Forward–Backward Averaging in the Presence of Array Manifold Errors

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***Abstract*— In this paper, we investigate the use of forward–backward (f/b) averaging for estimating the covariance matrix used for adaptive beamforming and space–time adaptive processing (STAP). We demonstrate that the estimation loss is reduced by the use of f/b averaging and, for some STAP cases, f/b averaging can even quadruple the available sample support. We also show that unknown array manifold errors have little effect on the effectiveness of f/b averaging. The gain from f/b averaging is demonstrated on data from the mountaintop database.**

***Index Terms*— Adaptive arrays.**

I. Introduction

FOR certain array geometries, there exists an invariance that allows each data vector to be used twice for covariance estimation in a process known as f/b averaging. Applications of f/b averaging include an improved linear prediction estimator [1] and, in conjunction with spatial smoothing, a scheme to decorrelate coherent signals incident on an array for direction finding [2]. The application considered here is to improve adaptive beamformer performance by providing extra snapshots for covariance estimation [4]. There are sample-starved cases, e.g., when training over a small clutter discrete in an adaptive radar scenario, where f/b averaging significantly improves performance.

By demonstrating how f/b averaging improves adaptive array performance, both in terms of the signal-to-interference-plus-noise ratio (SINR) and the sidelobe level in the adapted array pattern, with the possibility of quadrupling the sample support in some scenarios, we extend previous results [4]. Furthermore, we prove that performance gains are still achievable with f/b averaging in the presence of unknown array manifold errors.

This paper is organized as follows. In Section II we briefly review the data model used for adaptive array processing, and introduce a model for array manifold errors. In Section III f/b averaging is reviewed and quadruple averaging appropriate for some space–time adaptive processing scenarios are introduced. In Section IV, the effect of array manifold errors on f/b averaging is analyzed. Section V contains examples of f/b averaging used on data from the mountaintop database [5] which demonstrate the utility of f/b averaging on real data. Conclusions are drawn in Section VI.

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II. Data Model

We assume the reader of this paper is familiar with the signal model used for array signal processing. This section is intended to briefly introduce the notation used later.

Consider a uniform linear array of N sensors with an interelement spacing d though the analysis and all the results given here may be extended to any array, which produces a persymmetric covariance matrix, i.e., a matrix $\mathbf{M}$ such that

$$\mathbf{M}^* = \mathbf{JMJ}, \qquad \text{where } \mathbf{J} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \cdot^{\cdot^{\cdot}} & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{1}$$

The matrix $\mathbf{J}$ is known as the exchange matrix. The transfer function between bearing θ and the output of the array is represented by the Vandermonde steering vector

$$\mathbf{a}(\theta) = \left[1, e^{j2\pi\, d\lambda^{-1} \sin(\theta)}, \cdots, e^{j2\pi(N-1)\, d\lambda^{-1} \sin(\theta)}\right]^T \tag{2}$$

where λ represents wavelength. Note that to within a constant phase term $\mathbf{a}(\theta) = \mathbf{J}\mathbf{a}(\theta)^*$. The data received at the array output is the sum of the K incident signals and the noise

$$\mathbf{x}(t) = \sum_{k=1}^{K} \alpha_k(t)\mathbf{a}(\theta_k) + \mathbf{n}(t) \tag{3}$$

where $\alpha_k(t)$ is the complex amplitude of the kth signal at snapshot (e.g. range sample) t and $\mathbf{n}(t)$ the noise vector at time t. We will define $\alpha_k(t)$ such that the signal to noise ratio (SNR) of the kth signal is $|\alpha_k(t)|^2$. The noise is assumed to be a zero-mean white complex Gaussian random process (though the Gaussian assumption is not a requirement for f/b averaging).

The steering vector $\mathbf{a}(\theta)$ just described may be viewed as representing either a desired array response for which the array was designed, or a *presumed* array manifold, which due to modeling errors is different from the *true* manifold. In the latter case, the true manifold $\mathbf{a}_\mathbf{h}(\theta)$ may be related to $\mathbf{a}(\theta)$ by taking the Hadamard product of the latter with a vector of error coefficients $\mathbf{h} = [h_1, \cdots, h_N]^T$, which preserves the output power of the steering vector, i.e.

$$\mathbf{a}_\mathbf{h}(\theta) = \mathbf{a}(\theta) \odot \mathbf{h} \tag{4a}$$

and

$$\mathbf{a}_\mathbf{h}(\theta)^H \mathbf{a}_\mathbf{h}(\theta) = \mathbf{a}(\theta)^H \mathbf{a}(\theta). \tag{4b}$$

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Except for the special case where $\mathbf{h} = \mathbf{J}\mathbf{h}^*$ for the true array manifold $\mathbf{a}_\mathbf{h}(\theta) \neq \mathbf{J}\mathbf{a}_\mathbf{h}(\theta)^*$ to within a constant phase term, unlike the presumed array manifold $\mathbf{a}(\theta)$. This inequality is assumed for the remainder of the paper. By inserting $\mathbf{a}_\mathbf{h}(\theta)$ into (3), data vectors for the actual array may be obtained. The elements of $\mathbf{h}$ are taken to be

$$h_i = c + g_i \tag{5}$$

where c is a real constant and g_i is an error term with a zero mean complex Gaussian distribution. Let the ratio of the variance of g_i to c be ξ^2. This model is used so that the expectation of some vector inner products can be constructed and used to give a "feel" for how performance varies as a function of ξ.

The estimated interference-plus-noise covariance matrix of the data over L samples from the array is

$$\hat{\mathbf{R}} = \frac{1}{L}\sum_{l=1}^{L} \mathbf{x}_l \mathbf{x}_l^H. \tag{6}$$

Adapted beamforming weights are computed using the inverse of the sample interference covariance matrix, i.e.,

$$\mathbf{w}(\theta) = \hat{\mathbf{R}}^{-1}\mathbf{a}(\theta). \tag{7}$$

SINR loss, the ratio of output SINR to the SNR that could be achieved in the absence of interference by a matched filter, will be used as the main performance metric. For a target at angle θ_t

$$\text{SINR loss} = \frac{|\mathbf{w}(\theta_t)^H \mathbf{a}(\theta_t)|^2}{N\mathbf{w}(\theta_t)^H \mathbf{R}\mathbf{w}(\theta_t)} \tag{8}$$

where $\mathbf{R}$ is the exact interference-plus-noise covariance matrix. In this paper, the radar scenario in which the target range gate may be excluded from the data used to estimate $\hat{\mathbf{R}}$ is assumed so that the interference-only covariance matrix may be estimated and used to compute the adaptive weights.

As an extension to the array processing problem, a space–time adaptive problem may also be formulated [6]. Given M temporal samples (e.g., coherent pulses) from an N-element array, the received space–time data at the tth snapshot is represented by the N by M matrix $\mathbf{X}(t)$. The temporal sampling is assumed to be uniform. Errors in this assumption are not dealt with here, though their effect may be analyzed in the same way as spatial errors.

III. FORWARDS–BACKWARDS AVERAGING

$F(\mathbf{x})$, the spatial power spectral density (PSD) of $\mathbf{x}$ (a single data vector), may be defined as

$$F(\mathbf{x}) = |\mathbf{a}(\theta)^H \mathbf{x}|^2 \tag{9}$$

for all values of θ (the dependence of $F(\mathbf{x})$ on θ is implicit here although it is not explicitly indicated in the notation). Given $\mathbf{x}^*$, the conjugate of $\mathbf{x}$, and $\mathbf{x}^r$, the reverse sequence of $\mathbf{x}$, i.e., $\mathbf{x}^r = \mathbf{J}\mathbf{x}$, then the following is true:

$$F(\mathbf{x}^*) = F(\mathbf{x}^r) = (F(\mathbf{x}))^r \quad \text{and thus} \quad F(\mathbf{x}^{\mathbf{r}^*}) = F(\mathbf{x}). \tag{10}$$

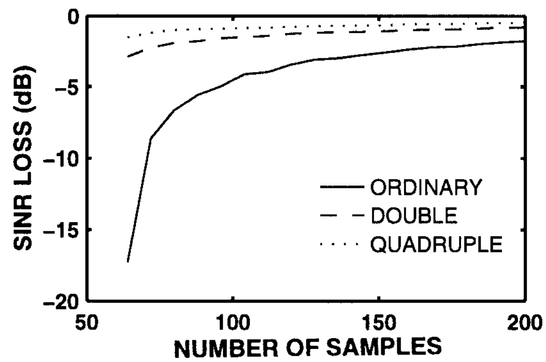

Fig. 1. SINR loss verses sample size for no, double, and quadruple f/b averaging computed using an eight-element eight-pulse STAP simulation.

Since $\mathbf{x}$ and $\mathbf{x}^{\mathbf{r}^*}$ have the same spatial PSD, either or both may be used for estimating the signals' spatial parameters—e.g., in (6). The use of both is referred to as f/b averaging (double averaging). If the noise is uncorrelated between sensors, then from the covariance estimation perspective $\mathbf{x}$ and $\mathbf{x}^{\mathbf{r}^*}$ contain uncorrelated noise samples. To prove this, take the noise vector of (3). The noise samples are uncorrelated if $\mathbf{E}\{\mathbf{n}(t)^{*\mathbf{r}}\mathbf{n}(t)^H\} = 0$ ($\mathbf{E}\{\cdot\}$ is the expectation operator). By simple algebra

$$\mathbf{E}\{\mathbf{n}(t)^{*\mathbf{r}}\mathbf{n}(t)^H\} = \mathbf{E}\{\mathbf{J}\mathbf{n}(t)\mathbf{n}(t)^T\}^* = 0 \tag{11}$$

for a zero mean complex Gaussian random uncorrelated process.

In the space–time case, if $\mathbf{X}^s$ and $\mathbf{X}^t$ are the reverse spatial and temporal sequences of $\mathbf{X}$, respectively, ($\mathbf{X}^s = \mathbf{J}\mathbf{X}$ and $\mathbf{X}^t = \mathbf{X}\mathbf{J}$) and $F_2(\mathbf{X})$ is the two-dimensional PSD of $\mathbf{X}$, then f/b averaging may be used for space–time adaptive processing (STAP) because

$$F_2(\mathbf{X}^{*_{st}}) = F_2(\mathbf{X}). \tag{12}$$

Furthermore, if there are only signals present with no temporal structure, (e.g., wide-band jamming in STAP) or with the same frequency structure either side of the center operating frequency (e.g., some wide-band adaptive beamformer structures employing tapped delay lines), then

$$F_2(\mathbf{X}^t) = F_2(\mathbf{X}^{*_s}) = F_2(\mathbf{X}). \tag{13}$$

Hence, in some scenarios it is possible to obtain three extra samples when using f/b averaging to estimate a space–time covariance matrix (quadruple averaging). The extra samples, all of which possess the same spatial PSD as $\mathbf{X}$, are $\mathbf{X}^{*_{st}}$, $\mathbf{X}^t$, and $\mathbf{X}^{*_s}$.

The improved convergence properties for STAP through the use of f/b averaging are demonstrated in Fig. 1 for an eight-element eight-pulse STAP simulation. Two jammers are present from bearings of 37° and 64°, each with an interference-to-noise-ratio (INR) per element per pulse of 50 dB. Fully adaptive STAP [6] was used and average SINR loss curves are generated for a broadside target. With no f/b averaging the SINR loss as a function of samples is as predicted in [3]

$$\text{SINR loss} = \frac{L + 2 - N}{L + 1}. \tag{14}$$

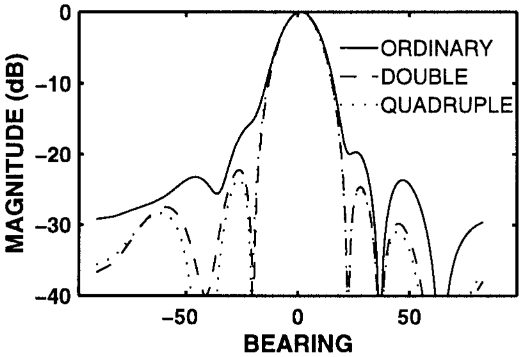

Fig. 2. Adapted responses obtained using 200 samples.

For STAP, N is replaced with NM. The use of double or quadruple f/b averaging improves the SINR loss performance to levels equivalent to doubling or quadrupling the ratio of L to N.

The normalized adapted response $p(\theta)$ of a spatial array is computed as

$$\hat{p}(\theta) = |\mathbf{w}(\theta_t)^H \mathbf{a}(\theta)|^2, \quad \text{and} \quad p(\theta) = \frac{\hat{p}(\theta)}{\max\{\hat{p}(\theta)\}}. \tag{15}$$

The sidelobe levels in the spatial response are a function of the number of samples [7]. Plots of the spatial adapted pattern for the example above are shown in Fig. 2 for a beam formed at broadside and zero Doppler with a 30-dB Chebychev taper in each dimension. The sidelobe levels of the adapted responses improve due to the addition of f/b averaging.

IV. Array Manifold Errors

The effects of the differences between the true and presumed array manifolds on f/b averaging should be addressed in order to quantify the practical usefulness of f/b averaging. Although the analysis in this section is for spatial adaptive beamforming, the results also apply to STAP. Two effects of the differences between the manifolds may be found.

The first effect appears when the spatial PSD is computed with the true array manifold. The true spatial PSD's of $\mathbf{x}_h$ (a single true data vector) and $\mathbf{x}_h^{*r}$ are different, i.e.,

$$F_h(\mathbf{x}_h) \neq F_h(\mathbf{x}_h^{*r}) \tag{16}$$

where

$$F_h(\mathbf{x}_h) = |\mathbf{a}_h(\theta)^H \mathbf{x}_h|^2 \tag{17}$$

over all θ. If f/b averaging is used with the *true* array mainfold, then, as a result of this inequality, the adaptive beamformer will adapt to a spatial PSD that does not correspond to the actual environment and performance will be degraded. However, in practice, the true array manifold is unknown, and presumed array manifold is used instead. If the spatial PSD's of $\mathbf{x}_h$ and $\mathbf{x}_h^{*r}$ are the same for the presumed array manifold, then this first effect will not appear when the presumed manifold is used.

Theorem: If a presumed array manifold where $\mathbf{a}(\theta) = \mathbf{J}\mathbf{a}(\theta)^*$ is used, then $F(\mathbf{x}_h) = F(\mathbf{x}_h^{*r})$, where $F(\mathbf{x}_h) = |\mathbf{a}(\theta)^H \mathbf{x}_h|^2$.

Proof:

$$\begin{aligned} F(\mathbf{x}_\mathbf{h}^{*\mathbf{r}}) &= |\mathbf{a}(\theta)^H \mathbf{x}_\mathbf{h}^{*\mathbf{r}}|^2 = |(\mathbf{a}(\theta)^{*\mathbf{r}})^H \mathbf{x}_\mathbf{h}|^2 \\ &= |\mathbf{a}(\theta)^H \mathbf{x}_\mathbf{h}|^2 = F(\mathbf{x}_\mathbf{h}). \end{aligned} \tag{18}$$

The second effect of array manifold errors is that a covariance matrix estimated from the true data vectors with f/b averaging may have a higher interference rank than one estimated without f/b averaging. This will limit the improvement available due to f/b averaging. The rank of the interference increases because the signals in $\mathbf{x}_h$ and $\mathbf{x}_h^{*\mathbf{r}}$ (the forward and backward samples) are not spatially identical and it is easily shown that $\mathbf{x}_\mathbf{h}(t)$ [from (3) and (4)] and $\mathbf{x}_\mathbf{h}^{*\mathbf{r}}(t)$ are uncorrelated. Applying f/b averaging to (3) results in

$$\mathbf{x}_\mathbf{h}(t)^{*\mathbf{r}} = \sum_{k=1}^{K} \alpha_k(t)^* \mathbf{J}\mathbf{a}_\mathbf{h}(\theta_k)^* + \mathbf{J}\mathbf{n}(t)^*. \tag{19}$$

Note that in (19) the modulating signal α_k has been conjugated. The correlation between α_k and α_k^* is

$$E\{\alpha_k(\alpha_k^*)^*\} = E\{\alpha_k \alpha_k\} = 0 \tag{20}$$

since the mean values of the in-phase and quadrature components of α_k are the same. Since each signal in $\mathbf{x}_h$ is uncorrelated with its equivalent in $\mathbf{x}_h^{*\mathbf{r}}$ and will be slightly different spatially, a covariance matrix produced using f/b averaging will have two signal subspace eigenvalues/eigenvectors for each signal present[1]. However, the second eigenvalue/eigenvector will only affect adaptive beamformer performance if it is above the noise floor.

Given two spatially similar but otherwise uncorrelated signals, it is possible to determine the eigenvalues of their covariance matrix. The following equation for the smaller of the two eigenvalues was derived by Hudson [8]:

$$\lambda_2 \approx \frac{N p_1 p_2 (1 - |\psi|^2)}{p_1 + p_2} \tag{21}$$

where p_1 and p_2 are the powers of the two signals, and $|\psi|^2$ is the "spatial cross-correlation" of the two signals. For the case of f/b averaging using the true array manifold with a single signal present

$$|\psi|^2 = \left| \frac{\mathbf{a}(\theta)_h^H \mathbf{a}(\theta)_h^{*r}}{\mathbf{a}(\theta)_h^H \mathbf{a}(\theta)_h} \right|^2 = \left| \frac{\mathbf{h}^H \mathbf{h}^{*r}}{\mathbf{h}^H \mathbf{h}} \right|^2. \tag{22}$$

Given the model of $\mathbf{h}$ from Section II, it can be shown that the expectation of the right-hand side (RHS) of (22) for small values of ξ can be approximated as

$$E\{|\psi|^2\} \approx \left(\frac{1}{1+\xi^2} \right)^2 = \frac{1}{1 + 2\xi^2 + \xi^4}. \tag{23}$$

Assuming that $\xi \ll 1$ the term ξ^4 in (23) may be ignored and the RHS of the equation may be approximated as $1 - 2\xi^2$. The term $2\xi^2$ may be viewed as the power in the error terms of the presumed array manifold. Due to the random nature of the errors modeled, on average the fraction N^{-1} of the error

[1] This is why f/b averaging has a decorrelating effect in the presence of coherent/correlated multipath [2].

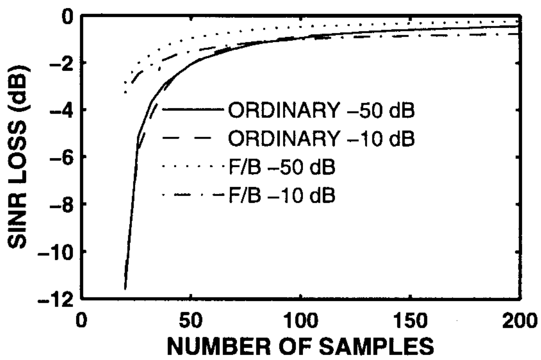

Fig. 3. SINR loss versus sample size with array errors.

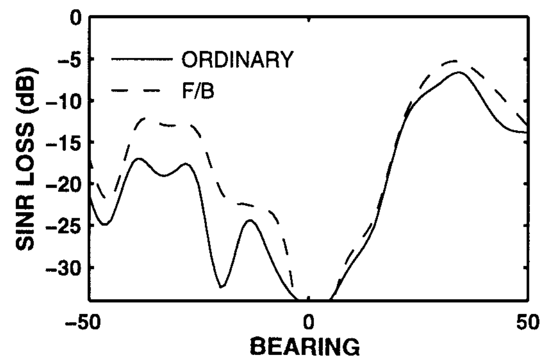

Fig. 4. Azimuth cut of SINR loss for file IDPCA65.

power is spatially correlated with the signal. Providing N is large this may be also be ignored. Hence, the approximation

$$|\psi|^2 \approx 1 - 2\xi^2 \tag{24}$$

may be made. Substituting this into (21) and noting that $p_1 = p_2 = 0.5|\alpha|^2$ for the f/b averaging case being considered here, the following equation is derived for the magnitude of the second eigenvalue, which appears due to f/b averaging

$$\lambda_2 \approx 2\xi^2 \left|\frac{1}{2}\alpha\right|^2 N = \frac{1}{2}\xi^2 |\alpha|^2 N. \tag{25}$$

If λ_2 is above the noise floor, then the performance of the adaptive beamformer will be compromised due to f/b averaging. From (25), it can be seen that when

$$\xi^2 < \frac{2}{|a|^2 N} \tag{26}$$

λ_2 is below the noise floor, and there will be no loss in adaptive beamformer performance.

Instead of quantifying the loss for the single signal case when λ_2 is above the noise floor, we will look at the more general case when f/b averaging causes the rank of the interference covariance matrix to increase from v to $v + r$, $(r \leq v)$. Here, v is the interference rank prior to f/b averaging and r the increase in rank due to f/b averaging in the presence of array manifold errors. Apart from being orthogonal to the rest of the signal subspace, each new signal subspace eigenvector will be random in nature (due to the random nature of the array manifold errors modeled). In this scenario, the following expression for the SINR loss due to the increased interference rank is derived in Appendix A:

$$\text{SINR loss} \approx \frac{N - v - r}{N - v}. \tag{27}$$

e.g., for $N = 20$ and $v = r = 2$ the SINR loss is $16/18$, which is about −0.5 dB. This loss will decrease as the number of array elements increases. From the combination of (14) (the gain due to improved sample support) and (27) (the loss due to array manifold errors) it may be determined when f/b averaging will improve performance.

Simulations for a 20-element array with array manifold errors of $\xi = 10^{-5}$ and $\xi = 10^{-1}$ were run. Two jammers from 37° and 64° are present, each with a 50-dB INR. Fig. 3 is a plot of mean SINR loss (averaged over the unjammed angles) versus the number of samples. For $\xi = 10^{-5}$ f/b averaging always improves performance, as would be expected from consideration of (26), (the additional eigenvalues due to f/b averaging are below the noise floor). However, with $\xi = 10^{-1}$ f/b averaging only improves performance below about 90 samples. Equation (26) shows that for $\xi = 10^{-1}$ the rank of the interference will double from two to four with f/b averaging. Hence, from consideration of (14) and (27) f/b averaging should improve performance below about 90 samples. Simulation and theory agree.

V. Real Data Examples

Results from the application of f/b averaging to data from the mountaintop radar data base are also presented here. A 14-element nominally uniform linear array is used. Although all of the receive channels were equalized, the array itself is not perfectly calibrated. The radar transmitter array is capable of simulating platform motion (see [5] for more details on the mountaintop system). PRI-staggered STAP [6] with a three-pulse sub-CPI was used to process the data.

The first data set (file IDPCA65 collected on 3/9/94) contains ground clutter returns, which form a ridge that is an unambiguous function of bearing and Doppler. 100 training samples were used to compute the adapted weights, with f/b averaging doubling the sample size. Fig. 4 shows an azimuth cut of the estimated[2] SINR loss at 64-km range and 0-Hz Doppler. Assuming no array manifold errors, the expected improvement due to averaging given by (14) is about 1.25 dB. Excluding the clutter null at broadside, the average improvement is 2.9 dB. This measured figure suggests that the range samples are not completely independent.

The second data set (file STAP2017 collected on 3/28/94) contains two wide-band jamming signals at 5° and −30° from broadside. Fifty training samples were used to compute the adapted weights with f/b type averaging quadrupling this figure. Fig. 5 shows an azimuth cut of the estimated SINR loss at 164-km and 0-Hz Doppler. Assuming no array manifold errors, the expected improvement from (14) is about 6 dB. The average improvement for the plot due to f/b averaging (excluding areas near the jammers) is 5.7 dB.

[2] Since for exprimental data the exact covariance matrix is unknown, an estimated covariance matrix was used in the denominator of (8) to compute the SINR loss.

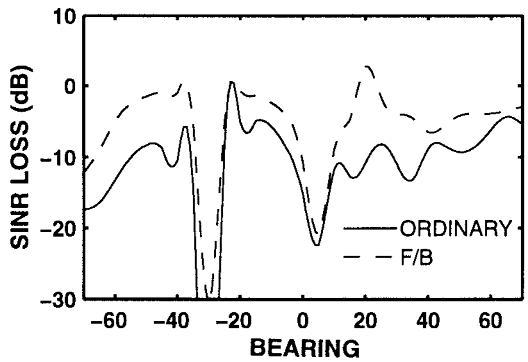

Fig. 5. Azimuth cut of SINR loss for file STAP2017.

VI. Conclusion

We have shown that for STAP f/b averaging can double or even quadruple the sample support. Furthermore, we have proven that there are many scenarios where f/b averaging improves adaptive beamformer performance *despite* the presence of *unknown*-array manifold errors. These cases are when a medium to large array with only poor or adequate sample support is used. The improvement due to f/b averaging was demonstrated by simulation and on experimental data from the mountaintop database.

Appendix A

SINR Loss Due to Random Eigenvectors: Noting that $\mathbf{R}\hat{\mathbf{R}}^{-1} \approx \mathbf{I}$ and $\mathbf{a}(\theta)^H\mathbf{a}(\theta) = N$, (8) for SINR loss becomes

$$\text{SINR loss} \approx \frac{\mathbf{a}(\theta)^H\hat{\mathbf{R}}^{-1}\mathbf{a}(\theta)}{\mathbf{a}(\theta)^H\mathbf{a}(\theta)}. \tag{28}$$

Providing the eigenvalues of all of the interference subspace eigenvectors are above the noise floor, then

$$\hat{\mathbf{R}}^{-1} \approx \mathbf{I} - \mathbf{E}\mathbf{E}^H \tag{29}$$

where $\mathbf{E} = [\mathbf{e}_1, \cdots, \mathbf{e}_{(v+r)}]$ is the matrix of interference subspace eigenvectors. This approximation will overestimate the influence of eigenvectors with eigenvalues near to the noise floor. However, simulation shows that this approximation is adequate for eigenvalues at least 10 dB above the noise floor.

By plugging (29) into (28), the following equation which assesses the effect of each interference subspace eigenvector on the SINR loss is produced:

$$\text{SINR loss} \approx 1 - \sum_{i=1}^{i=r+v} \frac{|\mathbf{a}(\theta)^H\mathbf{e}_i|^2}{\mathbf{a}(\theta)^H\mathbf{a}(\theta)}. \tag{30}$$

It is instructive to first consider the case of an interference subspace of rank 1 where the interference eigenvector is "random." By consideration of two vectors lying on an N-dimensional sphere, it is easily shown that the expected value of the quotient on the RHS of (30) is

$$E\left\{\frac{|\mathbf{a}(\theta)^H\mathbf{e}_i|^2}{\mathbf{a}(\theta)^H\mathbf{a}(\theta)}\right\} = \frac{1}{N}. \tag{31}$$

Thus, the expected value of the SINR loss in this scenario is about $1-1/N$, which for large N becomes insignificant. Now consider the case of r random eigenvectors and v nonrandom eigenvectors, (as may exist after f/b averaging in the presence of random array manifold errors). Since all of the eigenvectors are orthogonal to one another the r random eigenvectors lie randomly in the $N-v$-dimensional vector subspace orthogonal to the v nonrandom eigenvectors. Then for each of the r random eigenvectors the expected value of the quotient on the RHS of (30) is

$$E\left\{\frac{|\mathbf{a}(\theta)^H\mathbf{e}_i|^2}{\mathbf{a}(\theta)^H\mathbf{a}(\theta)}\right\} = \frac{1}{N-v}. \tag{32}$$

Extending this to the case of r random eigenvectors results in an SINR loss of

$$\text{SINR loss} \approx 1 - \frac{r}{N-v} = \frac{N-v-r}{N-v}. \tag{33}$$

References

[1] S. W. Lang and J. H. McClennan, "Frequency estimation with maximum entropy spectral estimators," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-28, pp. 850–861, 1980.
[2] S. U. Pillai and B. H. Kwon, "Forward/backward spatial smoothing for coherent signal identification," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 37, pp. 8–15, 1989.
[3] I. S. Reed, J. D. Mallet, and L. E. Brennan, "Rapid convergence in adaptive arrays," *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-9, pp. 853–863, 1973.
[4] R. Niztberg, "Application of ML estimation of persymmetric covariance matrices to adaptive processing," *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-16, pp. 124–127, 1980.
[5] G. W. Titi and D. Marshall, "The ARPA/NAVY mountaintop program—Adaptive signal processing for airborne early warning radar," in *Proc. ICASSP'96*, Atlanta, GA, May 1996, pp. 1165–1168.
[6] J. Ward, "Space-time adaptive processing for airborne radar," MIT Lincoln Lab. Tech. Rep., vol. 1015, 1994.
[7] B. D. Carlson, "Covariance matrix estimation and diagonal loading in adaptive arrays," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 24, pp. 397–401, 1988.
[8] J. E. Hudson, *Adaptive Array Principles*. London, U.K.: Peter Peregrinus, 1981.

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