A Functional for Dynamic Finite-Element Solutions in Electromagnetics

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***Abstract*— A new functional is introduced that satisfies of Maxwell's equations, provides minimization, and eliminates spurious solutions. An analytical method is developed that provides a means of evaluating functional forms. The analytical method confirms the effective functional form as the fundamental cause underlying the difficulties with spurious solutions that are not completely eliminated under all circumstances. It is shown that the curl–curl "functional" form allows for the existence of an improper gradient behavior in a general field expansion. The new functional is shown to be self adjoint and positive definite, thus providing an error minimization. Numerical results are obtained that demonstrate the effectiveness of the new functional to prevent spurious solutions using node-based elements.**

***Index Terms*—Finite-element methods, spurious solutions.**

I. Introduction

The electromagnetic analysis of microwave structures has expanded significantly with the introduction of computers to assist in the approximate numerical solution of problems considered unsolvable by standard analytical methods. The researcher is often forced to choose a numerical approach for problems involving scattering or guided propagation whenever the geometry under consideration does not coincide with a separable coordinate system. Numerical approaches are also required when high-frequency asymptotic or low-frequency quasi-static techniques cannot be used. The application of numerical methods for solving differential equations in electromagnetics has taken several paths in the last decade, from early applications of the finite-difference method [1], transmission-line matrix methods [2], and spectral-domain methods [3] to the finite-difference time-domain method that has come into maturity [4]. Finite elements have also been widely applied to problems in electromagnetics, but early applications were handicapped by the appearance of spurious solutions.

The causes of spurious solutions that result from applying finite elements have been attributed to several difficulties, including the failure of node-based elements to properly model the null space of the curl operator [6] and the improper handling of natural boundary conditions [7]. Several researchers [7], [8], [12] suggest that the node-based finite-element basis functions are improper for application in the curl–curl form. It is suggested that when Maxwell's equations are recast into the wave equation an infinitely degenerate zero eigenvalue and eigenspace results [6] and that spurious solutions result from the inability of node-based elements to properly model this eigenspace. An alternate approach departs from the typical nodal expansion function (a pyramidal basis function of compact support defined for each node) and defines a basis set in terms of the tangential field along the edges that will properly model the degenerate eigenspace of the wave equation under certain circumstances. This approach (known as the edge element method [8]), by definition, disallows divergent solutions within the element though a divergence may exist at the boundary of the elements for first order elements. Edge elements (or vector-finite elements) [12] have become the method of choice by many researchers for the elimination of spurious solutions.

The use of edge elements [8] have eliminated spurious solutions for a great number of problems, but edge elements are no panacea. Lee [16] reported that edge elements did not eliminate spurious solutions for the problems formulated in the cylindrical coordinate system and required an appropriate change of variables. Another important conclusion of Lee's [16] work is that the spurious solutions are eliminated by edge elements are reduced to zero eigenvalues. Polstyanko [17] suggested that the incomplete elimination slowed the convergence of the desired eigenvalues. Polstyanko addressed the incomplete elimination of the spurious solutions by introducing an additional constrain equation, thus increasing the computational overhead.

The question raised in this paper follows a more fundamental line: if a method is variationally based, should not the choice of poor, but admissible, expansion functions yield proper, though possibly inaccurate, results? Motivated by the failure of many finite-element schemes to properly enforce the correct divergence criterion on the problem to be solved in a robust manner, the primary purpose of this paper is to introduce a new functional whose minimum corresponds to a solution of Maxwell's equations, eliminates spurious solutions, and provides a mapping from the complex space to the real line. This functional will be consistent with the rigorous requirements of a variational scheme.

Another emphasis of this paper is to establish a methodology for evaluating the "functional" representations in order to present a new functional based on the insights gained. The method to be developed directs the focus on the functional representation to enable identification of the fundamental cause underlying the difficulties with spurious solutions as initially

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explored by the authors [11]. This analysis may be performed using an expansion that contains both divergenceless terms and gradient terms (consistent with Helmholtz theorem) that span the entire solution domain for a given problem without *a priori* assumptions. By using entire domain expansion functions that satisfy the boundary conditions, the difficulties that may be introduced by discretizing the domain for finite elements are completely avoided. This approach yields significant insight into the nature of a given functional and provides clues regarding the source of spurious solutions. This analytical approach will be applied to the commonly used weighted residual form used in the penalty method with a special case being the generic full-field "functional." By using analytical rather than numerical means, it will be shown that the solution form allows for the existence of an improper gradient behavior in a general expansion function. It will also be shown that the parameter in the penalty method [5], which exhibits a linear behavior in numerical solutions, is also predicted to have the same linear behavior by the exact analytical approach.

Section II will discuss the analytical method used to evaluate the functional forms and apply it to the penalty method and the curl–curl form. Section III will introduce the new functional and discuss several important characteristics that distinguish the new approach from others. The analytical approach will be applied to the new functional, which shall be denoted the electric and magnetic (EH) functional since both fields are explicitly represented. It will be shown that the new functional eliminates the gradient part of a general vector expansion function, thus eliminating divergent solutions. Section IV will demonstrate the robust character of the EH functional within a finite-element implementation.

II. Analytical Method for Analyzing Functional Forms

This section will examine the functional forms commonly used in solving dynamic problems in electromagnetics by using an analytical approach. By using an analytical approach, the nature of the spurious solutions may be observed independent of the mesh structure and the numerical basis functions used. It is to be emphasized that the analytical approach to be described is not intended to replace finite elements, but used only to demonstrate the potential difficulties with a functional form. The domain will be considered to be a separable geometry so that entire domain expansion functions may be used to eliminate the discretization error, errors in interpolation, and errors in boundary condition implementation. A rectangular waveguide example will be used in this paper for both the analytical and finite-element techniques since the solutions are well known. By a judicious choice of expansion functions that exactly satisfy the boundary conditions and provide orthogonality, we can use an expansion form of an infinite series that will provide an exact answer. A convenient form for the expansion functions is composed of the functions that describe the classic modal fields in the separable geometry.

An important aspect of the analytical approach is allowing for *all* possible vector field descriptions. Since we cannot assume that the computer "knows" what form of solution will

TABLE I
PENALTY METHOD RESULTS FOR A 16-NODE MESH OF RECTANGULAR WAVEGUIDE. THE RESULTS FOR $p = 0$ CORRESPOND TO THE FULL-FIELD FUNCTIONAL

p = 0	p = 1	Exact
2.849202E-03	3.187462	3.1416
2.104455	6.671525	6.2832
2.691510	6.900833	6.2832
3.174411	7.804257	7.0248
5.415371	8.167882	7.0248
6.599501	8 .247024	8.8858
6.829937	10.382930	8.8858
7.639579	10.672380	9.4248

be required in either analytical or numerical processes, the analytical approach applies the basic idea of the Helmholtz theorem, which specifies that any vector can be represented in terms of a irrotational part and a solenoidal part. In other words, any vector field can be completely specified by a description that contains a gradient and a curl. A robust algorithm should impose the necessary restrictions without human intervention for the desired vector field; that is, eliminating the solenoidal part typically ascribed to spurious solutions.

The functional used in the penalty method will be examined in detail with the curl–curl form as a special case, which is also used in the edge-element approach [8]. A comparison will be made with the numerical results of Rahman [5] to demonstrate the parameter dependence of the wavenumber with the penalty method. A typical expression that implements the penalty method is given by

$$\int [(\nabla \times \overrightarrow{E}) \cdot (\nabla \times \overrightarrow{E}_n)^* + p(\nabla \cdot \overrightarrow{E})(\nabla \cdot \overrightarrow{E}_n^*) - k^2 \overrightarrow{E} \cdot \overrightarrow{E}_n^*]\, d\Omega = 0. \tag{1}$$

A boundary term resulting from the application of the divergence theorem to the $\nabla \times \nabla \times \overrightarrow{E}$ term has been set to zero, enforcing $\hat{n} \times \overrightarrow{E} = 0$ on the boundary for the perfect electric conducting (PEC) waveguides considered in this paper. When the penalty term denoted by the letter p is set to zero, the curl–curl form is recovered. When p is set to unity, the "functional" corresponds to the Helmholtz equation. Numerically, applying finite-element analysis to a rectangular waveguide results in the data of Table I, graphically depicted in Fig. 1. The first three entries in the $p = 0$ column are spurious solutions. The first few modes of the $p = 1$ column correspond to physical modes, but later, entries have nonzero divergence.

1) Application of an Analytical Approach to the Penalty Method: The weighted residual form that serves as the starting point for applying the penalty method is

$$\int_V \overrightarrow{E}_n^* \cdot [\nabla \times \nabla \times \overrightarrow{E} - k^2 \overrightarrow{E} - p\nabla(\nabla \cdot \overrightarrow{E})] = 0 \tag{2}$$

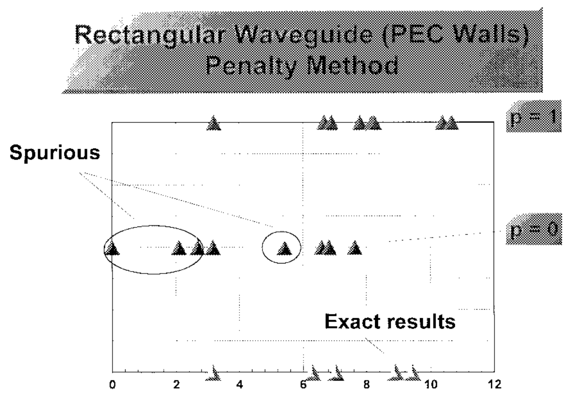

Fig. 1. Graphical depiction of the numerical solution of the data in Table I using the penalty method for a 16-node rectangular waveguide model.

or equivalently

$$\int_V \{(\nabla \times \overrightarrow{E}) \cdot (\nabla \times \overrightarrow{E}_n^*) - k^2 \overrightarrow{E} \cdot \overrightarrow{E}_n^* + p(\nabla \cdot \overrightarrow{E})(\nabla \cdot \overrightarrow{E}_n^*)\}\, dv - \oint_{cV} \{\overrightarrow{E}_n^* \times \nabla \times \overrightarrow{E}_n^* + p(\nabla \cdot \overrightarrow{E})\overrightarrow{E}_n^*\} \cdot \hat{n}\, ds = 0. \quad (3)$$

Making use of the Helmholtz theorem to generalize the possible values of electric field suggests that $\overrightarrow{E}$ can be written as the sum of a gradient term and a curl expression given by a sum of TE and TM modes in a waveguide such as

$$\overrightarrow{E} = [\nabla(f e^{-\gamma z}) - \nabla \times (\hat{z}\psi_h e^{-\gamma z}) + \nabla \times \nabla \times (\hat{z}\psi_e e^{-\gamma z})]. \quad (4)$$

The choice of expansion functions for ψ_e, ψ_h, and f is motivated by the desire to satisfy the boundary condition that was assumed in omitting the integration over the waveguide wall and to provide orthogonality in performing the integration over the cross section. Convenient expansion functions are the modal fields given by

$$\psi_e(x,y) = \sum_m \sum_n A_{mn} \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \quad (5)$$

$$\psi_h(x,y) = \sum_m \sum_n B_{mn} \cos\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) \quad (6)$$

and

$$f(x,y) = \sum_m \sum_n C_{mn} \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right). \quad (7)$$

Applying the method of weighted residuals to (2), and making use of orthogonality, three equations for A_{mn}, B_{mn}, and C_{mn} are obtained as given by the matrix expression of the form $\mathbf{Z}\overrightarrow{x} = 0$ with (8), shown at the bottom of the page, and $\overrightarrow{x} = \{A_{mn}, B_{mn}, C_{mn}\}^T$, where k_c is the cutoff wavenumber given by $k_c^2 = (\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2$ and k is the wavenumber $\frac{2\pi}{\lambda} = \omega\sqrt{\mu\varepsilon}$. For a nontrivial solution the determinant must vanish, leading to the eigenvalue equation

$$\det\{\mathbf{Z}\} = (k_c^4)(k_c^2 - \gamma^2 - k^2)^2 [p(k_c^2 - \gamma^2) - k^2] \times \{[k_c^2 - (\gamma^*)^2](k_c^2 - \gamma^2)\} = 0. \quad (9)$$

Examining the possible solutions that can provide a solution to (9) provides three possible cases to consider:

1) the classical solution $(k_c^2 - \gamma^2 - k^2) = 0$;
2) the solution $(k_c^2 - \gamma^2) = 0$;
3) the solution $[p(k_c^2 - \gamma^2) - k^2] = 0$.

For the classical solution where $(k_c^2 - \gamma^2 - k^2) = 0$, the fields are divergence free for $p \neq 1$ (since $C_{mn} = 0$) and the vector components are completely decoupled for $p = 1$, where the gradient term is represented by C_{mn} and may be nonzero. This decoupling is characteristic of the Helmholtz equation that cannot guarantee solenoidal fields. The second case of $(k_c^2 - \gamma^2) = 0$ allows the presence of a solution at $k = 0$ and also permits a TM-type solution with a gradient term to exist, specifically $C_{mn} = \gamma A_{mn}$.

The more interesting example of the pervasive spurious solutions is the case of $[p(k_c^2 - \gamma^2) - k^2] = 0$, which permits a gradient solution since the C_{mn} assume any value. For $p \neq 1$ the parameter dependence of the problem does not allow for the elimination the spurious solutions. Consider the numerical problem with a 16-node mesh model for a rectangular waveguide with PEC walls. The plot of Fig. 2 depicts the finite-element solution for the cutoff wavenumber squared (k_c^2) for varying penalty parameter p as depicted by the hollow circles. It can be seen that as the parameter p is varied there are some numerical solutions for k_c^2, which have an obvious linear dependence. The analytical approach is able to predict this linear behavior of the penalty method as reported by Rahman [5]. Consider (9) with $\gamma = 0$ to obtain

$$k^2 = \begin{cases} pk_c^2 \\ k_c^2. \end{cases} \quad (10)$$

The solid line is a plot of pk_c^2 for the mode corresponding to the spurious S_{31} mode and the dashed line is a plot for the S_{22} mode. Though this expression does not provide the y intercepts of the plot of Fig. 2, the slope very accurately approximates the slope for k_c^2 as computed by the numerical finite-element solutions.

It has been suggested by Rahman [5] that the spurious solutions are those which are dependent on the penalty parameter. By increasing the penalty term, the spurious eigenvalues are pushed out of the search window of iterative eigenvalue schemes consistent with the parameter choice of Reddy [9].

III. The New Functional (EH) and Related Properties

The emphasis on the functional as a key to the robust elimination of spurious solutions was demonstrated by the

$$\mathbf{Z} = \begin{bmatrix} (k_c^2)(k_c^2 + \gamma\gamma^*)(k_c^2 - \gamma^2 - k^2) & 0 & -(k_c^2)(\gamma^* + \gamma)[p(k_c^2 - \gamma^2) - k^2] \\ 0 & (k_c^2)\{k_c^2 - \gamma^2 - k^2\} & 0 \\ -(\gamma + \gamma^*)(k_c^2)(k_c^2 - \gamma^2 - k^2) & 0 & (k_c^2 + \gamma\gamma^*)[p(k_c^2 - \gamma^2) - k^2] \end{bmatrix} \quad (8)$$

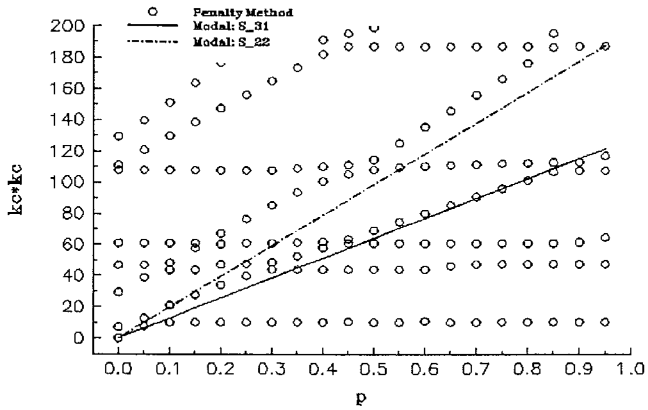

Fig. 2. Penalty method applied to a 16-node mesh of a rectangular waveguide.

analytical approach applied to the penalty method and the special case of a full-field functional. Most formulations including edge elements begin with Maxwell's equations and then perform a substitution to obtain a vector-wave equation that is either in terms of the electric field or the magnetic field typical of the curl–curl form. In order to obtain the weighted residual form typically used in finite-element approaches for these second-order systems, a residual is formed by integrating the product of the equation and a weighting function over the domain. These approaches essentially constitute an application of the method of weighted residuals or the method of moments [15] since the "functionals" used do not provide a mapping to the real line and *do not* provide a minimization. The new functional that provides a mapping to the real line is given by

$$\rho(\vec{E},\vec{H}) = \int_V \{C_1|\nabla\times\vec{H} - j\omega\varepsilon\vec{E} - \vec{J}|^2 + C_2|\nabla\times\vec{E} + j\omega\mu\vec{H} + \vec{M}|^2\}\,dV \tag{11}$$

where the constants C_1 and C_2 are real valued and strictly positive. Examining (11) it will be noted that the functional is dependent upon both $\vec{E}$ and $\vec{H}$, obtained by taking the magnitude squared of each of Maxwell's equations. The magnitude squared is consistent with the definition of an inner product over a complex space. Recall that the divergence information is contained within the curl equations for nonzero frequencies, so that divergence need not be explicitly expressed. The constraint on the constants and the use of the magnitude squared yields a functional that is positive definite and renders a mapping from a complex space to the real line, thus providing a functional that is useful in the application of variational techniques.

The first variation can be examined by considering a set of complex expansion functions expressed as

$$\vec{E} = \sum_n E_n \vec{e}_n \qquad \vec{H} = \sum_n H_n \vec{h}_n \tag{12}$$

with complex coefficients E_n and H_n given by $E_n = a_n + jb_n$ and $H_n = c_n + jd_n$. The first variation of (11) may be combined simply as a function of complex frequency as

$$\frac{\partial\rho}{\partial a_n} - \frac{\partial\rho}{\partial_j b_n} = \int_V \{C_1[(\nabla\times\vec{H} + s\varepsilon\vec{E} - \vec{J})\cdot(-s^*\varepsilon^*\vec{e}_n^{\,*})] + C_2[(\nabla\times\vec{E} - s\mu\vec{H} + \vec{M})\cdot(\nabla\times\vec{e}_n^{\,*})]\}\,dV = 0 \tag{13}$$

and

$$\frac{\partial\rho}{\partial c_n} - \frac{\partial\rho}{\partial_j d_n} = \int_V \{C_1[(\nabla\times\vec{H} - s\varepsilon\vec{E} - \vec{J})\cdot(\nabla\times\vec{h}_n^{\,*})] + C_2[(\nabla\times\vec{E} + s\mu\vec{H} + \vec{M})\cdot(s^*\mu^*\vec{h}_n^{\,*})]\}\,dV = 0 \tag{14}$$

where the conjugate of (13) and (14) would have been obtained if a summation were used in place of the difference in the left-hand side, providing individual variations with respect to all parameters. Note that the first variation results in a set of coupled equations.

The second variations are given by

$$\frac{\partial^2\rho}{\partial(a_n)^2} = \frac{\partial^2\rho}{\partial(b_n)^2} = 2\int_V \{C_1(|s\varepsilon\vec{e}_n|^2) + C_2|\nabla\times\vec{e}_n|^2\}\,dV > 0$$

and

$$\frac{\partial^2\rho}{\partial(c_n)^2} = \frac{\partial^2\rho}{\partial(d_n)^2} = 2\int_V \{C_1(|s\mu\vec{h}_n|^2) + C_2|\nabla\times\vec{h}_n|^2\}\,dV > 0.$$

It can be seen that since the second variation is always positive for nontrivial solutions, the numerical solution to (11) corresponds to a minimum, and represents an upper bound to the exact solution.

The Euler equation for the EH functional can be obtained by considering (13) and (14) with appropriate boundary conditions to yield

$$\int_V \vec{e}_n^{\,*}\cdot[C_1(-s^*\varepsilon^*)(\nabla\times\vec{H} - s\varepsilon\vec{E} - \vec{J}) + C_2\nabla\times(\nabla\times\vec{E} + s\mu\vec{H} + \vec{M})]\,dV = 0$$

and

$$\int_V \vec{h}_n^{\,*}\cdot[C_1\nabla\times(\nabla\times\vec{H} - s\varepsilon\vec{E} - \vec{J}) + C_2(s^*\mu^*)(\nabla\times\vec{E} + s\mu\vec{H} + \vec{M})]\,dV = 0.$$

This system of equations can be written in the least-squares operator form $L^a(Lu - f) = 0$ where for the EH functional

$$L = \begin{bmatrix} \sqrt{C_1}\nabla\times & -s\varepsilon\sqrt{C_1} \\ s\mu\sqrt{C_2} & \sqrt{C_2}\nabla\times \end{bmatrix}, \quad u = \begin{Bmatrix} \vec{H} \\ \vec{E} \end{Bmatrix}, \quad \text{and}$$

$$f = \begin{Bmatrix} \vec{J}\sqrt{C_1} \\ -\vec{M}\sqrt{C_2} \end{Bmatrix}.$$

The expression for the adjoint L^a is simply the conjugate transpose of L, providing the conjugate form of the source free Maxwell's equations.

A. Recapturing the Curl–Curl Form

Recall that the coefficients in (11) were specified as positive. An interesting question that may be explored is: what if the coefficients are incorrectly set as

$$\frac{C_1}{C_2} = -\frac{\mu}{\varepsilon} \tag{15}$$

where the material parameters ε and μ are positive real? Performing this substitution in (13) yields

$$\int_V C_2[(\nabla \times \nabla \times \vec{E} - k^2\vec{E} - \vec{J} + \nabla \times \vec{M}) \cdot \vec{e}_n^*]\, dV + \oint_{\partial V} C_2[(\nabla \times \vec{E} + s\mu\vec{H} + \vec{M}) \cdot (\hat{n} \times \vec{e}_n^*)]\, dS = 0. \tag{16}$$

where $k^2 = -s^2\mu\varepsilon$. Note that the first integral corresponds to the standard curl–curl form, now with sources included. The surface integral can be easily removed if perfect conductors bound the domain via application of $\hat{n} \times \vec{e}_n^* = 0$. Recapturing the curl–curl equation by imposing the constraints on the coefficients as described by (15) is suggestive of a fundamental difficulty associated with the curl–curl equation for numerical applications. The hypothesized constraint of (15) represents a violation of the original constraints for positive definiteness and minimization on the EH functional of (11) where C_1 and C_2 are strictly positive.

B. Application of the Analytical Approach to the EH Functional for Waveguide Propagation

For the electric and magnetic fields, let us assume the modal form (4) with the inclusion of a gradient term to allow the proper form for any general vector as required by the Helmholtz theorem. The general field expressions for both electric and magnetic fields are given by

$$\vec{E} = \vec{e}(x,y)e^{-\gamma z} = [\nabla(fe^{-\gamma z}) - \nabla \times (\hat{z}\psi_h e^{-\gamma z}) + \nabla \times \nabla \times (\hat{z}\psi_e e^{-\gamma z})] \tag{17}$$

and

$$\vec{H} = \vec{h}(x,y)e^{-\gamma z} = [\nabla(ge^{-\gamma z}) - \nabla \times (\hat{z}\varphi_e e^{-\gamma z}) + \nabla \times \nabla \times (\hat{z}\varphi_h e^{-\gamma z})] \tag{18}$$

giving

$$\vec{e} = \nabla_t(f - \gamma\psi_e) + \hat{z} \times \nabla_t\psi_h - \hat{z}\big(\gamma f + \nabla_t^2\psi_e\big) \tag{19}$$

and

$$\vec{h} = \nabla_t(g - \gamma\varphi_h) + \hat{z} \times \nabla_t\varphi_e - \hat{z}\big(\gamma g + \nabla_t^2\varphi_h\big) \tag{20}$$

with ∇_t denoting the portion of ∇ transverse to $\hat{z}$. The definitions for the field which provide orthogonality and the appropriate boundary behavior for a rectangular waveguide are

$$\begin{aligned} \psi_e &= \sum_m \sum_n A_{mn} \sin\Big(\frac{m\pi}{a}x\Big) \sin\Big(\frac{n\pi}{b}y\Big) \\ \psi_h &= \sum_m \sum_n B_{mn} \cos\Big(\frac{m\pi}{a}x\Big) \cos\Big(\frac{n\pi}{b}y\Big) \end{aligned} \tag{21}$$

$$\begin{aligned} \varphi_h &= \sum_m \sum_n D_{mn} \cos\Big(\frac{m\pi}{a}x\Big) \cos\Big(\frac{n\pi}{b}y\Big) \\ \varphi_e &= \sum_m \sum_n E_{mn} \sin\Big(\frac{m\pi}{a}x\Big) \sin\Big(\frac{n\pi}{b}y\Big) \end{aligned} \tag{22}$$

and

$$\begin{aligned} f &= \sum_m \sum_n C_{mn} \sin\Big(\frac{m\pi}{a}x\Big) \sin\Big(\frac{n\pi}{b}y\Big) \\ g &= \sum_m \sum_n F_{mn} \cos\Big(\frac{m\pi}{a}x\Big) \cos\Big(\frac{n\pi}{b}y\Big). \end{aligned} \tag{23}$$

The system of equations that results from substituting (21)–(23) in (11) and obtaining the first variation results in a system that can be expressed by

$$(\mathbf{Z})\{e_p\} = 0 \tag{24}$$

where

$$\{e_p\} = \{A_{mn}, B_{mn}, C_{mn}, D_{mn}, E_{mn}, F_{mn}\}^T$$

and (25), shown at the bottom of the page, where the matrix entries are $k_c^2 = (\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2$, $B = (k_c^2 + \gamma\gamma^*)$, $C = (\gamma + \gamma^*)$, and $D = (k_c^2 - \gamma^2)$.

The desired eigenvalues are obtained by setting $|\mathbf{Z}| = \mathbf{0}$. The form of the zero locations in the matrix of (25) suggests that there are two independent solutions, the first eigenvalue equation for $\{B_{mn}, D_{mn}, F_{mn}\}$ or $\{\psi_h, \varphi_h, g\}$ characteristic of TE modes is

$$c_1(c_2)^2 k_c^4 |s|^2 \mu\mu^* \big|k_c^2 - \gamma^2\big|^2 \big|k^2 - k_c^2 + \gamma^2\big|^2 = 0 \tag{26}$$

The remaining part of the determinant is the eigenvalue equation for $\{A_{mn}, C_{mn}, E_{mn}\}$ or $\{\psi_e, f, \varphi_e\}$ characteristic of TM modes is

$$(c_1)^2 c_2 k_c^4 |s|^2 \varepsilon\varepsilon^* \big|k_c^2 - \gamma^2\big|^2 \big|k^2 - k_c^2 + \gamma^2\big|^2 = 0. \tag{27}$$

These eigenvalue forms require either

1) $k^2 - k_c^2 + \gamma^2 = 0$;
2) $k_c^2 - \gamma^2 = 0$;
3) $k_c^2 = 0$.

$$\mathbf{Z} = k_c^2 \begin{bmatrix} C_1|s|^2\varepsilon\varepsilon^* B + C_2 DD^* & 0 & -C_1|s|^2\varepsilon\varepsilon^* C & 0 & C_1 s^*\varepsilon^* B - C_2 s\mu D^* & 0 \\ 0 & C_1|s|^2\varepsilon\varepsilon^* + C_2 B & 0 & C_1 s^*\varepsilon^* D - C_2 s\mu B & 0 & C_2 s\mu C \\ -C_1|s|^2\varepsilon\varepsilon^* C & 0 & C_1|s|^2\varepsilon\varepsilon^* k_c^{-2} B & 0 & -C_1 s^*\varepsilon^* C & 0 \\ 0 & C_1 s\varepsilon D^* - C_2 s^*\mu^* B & 0 & C_1 DD^* + C_2|s|^2\mu\mu^* B & 0 & -C_2|s|^2\mu\mu^* C \\ C_1 s\varepsilon B - C_2 s^*\mu^* D & 0 & -C_1 s\varepsilon C & 0 & C_1 B + C_2|s|^2\mu\mu^* & 0 \\ 0 & C_2 s^*\mu^* C & 0 & -C_2|s|^2\mu\mu^* & 0 & C_2|s|^2\mu\mu^* k_c^{-2} B \end{bmatrix} \tag{25}$$

The solutions of (24) corresponding to the *minimum* of the variational expression given by the EH functional (11), which decouple into a set of TE and TM modes as demonstrated by (26) and (27), respectively. The solution is independent of the constants C_1 and C_2. The solution of $|k_c^2 - \gamma^2|^2 = 0$ and $k_c^2 = 0$ actually forces all fields to be zero, thus being a trivial (and valid) solution of the full problem.

This section utilized an analytical approach to test a given functional expression for the appropriate solutions to a prescribed boundary value problem. By applying the concepts of the Helmholtz theorem to allow for the possibility of a gradient-type solution to exist, the analytical approach does not constrain the vector nature of the problem. It must be kept in mind that the computer cannot possibly know what form is desired, requiring a complete generalization of the vector field in the test process. The undesirable gradient behavior was completely eliminated by the introduction of the least-squares variational expression. In the next section, an example of applying finite elements and demonstrating the robustness of the new functional will be performed on the rectangular waveguide.

IV. The EH Functional-Finite-Element Implementation

This section details the nodal-based finite-element implementation of the EH functional introduced in Section III. The purpose of this section is primarily that of demonstrating the robust nature of the EH functional in eliminating spurious solutions with node-based elements. The first variation will be recast in terms of the wavenumber and the related matrix representation will then be examined. A least-squares matrix approach will be employed that will yield results that are free from the spurious solutions that plague the full-field and penalty approaches. This functional will be shown to eliminate spurious solutions in finite-element applications and provide a solution which converges as the node density of the mesh used to discretize the domain is increased.

To set up the element matrices for a homogeneous media, we normalize the constants C_1 and C_2 from the first variation of (13) and (14) and replace the complex radian frequency s with the wavenumber k and the wave impedance η to obtain

$$\int_s \{c_1 jk^*[(\eta\nabla_t \times \vec{h} - \gamma\eta\hat{z} \times \vec{h} - jk\vec{e}) \cdot \vec{e}_n^*] + c_2[(\nabla_t \times \vec{e} - \gamma\hat{z} \times \vec{e} + jk\eta\vec{h}) \cdot (\nabla_t \times \vec{e}_n^* - \gamma^*\hat{z} \times \vec{e}_n^*)]\}\, dS = 0 \tag{28}$$

and

$$\int_s \{c_1[(\eta\nabla_t \times \vec{h} - \gamma\eta\hat{z} \times \vec{h} - jk\vec{e}) \cdot (\nabla_t \times \vec{h}_2^* - \gamma^*\hat{z} \times \vec{h}_n^*)] - c_2 jk^*[(\nabla_t \times \vec{e} - \gamma\hat{z} \times \vec{e} + jk\eta\vec{h}) \cdot \vec{h}_n^*]\}\, dS = 0. \tag{29}$$

Equations (28) and (29) may be written in matrix form as

$$jk^* c_1\{(\mathbf{T_1} - \gamma\mathbf{T_2})\eta\vec{h} - jk\mathbf{M_e}\vec{e}\} + c_2\{(\mathbf{K_{e1}} - \gamma\mathbf{K_{e2}} - \gamma^*\mathbf{K}_{\mathbf{e2}}^\dagger + |\gamma|^2\mathbf{K_{e3}})\vec{e} + jk(\mathbf{S_1} - \gamma^*\mathbf{S_2})\eta\vec{h}\} = 0 \tag{30}$$

and

$$c_1\{(\mathbf{K_{h1}} - \gamma\mathbf{K_{h2}} - \gamma^*\mathbf{K}_{\mathbf{h2}}^\dagger + |\gamma|^2\mathbf{K_{h3}})\eta\vec{h} - jk(\mathbf{T}_\mathbf{1}^\dagger - \gamma^*\mathbf{T}_\mathbf{2}^\dagger)\vec{e}\} - jk^* c_2\{(\mathbf{S}_\mathbf{1}^\dagger - \gamma\mathbf{S}_\mathbf{2}^\dagger)\vec{e} - jk\mathbf{M_h}\eta\vec{h}\} = 0 \tag{31}$$

where $\vec{e}$ and $\vec{h}$ are vectors representing the coefficients of the electric and magnetic field expansions. In the analytical development, we found that the solution was independent of c_1 and c_2. Thus, we are able to write the four parts of (30) and (31) independently equal to zero as

$$(\mathbf{T_1} - \gamma\mathbf{T_2})\eta\vec{h} - jk\mathbf{M_e}\vec{e} = 0 \tag{32}$$

$$(\mathbf{K_{e1}} - \gamma\mathbf{K_{e2}} - \gamma^*\mathbf{K}_{\mathbf{e2}}^\dagger + |\gamma|^2\mathbf{K_{e3}})\vec{e} + jk(\mathbf{S_1} - \gamma^*\mathbf{S_2})\eta\vec{h} = 0 \tag{33}$$

$$(\mathbf{K_{h1}} - \gamma\mathbf{K_{h2}} - \gamma^*\mathbf{K}_{\mathbf{h2}}^\dagger + |\gamma|^2\mathbf{K_{h3}})\eta\vec{e}_n - jk(\mathbf{T}_\mathbf{1}^\dagger - \gamma^*\mathbf{T}_\mathbf{2}^\dagger)\vec{e} = 0 \tag{34}$$

and

$$(\mathbf{S}_\mathbf{1}^\dagger - \gamma\mathbf{S}_\mathbf{2}^\dagger)\vec{e} - jk\mathbf{M_h}\eta\vec{h} = 0. \tag{35}$$

We may also simply set c_1 and c_2 to unity (or other values) and solve (30) and (31).

There are three basic problems that can be examined in the context of (32)–(35). The first problem is one which considers the propagation constant γ for a prescribed frequency of operation ω (embedded in the definition of k). The second problem, useful in material characterization, involves the determination of k for a measured propagation constant γ at a specified frequency, thus providing the product of μ and ε. The third useful problem examined in this paper is the determination of the cutoff wavenumber k_c or frequency ω_c with $\gamma = 0$.

The resultant overdetermined system may be handled by the application of the method of least squares [10] by incorporating the γ in the matrices and writing the combined matrix as

$$\begin{bmatrix} jk\mathbf{M_e} & -T \\ \mathbf{K_e} & jk\mathbf{S} \\ jk\mathbf{T}^\dagger & -\mathbf{K_h} \\ -\mathbf{S}^\dagger & jk\mathbf{M_h} \end{bmatrix} \begin{pmatrix} \vec{e} \\ \eta\vec{h} \end{pmatrix} = 0 \tag{36}$$

or equivalently

$$jk\begin{bmatrix} \mathbf{M_e} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_1 \\ \mathbf{T}_1^\dagger & \mathbf{0} \\ \mathbf{0} & \mathbf{M_h} \end{bmatrix} \begin{pmatrix} \vec{e} \\ \eta\vec{h} \end{pmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{T}_1 \\ -\mathbf{K}_{\mathrm{e}1} & \mathbf{0} \\ \mathbf{0} & \mathbf{K_{h1}} \\ \mathbf{S}_1^\dagger & \mathbf{0} \end{bmatrix} \begin{pmatrix} \vec{e} \\ \eta\vec{h} \end{pmatrix}. \tag{37}$$

Applying a least-squares matrix approach, (37) is multiplied by the transpose conjugate of the matrix on the left side of (37) to obtain

$$jk\begin{bmatrix} \mathbf{M}_\mathbf{e}^\dagger\mathbf{M_e}\mathbf{T}_1\mathbf{T}_1^\dagger & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_1^\dagger\mathbf{S}_1 + \mathbf{M}_\mathbf{h}^\dagger\mathbf{M_h} \end{bmatrix} \begin{pmatrix} \vec{e} \\ \eta\vec{h} \end{pmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{M}_\mathbf{e}^\dagger\mathbf{T}_1 + \mathbf{T}_1\mathbf{K_{h1}} \\ -\mathbf{S}_1^\dagger\mathbf{K}_{\mathrm{e}1} + \mathbf{M}_\mathbf{h}^\dagger\mathbf{S}_1^\dagger & \mathbf{0} \end{bmatrix} \begin{pmatrix} \vec{e} \\ \eta\vec{h} \end{pmatrix}. \tag{38}$$

It is important to note that (38) is in the form of a generalized eigenvalue problem $Ax = \lambda Bx$ so that all the eigenvalues and

TABLE II
CUTOFF WAVENUMBERS FOR THE EH FUNCTIONAL USING THE LEAST-SQUARES MATRIX APPROACH FOR THE 16- AND 51-NODE MESH

EH Functional - 16 nodes	EH Functional - 51 nodes	Exact
3.2213	3.1735	3.1416
7.1625	6.4754	6.2832
7.8256	6.5890	6.2832
8.4975	7.2982	7.0248
8.6236	7.3209	7.0248
12.224	9.6666	8.8858
12.483	9.7231	8.8858
12.890	10.545	9.4248

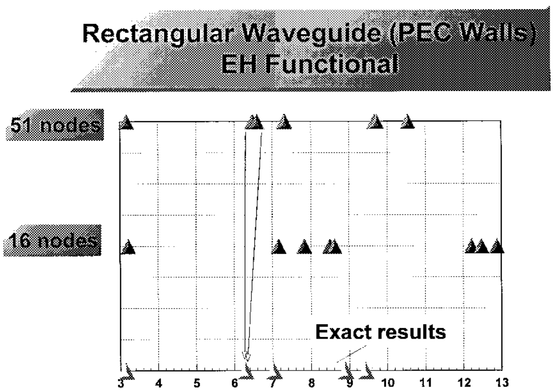

Fig. 3. Graphical depiction of the numerical solution for the EH functional of the data in Table II using the least-squares matrix approach for the 16- and 51-node rectangular waveguide model.

eigenvectors can be computed. All eigenvalue computations in this paper were performed in MATLAB. Table II details the eigenvalues corresponding to the cutoff wavenumbers for the mesh of a 16-node rectangular waveguide model. Once again a graphical approach to the presentation of the data of Table II may prove valuable in the interpretation of the results obtained. Consider the plot of Fig. 3. Note that the approximate numerical modes, though not accurate at higher orders, can be brought into correspondence with the exact solutions.

It is significant to note that the eigenvalues are free from the appearance of the spurious solutions that plagued the full-field and penalty method approaches. The higher order modes are not very accurate for the coarse discretization. The ideas of convergence can be demonstrated by considering a more finely discretized mesh where the rectangular waveguide is modeled with 80 elements and 51 nodes. The results for the cutoff wavenumbers are also given by Table II. The more finely discretized mesh provides improved results appearing to converge toward the exact results. These results demonstrate that even the higher order modes are adequately modeled by the more finely discretized mesh. Once again, there are no spurious solutions generated. It is possible to now bring the solutions of the numerical problem into a one-to-one correspondence with the exact problem. The higher order modes are accurately modeled as well. These numerical results support the claim that nodal elements are useful in used in a properly posed functional form. One disadvantage of the use of (38) is that the some of the sparsity is sacrificed in the matrix multiplication. The use of more restrictive bases such as edge elements or potential forms may provide further improvement.

V. CONCLUSION

The functional representation of the curl–curl form allows divergent solutions to exist as long as a gradient solution is allowed in the solution space. Several approaches have been taken to model the solution space with varied degrees of success. The results of the analytical approach presented in this paper confirm that the functional is the fundamental cause of spurious solutions and may not be completely addressed by the choice of the expansion functions as suggested by the application of edge elements. Through an analytical approach the functional has been found to allow for the existence of an improper gradient behavior in an expansion consistent with the Helmholtz theorem. It has also been shown that the linear behavior of the parameter in the penalty method can be predicted in an exact formulation by the analytical approach.

The key contributions of this work have been the introduction of a new functional (EH) forcing the satisfaction of Maxwell's equations, providing a minimization, and eliminating spurious solutions. The EH functional has been successfully implemented in a finite-element scheme after demonstrating the ability to eliminate the improper gradient behavior in the exact analytical approach. The resulting numerical solution for the cutoff wavenumber demonstrated the robustness of the functional in that spurious solutions were eliminated. The convergence was also briefly examined with the numerical solutions appearing to closely approximate even the higher order modes. This paper has demonstrated a robust functional that will eliminate spurious solutions, satisfy Maxwell's equations, and provide a minimum for *any* reasonable basis set.

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