Green's Function for an Unbounded Biaxial Medium in Cylindrical Coordinates

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***Abstract*— The dyadic Green's function for an unbounded biaxial medium is treated analytically in the Fourier domain. The Green's function is initially expressed as a triple Fourier integral, which is next reduced to a double one by performing the integration over the longitudinal Fourier variable. A delta-type source term is extracted, which is dependent on the particular coordinate system.**

***Index Terms*—Biaxial medium, dyadic Green's function.**

I. Introduction

RADIATION of electromagnetic waves in unbounded anisotropic media is a problem of great interest from both a theoretical and a practical point of view. In treating such problems, the well-known Sommerfeld radiation condition is not applicable and under certain conditions incoming waves appear. Several methods have been proposed to treat radiation in anisotropic media. Arbel and Felsen [1] have proposed the energy radiation condition based on the requirement that the waves transport energy away from the source. Seshadri and Wu [2] made use of the principle of causality, which requires the absence of response before the starting of the source. Lee and Lo [3] and Cottis and Kondylis [4] made use of the limiting absorption principle to obtain a unique solution in anisotropic lossless media. Further treatment of wave propagation in anisotropic media can be found in [5]–[12]. In [13], Green's dyadics in bianisotropic media are treated by means of a Fourier transform approach and asymptotic expressions are derived for the far- and near-field regions.

In the present work, the dyadic Green's function for an unbounded biaxial anisotropic medium is derived in the Fourier transform domain in a cylindrical coordinate system. Such a coordinate system is well suited to configurations such as cylindrical waveguides, two-dimensional problems, and so on. We consider a medium characterized by a real diagonal relative permittivity tensor $\bar{\underline{\varepsilon}}$ of the form

$$\bar{\underline{\varepsilon}} = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} = \varepsilon_1 \hat{x}\hat{x} + \varepsilon_2 \hat{y}\hat{y} + \varepsilon_3 \hat{z}\hat{z} \tag{1}$$

where ε_i, $i = 1, 2, 3$, are real positive quantities. Such a form can cover a wide variety of media (namely, those with a real symmetric permittivity tensor) by means of a coordinate rotation, as explained in [11]. The medium is magnetically isotropic with a magnetic permeability $\mu_0 = 4\pi \times 10^{-7}$ H/m. An $\exp[-j\omega t]$ time dependence is assumed throughout the text.

The Green's function is first determined as a triple inverse Fourier integral. Then, the integration over the longitudinal Fourier variable λ is performed and the far-field behavior of the two kinds of constituent waves that appear is studied in the spectral domain. A delta-type source term is extracted, which is dependent on the specific application of Fourier integration. It is interesting to notice that the cylindrical coordinate system provides the opportunity to point out the different forms of the delta source term according to the different principal volume implicitly assumed; this is in contrast to the spherical coordinate system [11], [14], where only the implicit choice of a spherical principal volume is possible within the framework of Fourier transform approach.

II. Formulation of the Problem

The dyadic Green's function due to a point source excitation located at $\underline{r'}$ inside a biaxial medium must satisfy the tensor Helmholtz equation

$$\nabla \times \nabla \times \bar{\underline{G}}(\underline{r}/\underline{r'}) - k_o^2 \bar{\underline{\varepsilon}} \cdot \bar{\underline{G}}(\underline{r}/\underline{r'}) = \bar{\underline{I}}\delta(\underline{r} - \underline{r'}) \tag{2}$$

where k_0 is the free-space wavenumber and $\bar{\underline{I}}$ is the unit dyadic. To solve (2) in cylindrical coordinates, $\bar{\underline{G}}(\underline{r}/\underline{r'})$ is represented through its inverse Fourier transform as

$$\bar{\underline{G}}(\underline{r}/\underline{r'}) = \iiint \bar{\underline{g}}(\underline{k}) \exp[j\underline{k} \cdot (\underline{r} - \underline{r'})]\, d\underline{k} \tag{3}$$

where $\bar{\underline{g}}(\underline{k})$ is the Fourier transform of $\bar{\underline{G}}(\underline{r}/\underline{r'})$ and

$$\underline{k} = k\hat{k} = p\hat{p} + \lambda\hat{z} \tag{4}$$

is the Fourier transform variable in cylindrical coordinates (p, φ_p, λ). The limits of integration in (3) run from $-\infty$ to $+\infty$ and are omitted for simplicity; this convention will be maintained in the following. The dielectric tensor of (1) is written in cylindrical coordinates as

$$\bar{\underline{\varepsilon}} = \begin{bmatrix} \varepsilon_1 \cos^2\varphi_p + \varepsilon_2 \sin^2\varphi_p & (\varepsilon_2 - \varepsilon_1)\sin\varphi_p\cos\varphi_p & 0 \\ (\varepsilon_2 - \varepsilon_1)\sin\varphi_p\cos\varphi_p & \varepsilon_1 \sin^2\varphi_p + \varepsilon_2\cos^2\varphi_p & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} \tag{5}$$

Substituting (3) and the corresponding Fourier integral expression for the delta function into (2), using (5), and applying the operator $\nabla\times\nabla\times$ with respect to the ρ, φ, z spatial cylindrical coordinates, the following matrix equation for the elements of $\bar{\underline{g}}(\underline{k})$ results:

$$\bar{\underline{A}}(\underline{k}) \cdot \bar{\underline{g}}(\underline{k}) = \frac{1}{(2\pi)^3}\bar{\underline{I}} \tag{6}$$

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where

$$\underline{\bar{A}}(\underline{k}) = k^2(\underline{\bar{I}} - \hat{k}\hat{k}) - k_0^2\underline{\bar{\varepsilon}}. \tag{7}$$

Solving (6) and substituting the solution into (3), $\underline{\bar{G}}(\underline{r}/\underline{r}')$ is obtained in the form of a triple Fourier integral

$$\underline{\bar{G}}(\underline{r}/\underline{r}') = \frac{1}{(2\pi)^3}\iiint \frac{\overline{\underline{a^{(p)}}}(\underline{k})}{D(\underline{k})}\exp[j\underline{k}\cdot(\underline{r}-\underline{r}')]\,d\underline{k} \tag{8}$$

where $D(\underline{k})$ is the determinant of $\underline{\bar{A}}(\underline{k})$ and $\overline{\underline{\alpha^{(P)}}}(\underline{k})$ its adjoint matrix given by

$$\overline{\underline{a^{(p)}}}(\underline{k}) = \begin{bmatrix} a_{pp} & a_{p\varphi_p} & a_{p\lambda} \\ a_{\varphi_p p} & a_{\varphi_p\varphi_p} & a_{\varphi_p\lambda} \\ a_{\lambda_p} & a_{\lambda_{\varphi_p}} & a_{\lambda\lambda} \end{bmatrix}. \tag{9}$$

The elements of $\overline{\underline{\alpha^{(p)}}}(\underline{k})$ are given in the Appendix.

To study the Green's function, one may regard $\underline{\bar{G}}(\underline{r}/\underline{r}')$ as a wave propagating either along the z axis or away from the z axis. Accordingly, the above integral should be written in a suitable form by expressing $D(\underline{k})$ as a biquadratic polynomial in either the λ Fourier variable or the p Fourier variable.

In the first case, one writes $D(\underline{k})$ as

$$D(\underline{k}) = -k_0^2\varepsilon_3\left(\lambda^2-\lambda_1^2\right)\left(\lambda^2-\lambda_2^2\right) \tag{10}$$

where $\pm\lambda_l = \pm\lambda_i(p,\varphi_p)$, $i=1,2$ are the four roots of the biquadratic in λ

$$-k_0^2\varepsilon_3[\lambda^4 + b_\lambda\lambda^2 + c_\lambda] = 0 \tag{11}$$

where

$$b_\lambda = b_\lambda(p,\varphi_p) = -k_0^2(\varepsilon_1+\varepsilon_2) + [1+\xi(\varphi_p)/\varepsilon_3]p^2 \tag{12}$$

$$c_\lambda = c_\lambda(p,\varphi_p) = [\xi(\varphi_p)/\varepsilon_3]\,p^4 - k_0^2[\varepsilon_1\varepsilon_2/\varepsilon_3 + \xi(\varphi_p)]\,p^2 + k_0^4\varepsilon_1\varepsilon_2 \tag{13}$$

$$\xi(\varphi_p) = \varepsilon_1\cos^2\varphi_p + \varepsilon_2\sin^2\varphi_p. \tag{14}$$

In the second case, $D(\underline{k})$ is written as

$$D(\underline{k}) = -k_0^2\xi(\varphi_p)\left(p^2-p_1^2\right)\left(p^2-p_2^2\right) \tag{15}$$

where $\pm p_i = \pm p_i(\lambda,\varphi_p)$, $i=1,2$ are the four roots of the biquadratic in p

$$-k_0^2\xi(\varphi_p)[p^4 + b_p p^2 + c_p] = 0 \tag{16}$$

where

$$b_p = b_p(\lambda,\varphi_p) = [\xi(\varphi_p)+\varepsilon_3]\left(\lambda^2 - k_0^2\varepsilon_3\frac{\xi(\varphi_p)+\varepsilon_1\varepsilon_2/\varepsilon_3}{\xi(\varphi_p)+\varepsilon_3}\right) \tag{17}$$

$$c_p = c_p(\lambda) = \varepsilon_3\left(\lambda^2-k_0^2\varepsilon_1\right)\left(\lambda^2-k_0^2\varepsilon_2\right). \tag{18}$$

To eliminate the integration over one Fourier variable, say over λ, use is made of the well-known expansion of a plane wave

$$\exp(j\underline{k}\cdot\underline{r}) = \exp(j\lambda z)\sum_{m=-\infty}^{+\infty} j^m J_m(p\rho)\exp[jm(\varphi-\varphi_p)].$$

Taking into account the above relations, $\underline{\bar{G}}(\underline{r}/\underline{r}')$ is written as

$$\begin{aligned}\underline{\bar{G}}(\underline{r}/\underline{r}') = &-\frac{1}{k_0^2(2\pi)^3}\int_0^{+\infty}p\,dp\int_0^{2\pi}d\varphi_p\int_{-\infty}^{+\infty}d\lambda \\ &\times\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} j^{m-n}\exp[j\lambda(z-z')] \\ &\cdot\exp[j(m\varphi+n\varphi')]\exp[-j(m+n)\varphi_p] \\ &\times\frac{J_m(p\rho)J_n(p\rho')}{D(\underline{k})}\underline{\bar{a}}(p,\varphi_p,\lambda)\end{aligned} \tag{19}$$

where $\underline{\bar{a}}(\underline{k})$ is the matrix resulting after the translation of $\overline{\underline{a^{(p)}}}(\underline{k})$ from the (p,φ_p,λ) cylindrical coordinate system to the (ρ,φ,z) cylindrical coordinate system, the unit vectors of which do not depend upon the integration variables. This is accomplished via the transformation

$$\underline{\bar{a}}(\underline{k}) = \underline{\bar{T}}^{-1}\cdot\overline{\underline{a^{(p)}}}(\underline{k})\cdot\underline{\bar{T}} \tag{20}$$

and

$$\underline{\bar{T}} = \begin{bmatrix}\hat{p}\cdot\hat{\rho} & \hat{p}\cdot\hat{\varphi} & \hat{p}\cdot\hat{z} \\ \hat{\varphi}_p\cdot\hat{\rho} & \hat{\varphi}_p\cdot\hat{\varphi} & \hat{\varphi}_p\cdot\hat{z} \\ \hat{z}\cdot\hat{\rho} & \hat{z}\cdot\hat{\varphi} & \hat{z}\cdot\hat{z}\end{bmatrix} = \begin{bmatrix}\cos(\varphi_p-\varphi) & \sin(\varphi_p-\varphi) & 0 \\ -\sin(\varphi_p-\varphi) & \cos(\varphi_p-\varphi) & 0 \\ 0 & 0 & 1\end{bmatrix} \tag{21}$$

with $\underline{\bar{T}}^{-1} = \underline{\bar{T}}^T$ due to the orthonormal nature of the cylindrical coordinate system.

III. Integration over the Longitudinal Fourier Variable

In this section, the case corresponding to $\underline{r} \neq \underline{r}'$ will be examined, while the case $\underline{r} \rightarrow \underline{r}'$ will be examined in Section IV. A study of the integrand function of (19) is necessary.

First, a careful examination of the integrand function with respect to the φ_p Fourier variable reveals that two kinds of integrals over φ_p appear; those related to the tensor elements $\alpha_{\rho\rho}$, $\alpha_{\rho\varphi}$, $\alpha_{\varphi\rho}$, $\alpha_{\varphi\varphi}$, and α_{zz}, that are zero when $m+n$ is odd and those related to $\alpha_{\rho z}$, $\alpha_{\varphi z}$, $\alpha_{z\rho}$, and $\alpha_{z\varphi}$ that are zero when $m+n$ is even. Taking this into account, (19) is written as

$$\begin{aligned}\underline{\bar{G}}(\underline{r}/\underline{r}') = &-\frac{1}{k_0^2(2\pi)^3}\int_0^{+\infty}p\,dp\int_0^{2\pi}d\varphi_p \\ &\times\int_{-\infty}^{+\infty}d\lambda\frac{\exp[j\lambda(z-z')]}{D(\underline{k})} \\ &\cdot\Bigg\{\sum_{\substack{m,n\\(e)}} j^{m-n}J_m(p\rho)J_n(p\rho')\exp[j(m\varphi+n\varphi')] \\ &\times\exp[-j(m+n)\varphi_p]\underline{\bar{a}_e}(p,\varphi_p,\lambda) \\ &+\sum_{\substack{m,n\\(o)}} j^{m-n}J_m(p\rho)J_n(p\rho')\exp[j(m\varphi+n\varphi')] \\ &\times\exp[-j(m+n)\varphi_p]\underline{\bar{a}_o}(p,\varphi_p,\lambda)\Bigg\}\end{aligned} \tag{22}$$

where

$$\underline{\overline{a_e}}(p,\varphi_p,\lambda)=\begin{bmatrix}a_{\rho\rho} & a_{\rho\varphi} & 0\\ a_{\varphi\rho} & a_{\varphi\varphi} & 0\\ 0 & 0 & a_{zz}\end{bmatrix}$$
$$\underline{\overline{a_o}}(p,\varphi_p,\lambda)=\begin{bmatrix}0 & 0 & a_{\rho z}\\ 0 & 0 & a_{\varphi z}\\ a_{z\rho} & a_{z\varphi} & 0\end{bmatrix} \tag{23}$$

and

$$\sum_{\substack{m,n\\(e)}}=\sum_{m=-\infty}^{+\infty}\sum_{\substack{n=-\infty\\m+n=\text{even}}}^{+\infty}\qquad \sum_{\substack{m,n\\(o)}}=\sum_{m=-\infty}^{+\infty}\sum_{\substack{n=-\infty\\m+n=\text{odd}}}^{+\infty}. \tag{24}$$

Having as our objective to perform the integration over the λ variable, we regard $\underline{\bar{G}}(\underline{r}/\underline{r'})$ as a wave propagating along the z axis and write $D(\underline{k})$ as in (10). Thus, four poles appear in the integrand function, namely $\pm\lambda_l=\pm\lambda_i(p,\varphi_p)$, $i=1,2$.

The integrations to be performed are of the form

$$\underline{\bar{I}}_\lambda=\int_{-\infty}^{+\infty}\frac{\underline{\bar{F}}(p,\varphi_p,\lambda)\exp[j\lambda(z-z')]}{(\lambda^2-\lambda_1^2)(\lambda^2-\lambda_2^2)}\,d\lambda. \tag{25}$$

It may easily be verified that the elements of $\underline{\bar{F}}(p,\varphi_p,\lambda)$ contain λ at some power from zero to three, except for the $\hat{z}\hat{z}$ element containing a λ^4 term. Therefore, the above integrals can be evaluated using residue theory for all values of z with the exception of the $\hat{z}\hat{z}$ term when $z=z'$. This case corresponds to the singularity source term and is examined in Section IV. The integrals given by (25) are broken into integrals of the form

$$\underline{\bar{I}}_\lambda^{(i)}=\int_{-\infty}^{+\infty}\frac{\lambda^i\exp[j\lambda(z-z')]}{(\lambda^2-\lambda_1^2)(\lambda^2-\lambda_2^2)}\,d\lambda,\quad i=0,1,2,3. \tag{26}$$

These integrals are evaluated by appropriate contour integration to yield the final result

$$\underline{\bar{I}}_\lambda^{(i)}=\frac{j\pi s}{(\lambda_2^2-\lambda_1^2)}\left[\lambda_1^{i-1}\exp[j\lambda_1|z-z'|]\right.$$
$$\left.-\lambda_2^{i-1}\exp[j\lambda_2|z-z'|]\right] \tag{27}$$

where

$$s=\begin{matrix}1, & z>z'\\ (-1)^i, & z<z'\end{matrix}.$$

Details about how the above result is derived and about the physical interpretation of the two types of waves that come up are given in [4]. It should be noted that under certain conditions, exponentially decreasing waves along the z axis are observed as well as incoming waves.

IV. Derivation of the Delta-Type Source Singularity

The singular behavior of Green's dyadics as the observation point traverses the source point has been extensively discussed over the past 30 years for the isotropic case (see, e.g., [15]–[22] and references therein). From the whole discussion, one may infer that in regions containing the source point, the Green's dyadic should be viewed as a singular distribution, i.e., one that cannot be identified to any ordinary function. In some approaches [19]–[20], the whole singular term is considered; in others [18], [21], an explicit delta-type term is extracted while a distributional behavior (not of the delta type) is also embedded in the remaining terms via a special limiting procedure (principal volume approach) or via the structure of the eigenfunction expansion in the source region.

By means of the Fourier representation approach adopted here, the delta-type source term may be derived in a fairly straightforward manner. It should be noted that the Fourier transform associating $\underline{\bar{G}}$ to $\underline{\bar{g}}$ and vice versa should be interpreted in the distributional sense; more precisely, given any scalar component g of $\underline{\bar{g}}$, the corresponding scalar component G of $\underline{\bar{G}}$ is a distribution defined via its "inner product" by the appropriate testing functions ψ

$$\langle G,\psi\rangle=\iiint G(\underline{r},\underline{r'})\psi(\underline{r})\,d\underline{r}$$
$$=\iiint\exp[-j\underline{k}\cdot\underline{r'}]\,g(\underline{k})\tilde{\psi}(\underline{k})\,d\underline{k} \tag{28}$$

where

$$\tilde{\psi}(\underline{k})=\frac{1}{(2\pi)^3}\iiint\tilde{\psi}(\underline{r})\exp[j\underline{k}\cdot\underline{r}]\,d\underline{r}. \tag{29}$$

This is the familiar Parseval's identity, extended to define the distributional Fourier transform [23]; this extension applies particularly when the "conventional" inverse Fourier transform imposed on g gives rise to divergent integrals, as in the case of delta distributions. The right-hand side of (28) converges provided $\tilde{\psi}(\underline{k})$ is sufficiently well behaved at infinity, i.e., $\psi(\underline{r})$ is sufficiently smooth.

Equations (28)–(29) show that the only components of $\underline{\bar{g}}$ yielding terms proportional to $\psi(\underline{r'})$, and, hence, corresponding to components of $\underline{\bar{G}}$ containing a $\delta(\underline{r}-\underline{r'})$ term are the ones that contain a constant term. This may be expressed in a more precise way by means of the concept of the support of a distribution. Upon inspection of (8)–(9), taking into account (10) and (15), it may be seen that such are the p^4 term in α_{pp} and the λ^4 term in $\alpha_{\lambda\lambda}$, i.e., the following terms in g_{pp} and $g_{\lambda\lambda}$, respectively,

$$\frac{p^4}{k_0^2\xi(\varphi_p)(p^2-p_1^2)(p^2-p_2^2)} \tag{30}$$
$$\frac{\lambda^4}{k_0^2\varepsilon_3(\lambda^2-\lambda_1^2)(\lambda^2-\lambda_2^2)}. \tag{31}$$

An analogous approach of considering the leading terms in a corresponding partial fraction expansion has been proposed in [24].

At this point, one could take into consideration [22] that when the inverse Fourier transform is imposed on $\underline{\bar{g}}$, the order of the integrations over the variables λ and p that extend up to infinity implies a certain principal volume procedure. More precisely, performing the integration over λ first implies that the dimension of the principal volume along the z axis vanishes first, corresponding to a pillbox-shaped volume [22]. Similarly, performing the integration over p first corresponds to a needle-shaped principal volume.

Applying this procedure to the terms (30)–(31) given above, one observes the following: If the integration over λ is performed first, the integral over λ resulting from the term (30) is convergent. Moreover, (12)–(14) show that when p goes to infinity, λ_1^2 and λ_2^2 go to $-\infty$, i.e., the roots λ_1 and λ_2 are purely imaginary. Hence, the integration over λ gives rise to an expression that is exponentially decaying with p, which subsequently renders the resulting integral over φ_p and p convergent. Thus, the term (30) does not produce a delta-type term in this case. An analogous situation holds for the term (31) when the integration over p is performed first. To summarize: in the case of a pillbox-shaped principal volume, only the term (31) contributes a delta-type singularity, while in the case of a needle-shaped principal volume only the term (30) does.

According to the previous considerations, the specific part of the term (31) that leads to the delta-type term when the integration over λ is performed first is

$$\frac{1}{k_0^2\varepsilon_3}$$

and, in view of (A.3), this term arises also in g_{zz}. Hence, after Fourier inversion, it is readily seen that the delta-type term is

$$\frac{1}{k_0^2\varepsilon_3}\delta(\underline{r}-\underline{r}')\hat{z}\hat{z}$$

in the case of a pillbox principal volume.

Likewise, in the case of a needle-shaped principal volume, the (30) term contributes to the delta-type term through

$$\frac{1}{k_0^2\xi(\varphi_p)} = \frac{1}{k_0^2(\varepsilon_1\cos^2\varphi_p+\varepsilon_2\sin^2\varphi_p)}.$$

Upon translation in the spherical cordinate system via (20)–(21) (see also (A.1) of the Appendix), this term appears in $g_{\rho\rho}$, $g_{\rho\varphi}$, $g_{\varphi\rho}$, $g_{\varphi\varphi}$, multiplied by

$$(\hat{p}\cdot\hat{\rho})^2 = \cos^2(\varphi_p-\varphi) = \cos^2\varphi\cos^2\varphi_p + \sin^2\varphi\sin^2\varphi_p + 2\cos\varphi\sin\varphi\cos\varphi_p\sin\varphi_p \tag{32}$$

$$(\hat{p}\cdot\hat{\varphi})^2 = \sin^2(\varphi_p-\varphi) = \sin^2\varphi\cos^2\varphi_p + \cos^2\varphi\sin^2\varphi_p - 2\cos\varphi\sin\varphi\cos\varphi_p\sin\varphi_p \tag{33}$$

$$\begin{aligned}(\hat{p}\cdot\hat{\rho})(\hat{p}\cdot\hat{\varphi}) &= \cos(\varphi_p-\varphi)\sin(\varphi_p-\varphi)\\ &= (\cos^2\varphi-\sin^2\varphi)\cos\varphi_p\sin\varphi_p\\ &\quad+\sin\varphi\cos\varphi(\sin^2\varphi_p-\cos^2\varphi_p).\end{aligned} \tag{34}$$

Hence, the following terms emerge:

$$\frac{\cos^2\varphi_p}{\varepsilon_1\cos^2\varphi_p+\varepsilon_2\sin^2\varphi_p} = \frac{1+\cos 2\varphi_p}{\alpha+\beta\cos 2\varphi_p} \tag{35a}$$

$$\frac{\sin^2\varphi_p}{\varepsilon_1\cos^2\varphi_p+\varepsilon_2\sin^2\varphi_p} = \frac{1-\cos 2\varphi_p}{\alpha+\beta\cos 2\varphi_p} \tag{35b}$$

$$\frac{\cos\varphi_p\sin\varphi_p}{\varepsilon_1\cos^2\varphi_p+\varepsilon_2\sin^2\varphi_p} = \frac{\sin 2\varphi_p}{\alpha+\beta\cos 2\varphi_p} \tag{35c}$$

where

$$\alpha = \varepsilon_1+\varepsilon_2 \quad \beta = \varepsilon_1-\varepsilon_2. \tag{36}$$

The constant parts contributed by the above terms may be found as the zero-order coefficients in their Fourier series expansion, as follows:

$$C_0^{(a)} = \frac{1}{2\pi}\int_0^{2\pi}\frac{1+\cos 2\varphi_p}{\alpha+\beta\cos 2\varphi_p}\,d\varphi_p \tag{37a}$$

$$C_0^{(b)} = \frac{1}{2\pi}\int_0^{2\pi}\frac{1-\cos 2\varphi_p}{\alpha+\beta\cos 2\varphi_p}\,d\varphi_p \tag{37b}$$

$$C_0^{(c)} = \frac{1}{2\pi}\int_0^{2\pi}\frac{\sin 2\varphi_p}{\alpha+\beta\cos 2\varphi_p}\,d\varphi_p. \tag{37c}$$

Obviously, $C_0^{(c)} = 0$ due to the odd symmetry of the integrand around the point $\varphi_p = \pi$, while the other two coefficients can be evaluated by means of the integrals

$$\begin{aligned}\int_0^{2\pi}\frac{1}{\alpha+\beta\cos 2\varphi_p}\,d\varphi_p &= 4\int_0^{\pi/2}\frac{1}{\alpha+\beta\cos 2\varphi_p}\,d\varphi_p\\ &= \frac{2\pi}{\sqrt{\alpha^2-\beta^2}}\\ \int_0^{2\pi}\frac{\cos 2\varphi_p}{\alpha+\beta\cos 2\varphi_p}\,d\varphi_p &= 4\int_0^{\pi/2}\frac{\cos 2\varphi_p}{\alpha+\beta\cos 2\varphi_p}\,d\varphi_p\\ &= \frac{2\pi}{\beta}-\frac{2\pi\alpha}{\beta\sqrt{\alpha^2-\beta^2}}\end{aligned}$$

which eventually yield

$$C_0^{(a)} = \frac{1}{\sqrt{\varepsilon_1}(\sqrt{\varepsilon_1}+\sqrt{\varepsilon_2})} \tag{38a}$$

$$C_0^{(b)} = \frac{1}{\sqrt{\varepsilon_2}(\sqrt{\varepsilon_1}+\sqrt{\varepsilon_2})}. \tag{38b}$$

From (19) and (20) and taking (38a) and (38b) into account, one concludes that the delta-type term is

$$\begin{aligned}&\big[\big(C_0^{(a)}\cos^2\varphi + C_0^{(b)}\sin^2\varphi\big)\hat{\rho}\hat{\rho}\\ &\quad+\big(C_0^{(a)}\sin^2\varphi + C_0^{(b)}\cos^2\varphi\big)\hat{\varphi}\hat{\varphi}\\ &\quad+\big(C_0^{(a)}-C_0^{(b)}\big)\sin\varphi\cos\varphi(\hat{\rho}\hat{\varphi}+\hat{\varphi}\hat{\rho})\big]\delta(\underline{r}-\underline{r}')\\ &\quad=\left[\left(\frac{\cos^2\varphi}{\sqrt{\varepsilon_1}}+\frac{\sin^2\varphi}{\sqrt{\varepsilon_2}}\right)\hat{\rho}\hat{\rho}+\left(\frac{\sin^2\varphi}{\sqrt{\varepsilon_1}}+\frac{\cos^2\varphi}{\sqrt{\varepsilon_2}}\right)\hat{\varphi}\hat{\varphi}\right.\\ &\qquad\left.+\frac{\sqrt{\varepsilon_1}-\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1\varepsilon_2}}\sin\varphi\cos\varphi(\hat{\rho}\hat{\varphi}+\hat{\varphi}\hat{\rho})\right]\\ &\qquad\times\frac{1}{\sqrt{\varepsilon_1}+\sqrt{\varepsilon_2}}\delta(\underline{r}-\underline{r}').\end{aligned}$$

In the isotropic case $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon$ the above terms reduce to

$$\frac{1}{2\varepsilon}(\hat{\rho}\hat{\rho}+\hat{\varphi}\hat{\varphi})\delta(\underline{r}-\underline{r}')$$

for a needle-shaped principal volume or

$$\frac{1}{\varepsilon}\hat{z}\hat{z}\delta(\underline{r}-\underline{r}')$$

for a pillbox-shaped principal volume. Both expressions coincide to those given in the literature [18].

V. Conclusion

The dyadic Green's function for an unbounded biaxial medium has been treated analytically in the Fourier domain. The initial triple Fourier integral is reduced to a double one by performing the integration over the longitudinal Fourier variable. The delta-type source term has been extracted for both a pillbox- and a needle-shaped principal volume. In the limit $\varepsilon_1 = \varepsilon_2 = \varepsilon_3$, this term reduces to the corresponding one of the isotropic case given in the literature.

Appendix

After some trivial algebra, the elements of the adjoint matrix $\overline{\underline{\alpha^{(p)}}}(\underline{k})$ featuring in (9) are found to be

$$\begin{aligned} a_{pp} &= p^4 + \left(\lambda^2 - B - k_0^2\varepsilon_3\right)p^2 - k_0^2\varepsilon_3(\lambda^2 - B) \\ a_{p\varphi_p} &= a_{\varphi_p p} = -Cp^2 + k_0^2\varepsilon_3 C \\ a_{p\lambda} &= \alpha_{\lambda p} = \lambda p^3 + \lambda(\lambda^2 - B)p \\ a_{\varphi_p\varphi_p} &= -Ap^2 - k_0^2\varepsilon_3(\lambda^2 - A) \\ a_{\varphi_p\lambda} &= a_{\lambda\varphi_p} - \lambda pC \\ a_{\lambda\lambda} &= (\lambda^2 - A)p^2 + (\lambda^2 - B)(\lambda^2 - A) - C^2 \end{aligned}$$

where

$$\begin{aligned} A &= A(\varphi_p) = k_0^2(\varepsilon_1\cos^2\varphi_p + \varepsilon_2\sin^2\varphi_p) \\ B &= B(\varphi_p) = k_0^2(\varepsilon_1\sin^2\varphi_p + \varepsilon_2\cos^2\varphi_p) \\ C &= C(\varphi_p) = k_0^2(\varepsilon_1 - \varepsilon_2)\sin\varphi_p\cos\varphi_p. \end{aligned}$$

After translation into the (ρ, φ, z) cylindrical coordinate system according to the transformation (20), the elements of $\underline{\overline{a}}(\underline{k})$ may readily be found through the relation

$$a_{ij} = \sum_{h,k}(\hat{h}\cdot\hat{i})(\hat{k}\cdot\hat{j})\,a_{hk} \quad i,j = \rho,\varphi,z \quad h,k = p,\varphi_p,\lambda \tag{A.1}$$

For example

$$\begin{aligned} a_{\rho\rho} &= (\hat{p}\cdot\hat{\rho})^2 a_{pp} + (\hat{p}\cdot\hat{\rho})(\hat{\varphi}_p\cdot\hat{\rho})\,a_{p\varphi_p} \\ &\quad + (\hat{\varphi}_p\cdot\hat{\rho})(\hat{p}\cdot\hat{\rho})\,a_{\varphi_p p} + (\hat{\varphi}_p\cdot\hat{\varphi}_p)^2\,a_{\varphi_p\varphi_p} \end{aligned} \tag{A.2}$$

$$a_{zz} = a_{\lambda\lambda} \tag{A.3}$$

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