho' d\phi' d\phi'. \quad (48) \end{aligned}$$

Substituting (47) into (48) and integrating over the ϕ' coordinate gives values of zero except for the case when $n = 0$. After completing the spatial integrations and noting

that $J'_0(k_\rho a) = -J_1(k_\rho a)$, the solution in both regions of the problem becomes

$$H_z(\bar{r}) = \mathcal{J}^0 \int_0^\infty k_\rho^2 \frac{e^{-jk_z|z|}}{j2k_z} J_0(k_\rho \rho) J_1(k_\rho a) dk_\rho \quad (49)$$

$$E_z(\bar{r}, \bar{r}') = 0. \quad (50)$$

Equation (34) may now be used to solve for the remaining transverse magnetic fields

$$H_\rho(\bar{r}) = \mathcal{J}^0 \int_0^\infty k_\rho \frac{e^{-jk_z z}}{2} J_1(k_\rho \rho) J_1(k_\rho a) dk_\rho \quad z \geq 0 \quad (51)$$

$$H_\rho(\bar{r}) = -\mathcal{J}^0 \int_0^\infty k_\rho \frac{e^{+jk_z z}}{2} J_1(k_\rho \rho) J_1(k_\rho a) dk_\rho \quad z \leq 0. \quad (52)$$

The transverse electric field is

$$E_\phi(\bar{r}) = -j\omega\mu_0 \mathcal{J}^0 \int_0^\infty k_\rho \frac{e^{-jk_z|z|}}{j2k_z} J_1(k_\rho \rho) J_1(k_\rho a) dk_\rho. \quad (53)$$

Using duality, these solutions match those found by Dudley [6] for a ring of magnetic current.

V. CONCLUSION

The problem of specifying the boundary conditions for the z -components of electric and magnetic fields at a planar boundary containing transverse sources has been addressed. The solution advanced in this paper is unique in its approach in that it decomposes the boundary conditions of the transverse fields in a manner consistent with the decomposition of the fields in the source-free homogeneous regions. Once accomplished, this decomposition makes it possible to determine E_z and H_z at the boundary. Since the transverse components of the fields are found differentiating the z components, an entire solution is known.

This procedure for meeting the boundary conditions outlined in this paper is found to be consistent with the vector transform approach. Furthermore, such transforms may be derived from the results of this paper. To illustrate, the cylindrical coordinate system was specified and the Vector Hankel transform is shown to originate from the limiting behavior of the z -directed fields. Finally, to demonstrate the simplicity of this approach, the ring-of-current problem is solved and the solution is found to match known analytical results.

APPENDIX A

Two useful 2-D vector identities are utilized in this paper. The derivations are easily accomplished using two well-known vector identities

$$\nabla \cdot (\bar{A} \times \bar{B}) = \bar{B} \cdot (\nabla \times \bar{A}) - \bar{A} \cdot (\nabla \times \bar{B}) \quad (A.1)$$

$$\begin{aligned} \nabla \times (\bar{A} \times \bar{B}) &= \bar{A}(\nabla \cdot \bar{B}) - \bar{B}(\nabla \cdot \bar{A}) \\ &\quad + (\bar{B} \cdot \nabla)\bar{A} - (\bar{A} \cdot \nabla)\bar{B}. \end{aligned} \quad (A.2)$$

If it is specified that $\bar{A} = \bar{z}_0$ and $\bar{B} = \bar{z}_0 B_z + \bar{B}_t$, (A.1) and (A.2) reduce to

$$\nabla_t \cdot (\bar{z}_0 \times \bar{B}_t) = -\bar{z}_0 \cdot (\nabla_t \times \bar{B}_t) \quad (A.3)$$

$$\nabla_t \times (\bar{z}_0 \times \bar{B}_t) = \bar{z}_0 (\nabla_t \cdot \bar{B}_t). \quad (A.4)$$

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