On a New Cylindrical Harmonic Representation for Spherical Waves

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***Abstract*—An exact series representation is presented for integrals whose integrands are products of cosine and spherical wave functions, where the argument of the cosine term can be any integral multiple n of the azimuth angle ϕ. This series expansion will be shown to have the following form:**

$$I(n) = \frac{e^{-jkR_0}}{R_0}\,\delta_{no} - jk\sum_{m=1}^{\infty} C(m,n)\,\frac{(k^2\rho\rho_0)}{m!}\,\frac{h_m^{(2)}(kR_0)}{(kR_0)^m}.$$

It is demonstrated that in the special cases $n = 0$ and $n = 1$, this series representation corresponds to existing expressions for the cylindrical wire kernel and the uniform current circular loop vector potential, respectively. A new series representation for spherical waves in terms of cylindrical harmonics is then derived using this general series representation. Finally, a closed-form far-field approximation is developed and is shown to reduce to existing expressions for the cylindrical wire kernel and the uniform current loop vector potential as special cases.

***Index Terms*— Circular loop antenna, cylindrical harmonics, cylindrical wire dipole, spherical wave expansions.**

I. Introduction

A family of integrals which frequently arise in antenna theory are considered in this paper. The general form of these integrals is defined by

$$I(n) = \frac{1}{2\pi}\int_0^{2\pi} \cos n\phi\,\frac{e^{-jkR}}{R}\,d\phi \tag{1}$$

where

$$R = \sqrt{\zeta^2 + \rho^2 + \rho_o^2 - 2\rho\rho_0\,\cos\phi} \tag{2}$$

and

$$\zeta = z - z_0. \tag{3}$$

It is of interest to point out here that the integral in (1) reduces to the well-known cylindrical wire kernel when $n = 0$ and, when $n = 1$, it is proportional to the vector potential for a circular loop antenna with a uniformly distributed current. More generally, integrals of the form (1) arise in the decomposition of spherical waves into cylindrical harmonics [1]. Although the special cases $n = 0$ and $n = 1$ of this integral have been treated elsewhere in the literature [2], [3], no general methodology for evaluating integrals of this form currently exists for arbitrary values of the parameter n. Treating the entire family of integrals given by (1) unifies these independent cases as part of a more general framework.

In this paper, we develop an exact series representation for such integrals which is valid for all integer values of the parameter n. The development of this series representation, based essentially upon an expansion for the exponential in terms of spherical Hankel functions of the second kind is given in Section II. In Section III, important immediate applications of this result are considered, including the special cases of the cylindrical wire kernel and the uniform current loop antenna mentioned above, as well as a new series expansion for spherical waves derived by means of a decomposition of these waves into cylindrical harmonics. Finally, a closed-form far-field approximation of (1) involving Bessel functions of the first kind is presented in Section IV.

II. Theoretical Development

In this section, an exact series representation for the general family of integrals defined in (1) will be derived. We begin this derivation by recognizing that (1) may be expressed in the convenient form

$$I(n) = \frac{j}{(2\pi k\zeta)}\,\frac{d}{d\zeta}\int_0^{2\pi} \cos n\phi e^{-jkR}\,d\phi. \tag{4}$$

which reduces the integrand to a product involving a simple exponential. A procedure has been described in [4], which makes use of Lommel expansions in order to derive an exact series representation for the exponential function which appears in the integrand of (4). The resulting series representation is given by

$$e^{-jkR} = e^{-jkR_0} + \sum_{m=1}^{\infty} \frac{(k^2\rho\rho_0)^m\,\cos^m\phi}{m!}\,\frac{h_{m-1}^{(2)}(kR_0)}{(kR_0)^{m-1}} \tag{5}$$

where

$$R_0 = \sqrt{\zeta^2 + \rho^2 + \rho_0^2} \tag{6}$$

and the $h_{m-1}^{(2)}$ are spherical Hankel functions of the second kind of order $m - 1$.

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Substituting expression (5) for the exponential into (4) and integrating term-by-term yields

$$\begin{aligned} I(n) = &\frac{j}{(2\pi k\zeta)} \left(\frac{d}{d\zeta} e^{-jkR_0} \right) \int_0^{2\pi} \cos n\phi \, d\phi \\ &+ \frac{j}{k\zeta} \sum_{m=1}^{\infty} \frac{(k^2\rho\rho_0)^m}{m!} \left(\frac{d}{d\zeta} \frac{h_{m-1}^{(2)}(kR_0)}{(kR_0)^{m-1}} \right) \\ &\cdot \frac{1}{2\pi} \int_0^{2\pi} \cos n\phi \, \cos^m \phi \, d\phi. \end{aligned} \tag{7}$$

The expressions on the right-hand side of (7) can be greatly simplified by carrying out the indicated operations. To evaluate the first term on the right-hand side of (7) we observe that

$$\frac{1}{2\pi} \int_0^{2\pi} \cos n\phi \, d\phi = \delta_{n0} \tag{8}$$

where δ_{n0} represents the Kronecker delta function defined by

$$\delta_{n0} = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

and also that

$$\frac{d}{d\zeta} e^{-jkR_0} = -\frac{jk}{R_0} \zeta e^{-jkR_0}. \tag{10}$$

Thus, by applying (8)–(10), it follows that the first term in (7) reduces to

$$\frac{j}{(2\pi k\zeta)} \left(\frac{d}{d\zeta} e^{-jkR_0} \right) \int_0^{2\pi} \cos n\phi \, d\phi = \frac{e^{-jkR_0}}{R_0} \delta_{n0}. \tag{11}$$

The integral appearing in the second term of (7) can be evaluated by a simple change of variables which yields [3]

$$\begin{aligned} &\frac{1}{2\pi} \int_0^{2\pi} \cos n\phi \, \cos^m \phi \, d\phi \\ &\quad = \frac{1}{\pi} \int_0^{\pi} \cos n\phi \, \cos^m \phi \, d\phi = C(m,n). \end{aligned} \tag{12}$$

where

$$C(m,n) = \begin{cases} \dfrac{1}{2^m} \dbinom{m}{p}, & \text{if } n \le m \text{ and } m-n = 2p \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

Using the recurrence relation

$$\frac{d}{dx} \left(\frac{h_n^{(2)}(x)}{x^n} \right) = -\frac{h_{n+1}^{(2)}(x)}{x^n} \tag{14}$$

along with (12), we can then express the second term of (7) as

$$\begin{aligned} &\frac{j}{k\zeta} \sum_{m=1}^{\infty} \frac{(k^2\rho\rho_0)^m}{m!} \left(\frac{d}{d\zeta} \frac{h_{m-1}^{(2)}(kR_0)}{(kR_0)^{m-1}} \right) \\ &\quad \cdot \frac{1}{2\pi} \int_0^{2\pi} \cos n\phi \, \cos^m \phi \, d\phi \\ &= -jk \sum_{m=1}^{\infty} C(m,n) \frac{(k^2\rho\rho_0)^m}{m!} \frac{h_m^{(2)}(kR_0)}{(kR_0)^m}. \end{aligned} \tag{15}$$

Finally, substituting (11) and (15) into (7) leads us to the following exact series representation for the integral $I(n)$:

$$\begin{aligned} I(n) = &\frac{e^{-jkR_0}}{R_0} \delta_{n0} - jk \sum_{m=1}^{\infty} C(m,n) \frac{(k^2\rho\rho_0)^m}{m!} \\ &\cdot \frac{h_m^{(2)}(kR_0)}{(kR_0)^m}. \end{aligned} \tag{16}$$

III. Applications

A. The Cylindrical Wire Kernel

When $n = 0$, the integral (1) reduces to the well-known cylindrical wire kernel [2]. In other words,

$$K(\zeta) = I(0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-jkR}}{R} d\phi \tag{17}$$

which, according to (16), has the following series representation:

$$K(\zeta) = \frac{e^{-jkR_0}}{R_0} - jk \sum_{m=1}^{\infty} C(m,0) \frac{(k^2\rho\rho_0)^m}{m!} \frac{h_m^{(2)}(kR_0)}{(kR_0)^m}. \tag{18}$$

Because $C(m,0)$ vanishes whenever m is odd, we can introduce the new summation index $n = \frac{1}{2}m$ and thereby recast (18) in the form

$$K(\zeta) = \frac{e^{-jkR_0}}{R_0} - jk \sum_{n=1}^{\infty} \frac{(k^2\rho\rho_0/2)^{2n}}{(n!)^2} \frac{h_{2n}^{(2)}(kR_0)}{(kR_0)^{2n}} \tag{19}$$

which is equivalent to the result obtained by Wang [5]. Algorithms based on this series representation of the cylindrical wire kernel have recently been developed for the efficient yet accurate evaluation of moment method impedance matrix integrals [6].

B. The Uniform Current Loop

The vector potential for a uniform current circular loop antenna can be written as [7]

$$\vec{A} = A_\phi \hat{\phi} \tag{20}$$

where

$$A_\phi = \frac{\rho_0 \mu I_0}{2} I(1) = \frac{\rho_0 \mu I_0}{4\pi} \int_0^{2\pi} \cos \phi \frac{e^{-jkR}}{R} d\phi. \tag{21}$$

It then follows from (16) that a series representation for (21) may be found which has the form

$$A_\phi = \frac{k\rho_0\mu I_0}{2j} \sum_{m=1}^{\infty} C(m,1) \frac{(k^2\rho\rho_0)^m}{m!} \frac{h_m^{(2)}(kR_0)}{(kR_0)^m}. \tag{22}$$

Because the terms of this series are nonzero only if m is odd, we can rewrite (22) as

$$\begin{aligned} A_\phi &= \frac{k\rho_0\mu I_0}{2j} \sum_{n=0}^{\infty} \binom{2n+1}{n} \frac{(k^2\rho\rho_0/2)^{2n+1}}{(2n+1)!} \frac{h_{2n+1}^{(2)}(kR_0)}{(kR_0)^{2n+1}} \\ &= \frac{k\rho_0\mu I_0}{2j} \sum_{n=1}^{\infty} \frac{(k^2\rho\rho_0/2)^{2n-1}}{n!(n-1)!} \frac{h_{2n-1}^{(2)}(kR_0)}{(kR_0)^{2n-1}} \end{aligned} \tag{23}$$

which is in agreement with the result obtained by Werner [3]. Thus, we see from (23) above, together with (19) from Section III-A that the series representation derived in Section I provides, as important special cases, exact expressions for the cylindrical wire kernel and the uniform current circular loop vector potential.

C. Series Expansion for a Spherical Wave

The decomposition of a spherical wave into cylindrical harmonics can be written as [1]

$$\frac{e^{-jkR}}{R} = \sum_{m=0}^{\infty} \varepsilon_m \cos m\phi \int_0^{\infty} \lambda J_m(\lambda\rho) J_m(\lambda\rho_0) \cdot \frac{e^{-|\zeta|\sqrt{\lambda^2-k^2}}}{\sqrt{\lambda^2-k^2}} d\lambda. \tag{24}$$

where ε_m is Neumann's number defined by

$$\varepsilon_m = \begin{cases} 1, & m=0 \\ 2, & m \geq 1. \end{cases} \tag{25}$$

It follows immediately upon multiplying both sides of this equation by $\cos n\phi$ and integrating with respect to ϕ from 0 to 2π that [1]

$$I(n) = \int_0^{\infty} \lambda J_n(\lambda\rho) J_n(\lambda\rho_0) \frac{e^{-|\zeta|\sqrt{\lambda^2-k^2}}}{\sqrt{\lambda^2-k^2}} d\lambda. \tag{26}$$

Hence, by substituting (26) into (24) and making use of (16), we arrive at the following exact series expansion for a spherical wave:

$$\begin{aligned}\frac{e^{-jkR}}{R} &= \sum_{n=0}^{\infty} \varepsilon_n I(n) \cos n\phi \\ &= \frac{e^{-jkR_0}}{R_0} - jk \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} C(m,n)\varepsilon_n \frac{(k^2\rho\rho_0)^m}{m!} \cdot \frac{h_m^{(2)}(kR_0)}{(kR_0)^m} \cos n\phi.\end{aligned} \tag{27}$$

This new spherical wave expansion is particularly useful in those situations, which are often encountered in electromagnetics where integrations of the form

$$\frac{1}{2\pi}\int_0^{2\pi} f(\phi) \frac{e^{-jkR}}{R} d\phi \tag{28}$$

are required. For these cases, term-by-term integration of the series would translate into much simpler integrals of the type

$$\frac{1}{2\pi}\int_0^{2\pi} f(\phi) \cos n\phi \, d\phi \tag{29}$$

which, for most cases of practical interest, can be evaluated in closed form. The double series given in (27) may be reduced to a single series by making use of the following identity:

$$\cos^m \phi = \sum_{n=0}^{\infty} C(m,n)\varepsilon_n \cos n\phi. \tag{30}$$

The resulting series representation is found to be

$$\frac{e^{-jkR}}{R} = \frac{e^{-jkR_0}}{R_0} - jk \sum_{m=1}^{\infty} \frac{(k^2\rho\rho_0)^m}{m!} \frac{h_m^{(2)}(kR_0)}{(kR_0)^m} \cos^m \phi. \tag{31}$$

IV. FAR-FIELD APPROXIMATIONS

If we choose a field point very far from the origin so that $kR \to \infty$, then it can be shown that [8]

$$h_m^{(2)}(kR_0) \to j^{m+1} \frac{e^{-jkr}}{kr} \tag{32}$$

where

$$r = \sqrt{\rho^2 + z^2}. \tag{33}$$

This same argument can be used to prove that

$$\frac{(k^2\rho\rho_0)^m}{m!} \frac{1}{(kR_0)^m} \to \frac{(k\rho_0 \sin\theta)^m}{m!} \tag{34}$$

where θ is the angle from the z-axis to the field point vector. Therefore, by combining (16), (32), and (34), we obtain the following far-field approximation for $I(n)$:

$$\begin{aligned}I(n) &\approx \frac{e^{-jkr}}{r}\delta_{n0} + \frac{e^{-jkr}}{r}\sum_{m=1}^{\infty} C(m,n) j^m \frac{(k\rho_0 \sin\theta)^m}{m!} \\ &= \frac{e^{-jkr}}{r}\sum_{m=0}^{\infty} C(m,n) j^m \frac{(k\rho_0 \sin\theta)^m}{m!}.\end{aligned} \tag{35}$$

At this point in the development, we choose to introduce a new summation index, namely $p = (m-n)/2$, which transforms (35) into

$$\begin{aligned}I(n) &\approx \frac{e^{-jkr}}{r}\sum_{p=0}^{\infty}\binom{2p+n}{p}\frac{j^{2p+n}}{2^{2p+n}}\frac{(k\rho_0 \sin\theta)^{2p+n}}{(2p+n)!} \\ &= \frac{e^{-jkr}}{r} j^n \sum_{p=0}^{\infty} (-1)^p \frac{(k\rho_0 \sin\theta/2)^{2p+n}}{p!(p+n)!} \\ &= \frac{e^{-jkr}}{r} j^n J_n(k\rho_0 \sin\theta)\end{aligned} \tag{36}$$

where the series representation for Bessel functions of the first kind was used in order to arrive at the final closed-form far-field representation in (36). This provides additional validation of the exact series representation for (1) given in (16) since the same result may be readily obtained by applying a standard far-field approximation directly to (1), i.e., replacing R by $r - \rho_0 \sin\theta \cos\phi$ and r in the phase term and amplitude term of the integrand in (1), respectively. It will be demonstrated below that this general far-field expansion reduces to known results for the two special cases considered above (i.e., $n = 0$ and $n = 1$).

A. Far-Field Expression for the Cylindrical Wire Kernel

By setting $n = 0$, we can use (17) and (36) to obtain the following far-field approximation for the cylindrical wire kernel:

$$K(\zeta) \approx \frac{e^{-jkr}}{r} J_0(k\rho_0 \sin\theta) \tag{37}$$

which agrees with the result reported in [9].

B. Far-Field Expression for the Uniform Current Loop

If we now set $n = 1$, we can use (21) and (36) to obtain a far-field expression for the uniform current loop vector potential. Combining these equations we arrive at

$$A_\phi \approx j \, \frac{\rho_0 \mu I_0}{2} \, \frac{e^{-jkr}}{r} \, J_1(k\rho_0 \, \sin\theta) \tag{38}$$

which is in agreement with the classical result [7].

V. Conclusion

An exact series representation for integrals of the form (1) has been presented in this paper. As special cases of this general series representation, the kernel integral for a cylindrical dipole and the vector potential for a uniform current circular loop were evaluated. In addition, the general series representation was used to develop a new and useful series expansion for a spherical wave in terms of cylindrical harmonics. A general closed-form far-field approximation was also developed and shown to reduce to the known results for the special cases of the cylindrical wire kernel and the uniform current loop vector potential.

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