Fractal Analysis of the Signal Scattered from the Sea Surface

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***Abstract*— This paper deals with the problem of the electromagnetic scattering from a sea fractal surface. The purpose of the paper is to demonstrate that the sea-scattered signal retains some fractal characteristic of the sea surface. In detail, we analyze the signal scattered from a sea surface modeled by the one-dimensional (1-D) sea-surface fractal profile proposed in [1]. The results show the graphs of the in-phase and quadrature components of the received signal are fractal curves with box-counting fractal dimension equal to the one of the sea profile. These results are validated by presenting and discussing some numerical examples.**

***Index Terms*—Fractal, scattering, sea.**

I. Introduction

THE electromagnetic scattering from the sea surface is a phenomenon on which researchers spent a lot of time for physical interpretation and analysis. An accurate study of the sea scattering provides profitable advantages in: 1) investigating phenomena like multipath and clutter, which heavily degrades the performance of telecommunication and radar systems; 2) improving and developing some applications in the field of the environmental monitoring like sea-surface characterization, sea SAR imaging, and sea-surface traffic monitoring; 3) interpreting some sea-surface physical phenomena like hydrodynamic evolution of the sea waves, energetic exchange at the sea-atmosphere interface, sea-surface current analysis. Up to date, the most important results on electromagnetic sea scattering are obtained from the statistical analysis of experimental data. Characterization of the sea in terms of the distribution of the scattered signal amplitude is achieved and used to developed statistical models.

Recently, fractal theory has been proposed as a mean for the sea-scattered signal analysis. Since natural surfaces have fractal characteristics [1], [2] it is reasonable to expect that the sea-scattered signal retains some fractal properties of the sea surface. First results in this direction were obtained by Haykin [3]. He showed that the box-counting fractal dimension of the amplitude of the sea-scattered signal for a given sea-state situation is a value of about 1.75. This result was attained trough an analysis of experimental data without giving any theoretical justification.

The purpose of this paper is to give a mathematical demonstration that the signal scattered from a sea fractal surface maintains the main fractal characteristics of the sea surface. To this end the sea surface is modeled by the one-dimensional (1-D) fractal profile proposed in [1] and the expression of sea-scattered signal is calculated in a closed form. The theoretical result shows that plots of the in-phase and quadrature components of the received signal against time are fractal curves with the same dimension of the sea surface. So, a mathematical proof that the signal scattered from the sea actually have a fractal behavior similar to the one of the sea surface is obtained.

The paper is organized as follows. We briefly recall the 1-D sea-surface fractal model proposed in [1]. This model agrees with the solution of the hydrodynamic differential equations [4] and it is based on the band-limited Weierstrass–Mandelbrot (WM) functions [5], [6]. We evaluate the scattering coefficient, which in turn is proportional to the received signal, by using the Kirchhoff method. We decompose the expression of the scattering coefficient as a sum of time-varying terms. We demonstrate that all terms have a box-counting fractal dimension less than or equal to the fractal dimension of the sea profile. By invoking a theorem of the fractal theory relevant to sum of fractal functions [7] we obtain the previous mentioned result: the real and imaginary parts of the received signal are fractals with dimension equal to the dimension of the sea profile. In the last section, we present some numerical simulations in order to validate the theoretical results.

II. Sea Fractal Model

In this section, we briefly recall the 1-D sea fractal model proposed in [1] without entering in a detailed description of the parameters (refer to [1]).

The analytical expression of the fractal sea-surface model with 1-D roughness is

$$f(x,t) = \sigma C \sum_{n=0}^{N_f-1} b^{(s-2)n} \sin\left\{ \frac{2\pi}{(\Lambda_0/b^n)} \times \left[x + \left(V + \frac{\Lambda_0}{b^n}\frac{\Omega_n}{2\pi} \right) t \right] + \Phi_n(t) \right\} \tag{1}$$

where Λ_0 is the fundamental spatial wavelength, b is the scale factor, s is the box-counting fractal dimension, V is the observer platform velocity, Ω_n is the angular frequency (it takes into account for the dispersion effect), $N_f \gg 1$ is the number of sinusoidal components, σ is the standard deviation of the amplitude and C is a normalization constant necessary to have a standard deviation of the sea height equal to σ.

Manuscript received December 31, 1997; revised July 13, 1998. This work was supported in part by the Italian Government through the Ministry of the University.

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Publisher Item Identifier S 0018-926X(99)03722-9.

Note that the graph of $f(x,t)$ for a fixed time instant t is a fractal curve in mathematical sense if $N_f = \infty$. In practical cases, if the number of components composing the sea is sufficiently greater than one we can state that sea profiles behave as a fractal up to a suitable scale depending on the value of N_f. This fact must be carefully considered in the fractal-dimension estimation algorithms to select the scale window under which the signal should be analyzed.

The phases $\Phi_n(t)$ are set according to the following relationship:

$$\Phi_n(t) = \begin{cases} \psi_n & 0 \le n \le n_0 - 1 \\ \vartheta_n(t) & n_0 \le n \le N_f - 1 \end{cases} \tag{2}$$

where ψ_n are initial arbitrary phases modeled as independent random variables uniformly distributed in the interval $[-\pi, \pi]$ and the time-varying phase $\vartheta_n(t)$ is considered as a white stochastic process independent from ψ_n and with a probability density function (pdf) of the first-order uniform in the interval $[-\pi, \pi]$.

The assumption of (2) has the following physical meaning.

1) The sinusoidal components of the sea-surface $f(x,t)$ at low spatial frequencies $(0 \le n \le n_0 - 1)$, which represent the sea long waves, simply translate.
2) The sinusoidal components of the sea-surface $f(x,t)$ at high spatial frequencies $(n_0 \le n \le N_f - 1)$, which represent the fine structure of the sea waves, have random behavior.

Furthermore, the use of (2) also provides a few benefits: 1) the computational algorithm for the generation of the sea profile is easy to implement; 2) the spatial correlation length Γ defined as the value for which the autocorrelation coefficient $\rho(\xi)$ becomes $(1/e)$ is easily to compute [1]; and 3) the normalization constant C can be computed in a closed form

$$C = \sqrt{\frac{2\left(1 - b^{2(s-2)}\right)}{1 - b^{2(s-2)N_f}}}. \tag{3}$$

As shown in the next section, the use of Kirchhoff method for the evaluation of the scattering coefficient requires low incident and scattering angles. To satisfy this requirement we refer to the case of an airborne or spaceborne radar. In this condition: 1) the platform velocity V is so high that sea wave velocities $\frac{\Lambda_0}{b^n}\frac{\Omega_n}{2\pi}$ can be neglected; 2) in the typical observation time of the order of one second or less we can assume that the sea wave height have a sinusoidal law against the time (the phases $\Phi_n(t)$ become independent on time). The sea model can be simplified as

$$f(x,t) = \sigma C \sum_{n=0}^{N_f-1} b^{(s-2)n} \sin\left\{\frac{2\pi}{(\Lambda_0/b^n)}[x + Vt] + \Phi_n\right\}. \tag{4}$$

For the sake of the simplicity, in the theoretical analysis we always refer to the model of (4).

The general sea model will be considered in the numerical analysis.

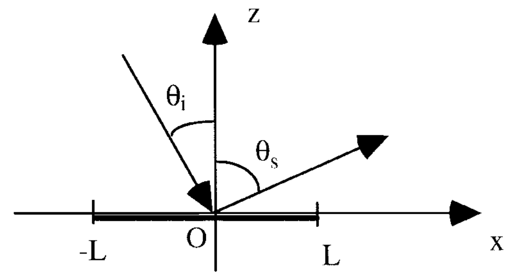

Fig. 1. Scattering geometry.

III. Scattering Coefficient Evaluation

In this section, we analyze the problem of the scattering from a sea fractal surface and we calculate the scattering coefficient in a closed form. Note that the scattering coefficient is proportional to the complex received signal, so the theoretical results we obtain from its analysis can be directly used to characterize the received signal.

Because of the complexity involved in the computation, in this section we only report the expression of the scattering coefficient and the conditions under which Kirchhoff approximation can be applied. For more details, the reader can refer to [1], [6], and [8].

Let us consider the geometry of Fig. 1 where θ_i and θ_s are the incident and the scattering angle, respectively, and $2L$ is the size of the patch illuminated by the antenna beam.

The scattering coefficient γ defined as the ratio between the actual electric scattered field and the electric field scattered in the specular direction from a smooth surface with infinite conducibility [9] can be calculated by applying Kirchhoff method. The resulting expression, obtained under the assumptions of infinite conducibility and vertical (V-V) polarization is

$$\gamma(t) = \sum_{\underline{m}} G(\underline{m}) \cdot e^{j\underline{m}^T \underline{\hat{\Phi}}(t)} \tag{5}$$

where $\underline{m} = (m_1, m_2, m_3, \cdots, m_{N_f-1})^T$ with $[\bullet]^T$ the transpose operator and

$$G(\underline{m}) = g(\theta_i, \theta_s) \cdot \mathrm{sinc}\left[\left(\upsilon_x + K_0 \sum_{n=0}^{N_f-1} m_n b^n\right) L\right] \cdot G_1(\underline{m}) \tag{6}$$

where $g(\theta_i, \theta_s) = \frac{1}{\cos(\vartheta_i)} \frac{1+\cos(\vartheta_i+\vartheta_s)}{\cos(\vartheta_i)+\cos(\vartheta_s)}$ and

$$\nu_x = \frac{2\pi}{\lambda}(\sin\vartheta_i - \sin\vartheta_s) \tag{7}$$

$$G_1(\underline{m}) = \prod_{n=0}^{N_f-1} J_{m_n}\left[C\nu_z\sigma \cdot b^{(s-2)n}\right] \tag{8}$$

$$\nu_z = -\frac{2\pi}{\lambda}(\cos\theta_i + \cos\theta_s). \tag{9}$$

Symbols in (5) have the following meaning: $\sum_{\underline{m}} = \sum_{m_0=-\infty}^{\infty} \sum_{m_1=-\infty}^{\infty} \cdots \sum_{m_{N_f-1}=-\infty}^{\infty}$, $\underline{\hat{\Phi}}(t) = [\hat{\Phi}_0(t), \hat{\Phi}_1(t), \cdots, \hat{\Phi}_{N_f-1}(t)]^T$, and $\hat{\Phi}_k(t) = \Phi_{k+K_0} V b^k t$, where $K_0 = 2\pi/\Lambda_0$ is the fundamental wave number. In (6), the sinc function is defined as $\mathrm{sinc}(x) = \sin(x)/x$ and in (7) and (9),

λ is the transmitted wavelength. Note that (5) for a fixed time instant t can be interpreted as the N_f-dimensional discrete Fourier transform of the function $G(\underline{m})$.

The scattered electric field can be calculated by (5) if Kirchhoff method is applicable. To determine the conditions under which Kirchhoff approximation can be used, we refer to the criterion proposed by Soto–Crespo and Nieto–Vesperinas [10]. They state that the electromagnetic field scattered from very rough stochastic surfaces with high-incident angle can be calculated by means of Kirchhoff approximation if the ratio between the total power flow of the scattering wave and the total power flow of the incident wave is less than 2%. For a general random rough surface like the proposed one, such a criterion can be summarized by the following two conditions [1]:

$$\frac{\Gamma}{\lambda} > 6 \tag{10}$$

$$\frac{\sigma}{\Gamma\cos(\theta_i)} \leq 0.2 \tag{11}$$

IV. Theoretical Analysis

The end of this section is to demonstrate that the sea-scattered signal (which, in turn, is proportional to the scattering coefficient) retains some fractal characteristic of the sea fractal surface. The problem is tackled from a theoretical point of view and the results show that the graphs of the real and imaginary parts of the scattering coefficient (in phase and in quadrature components of the received signal) are fractal curves with the same fractal dimension as the sea profile. The demonstration is performed by following these steps: 1) rewrite the expression of $\gamma(t)$ as a sum of a finite number of terms; 2) demonstrate that each term of the expression of $\gamma(t)$ is a fractal function whose dimension is less than or equal to the dimension of the sea profile, i.e., s; and 3) by using the theorem of the fractal theory under which the sum of fractal functions with dimension less than or equal to s is a fractal with dimension s [7], we conclude that real and imaginary parts of $\gamma(t)$ are fractals with dimension s, i.e., its dimension is equal to the one of the sea profile.

To develop point 1), let us distinguish vector $\underline{m}$ in the following N_f classes.

Class 1: $\underline{m}^1$; vectors containing α_1 elements of values $\pm a_1$ (a_1 is a positive integer) and $(N_f - \alpha_1)$ zeros.

Class 2: $\underline{m}^2$; vectors containing α_1 elements of values $\pm a_1$ (a_1 is a positive integer), α_2 elements of values $\pm a_2$ (a_2 is a positive integer and $a_1 \neq a_2$), and $(N_f - \alpha_1 - \alpha_2)$ zeros.

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Class K $(1 \leq K \leq N_f)$: $\underline{m}^K$; vectors with α_1 elements of values $\pm a_1$ (a_1 is a positive integer), α_2 elements of values $\pm a_2$ (a_2 is a positive integer and $a_1 \neq a_2$), α_3 elements of values $\pm a_3$ (a_2 is a positive integer and $a_1 \neq a_2 \neq a_3$), $\cdots$, α_K elements of values $\pm a_K$ (a_K is a positive integer and $a_i \neq a_j$ $(\forall i \neq j)$), and $(N_f - \alpha_1 - \alpha_2 - \cdots - \alpha_K)$ zeros.

Let us define the following vectors: $\underline{a}_K \equiv [a_i]_{i=1}^K$ $(K = 1, 2, \cdots, N_f)$; $\underline{\alpha}_K \equiv [\alpha_i]_{i=1}^K$ $(K = 1, 2, \cdots, N_f)$ and denote as $\gamma^{(K)}(t)$ $(K = 1, 2, \cdots, N_f)$, the signal obtained by summing up terms of sum (5) corresponding with vectors $\underline{m}$ of the class $\underline{m}^K$ selected by fixing $\underline{a}_K$ and $\underline{\alpha}_K$.[1] Note that once $\underline{a}_K$ and $\underline{\alpha}_K$ are chosen, vectors $\underline{m}$ of the class $\underline{m}^K$ differ one to the other for the order and signum of the elements different from zero. The expression of $\gamma(t)$ can be written as

$$\gamma(t) = G[\underline{0}] + \sum_{K=1}^{N_f} \sum_{\underline{a}_K} \sum_{\underline{\alpha}_K} \gamma^{(K)}(t) \tag{12}$$

where $\sum_{\underline{a}_K} = \sum_{a_1=1}^{\infty} \sum_{a_2=1}^{\infty} \cdots \sum_{a_K=1}^{\infty}$, $\sum_{\underline{\alpha}_K} = \sum_{\alpha_1=1}^{N_f-K+1} \sum_{\alpha_2=1}^{N_f-\alpha_1-K+2} \cdots \sum_{\alpha_K=1}^{N_f-\sum_{j=1}^{K}\alpha_j}$ and $\underline{0}$ is a vector with N_f zeros.

To determine the fractal characteristics of $\gamma(t)$ we have to analyze each term of the sum (12). The goal of this analysis is to demonstrate that the terms $\gamma^{(K)}(t)$ $(K = 1, 2, \cdots, N_f)$ are fractal functions with dimension less than or equal to the dimension s of the sea profile. Since the theoretical manipulations that lead to the result are quit complicated, we consider as first cases $K = 1$ and $K = 2$ and then we generalize the result for $1 \leq K \leq N_f$.

A. Fractal Analysis of the Term $\gamma^{(1)}(t)$

As just mentioned above the signal $\gamma^{(1)}(t)$ is obtained by summing up terms of sum (5) corresponding with vectors $\underline{m}$ of the class $\underline{m}^1$ selected by fixing a_1 and α_1. However, to identify a single vector of the class $\underline{m}^1$ is not sufficient to provide the values of a_1 and α_1, the order and signum of the elements of $\underline{m}$ different from zero must be specified too. To this purpose let us define the following vectors.

$$\underline{p}^1 = \{p_{q_1}^1\}_{q_1=1}^{\alpha_1}; \quad 1 \leq p_{q_1}^1 \leq N_f - 1; \quad p_{q_1-1}^1 < p_{q_1}^1; \quad 1 \leq \alpha_1 \leq N_f$$

$\underline{p}^1$ is a pointer vector whose element $p_{q_1}^1$ gives the position of the q_1th element $\pm a_1$ in the vector. For example $p_5^1 = 7$ means the fifth element $\pm a_1$ is located in position seven within the vector $\underline{m}$.

$$\underline{k}^1 = \{k_{\nu_1}^1\}_{\nu_1=1}^{\alpha_1^+}; \quad 1 \leq k_{\nu_1}^1 \leq \alpha_1; \quad k_{\nu_1-1}^1 < k_{\nu_1}^1; \quad 0 \leq \alpha_1^+ \leq \alpha_1$$

where α_1^+ is the number of positive elements $+a_1$.

The vector $\underline{k}^1$ is a pointer to vector $\underline{p}^1$ and it locates the positions of positive elements in the vector $\underline{m}$. Specifically, the value of $k_{\nu_1}^1$ gives the element of vector $\underline{p}^1$ containing the position of the ν_1th positive element $+a_1$ in the vector. As an example, $k_3^1 = 5$ means that the third positive element $+a_1$ is located in the position given by the value of p_5^1. If $p_5^1 = 8$ this element is in position eight in the vector $\underline{m}$. It is clear that when $\alpha_1^+ = 0$ the elements of $\underline{m}$ different from zero are negative and the vector $\underline{k}^1$ does not exist.

[1] A most suitable symbol for this signal should be $\gamma^K(t, \underline{a}_K, \underline{\alpha}_K)$ $(K = 1, 2, \cdots, N_f)$ to put forward the dependence on $\underline{a}_K$ and $\underline{\alpha}_K$. We prefer the use of the symbol $\gamma^K(t)$ to simplify the notation.

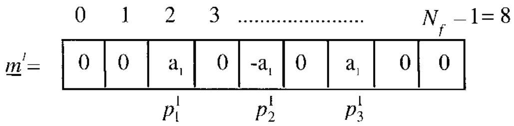

Fig. 2. Example of a vector belonging to class $\underline{m}^1$.

To better understand the vector notation introduced above, let us consider the example given in Fig. 2. The vector $\underline{m}$ is selected by taking $N_f = 9$, $\alpha_1 = 3$, $\alpha_1^+ = 2$, $\underline{p}^1 = [2, 4, 6]$, $\underline{k}^1 = [1, 3]$.

Let us denote as $\underline{m}^1(\underline{p}^1, \underline{k}^1)$ the vector of the first class obtained by fixing a_1, α_1 with a suitable choice of vectors $\underline{p}^1, \underline{k}^1$ and let $\gamma^{(1)}(t, \underline{p}^1, \underline{k}^1)$ be the term of (5) corresponding to $\underline{m}^1(\underline{p}^1, \underline{k}^1)$. The signal $\gamma^{(1)}(t)$ can be rewritten as

$$\gamma^{(1)}(t) = \sum_{\alpha_1^+=0}^{\alpha_1} \sum_{\underline{k}^1} \sum_{\underline{p}^1} \gamma^{(1)}(t, \underline{p}^1, \underline{k}^1) \tag{13}$$

where $\sum_{\underline{p}^1} = \sum_{p_1^1} \sum_{p_2^1} \cdots \sum_{p_{\alpha_1}^1}$, $\sum_{\underline{k}^1} = \sum_{k_1^1} \sum_{k_2^1} \cdots \sum_{k_{\alpha_1^+}^1}$. Note that when $\alpha_1^+ = 0$ the elements of $\underline{m}$ different from zero are negative and the vector $\underline{k}^1$ does not exist. In this case, the second sum of (13) must be ignored.

As a first step, we determine $\gamma^{(1)}(t, \underline{p}^1, \underline{k}^1)$ in a closed form. Let us rewrite $G_1(\underline{m}^1(\underline{p}^1, \underline{k}^1))$ for the vector $\underline{m}^1(\underline{p}^1, \underline{k}^1)$ by using (8)

$$G_1[\underline{m}^1(\underline{p}^1, \underline{k}^1)] = (\pm)\left\{\prod_{q_1=1}^{\alpha_1} J_{a_1}\left[C\nu_z\sigma \cdot b^{(s-2)p_{q_1}^1}\right]\right\} \cdot \left\{\prod_{z=0}^{N_f-\alpha_1} J_0\left[C\nu_z\sigma \cdot b^{(s-2)p_z^0(\underline{p}^1)}\right]\right\} \tag{14}$$

where $\underline{p}^0(\underline{p}^1) = \{p_z^0(\underline{p}^1)\}_{z=1}^{N_f-\alpha_1}$ is a pointer vector that gives the position of the zeros in the vector $\underline{m}$. We assume $p_0^0(\underline{p}^1) = \infty$ so that when $\alpha_1 = N_f$ the product term of (14) involving the $J_0(\bullet)$ Bessel function becomes one. (Note that $(s-2) < 0$.) As an example the vector $\underline{p}^0(\underline{p}^1)$ relevant to the situation represented in Fig. 2 is $\underline{p}^0(\underline{p}^1) = [0, 1, 3, 5, 7, 8]$. When the positions of the elements $\pm a_1$ is different (i.e., the vector $\underline{p}^1$ is changed), the vector $\underline{p}^0(\underline{p}^1)$ is changed too. By remembering that $J_{-a_1}(x) = (-1)^{a_1} \cdot J_{a_1}(x)$ we can argue that the signum minus in (14) only occurs when a_1 and $\alpha_1^- = \alpha_1 - \alpha_1^+$ are odd. Note that apart from a signum, the quantity $G_1(\underline{m}^1(\underline{p}^1, \underline{k}^1))$ only depends on the positions of the elements $\pm a_1$, i.e., on the vector $\underline{p}^1$. To simplify the notation let us define x_{q_1} as

$$x_{q_1} = C\nu_z\sigma b^{(s-2)p_{q_1}^1}. \tag{15}$$

In (14), the Bessel function $J_{a_1}(x_{q_1})$ can be expanded in ascending series as

$$J_{a_1}(x_{q_1}) = \left(\frac{x_{q_1}}{2}\right)^{a_1} \rho_{a_1}(q_1) \tag{16}$$

where

$$\rho_{a_1}(q_1) = \left\{\sum_{r=0}^{\infty} \frac{(-1)^r (x_{q_1}/2)^{2r}}{r!(a_1+r)!}\right\}. \tag{17}$$

By substituting (16) in (14) and after some manipulations we obtain

$$G_1[\underline{m}^1(\underline{p}^1, \underline{k}^1)] = \mu(a_1, \alpha_1)\mu_0(\underline{p}^1)\left\{\prod_{q_1=1}^{\alpha_1} b^{a_1(s-2)p_{q_1}^1}\right\} \tag{18}$$

$$\mu(a_1, \alpha_1) = (\pm)\left(\frac{C\nu_z\sigma}{2}\right)^{a_1\alpha_1} \tag{19}$$

and

$$\mu_0(\underline{p}^1) = \left\{\prod_{z=0}^{N_f-\alpha_1} J_0\left[C\nu_z\sigma \cdot b^{(s-2)p_z^0(\underline{p}^1)}\right]\right\} \cdot \left\{\prod_{q_1=1}^{\alpha_1} \rho_{a_1}(q_1)\right\}. \tag{20}$$

By inserting (18) in (6) and by using (5) we have the expression of the term $\gamma^{(1)}(t, \underline{p}^1, \underline{k}^1)$

$$\gamma^{(1)}(t, \underline{p}^1, \underline{k}^1) = \mu(a_1, \alpha_1)A(\underline{p}^1, \underline{k}^1)\left\{\prod_{q_1=1}^{\alpha_1} b^{a_1(s-2)p_{q_1}^1}\right\} \times \exp\left[j\left\{a_1 \sum_{q_1=1}^{\alpha_1} \operatorname{sgn}\left[m_{p_{q_1}^1}^1\right]\left(K_0 V b^{p_{q_1}^1} t + \Phi_{p_{q_1}^1}\right)\right\}\right] \tag{21}$$

where $m_{p_{q_1}^1}^1$ is the $p_{q_1}^1$ elements of the vector $\underline{m}^1(\underline{p}^1, \underline{k}^1)$, $\operatorname{sgn}(\bullet)$ is the signum function, and with

$$A(\underline{p}^1, \underline{k}^1) = g(\theta_i, \theta_s)\mu_0(\underline{p}^1) \times \operatorname{sinc}\left[\left(\upsilon_x + K_0 a_1 \sum_{q_1=1}^{\alpha_1} b^{p_{q_1}^1} \operatorname{sgn}\left[m_{p_{q_1}^1}^1\right]\right)L\right]. \tag{22}$$

At first, let us consider the case $\alpha_1 = 1$ and focus the attention on the real part $\gamma_R^{(1)}(t, \underline{p}^1, \underline{k}^1)$ of (21)

$$\gamma_R^{(1)}(t, p_1^1, k_1^1) = \mu(a_1, 1)A(p_1^1, k_1^1)b^{a_1(s-2)p_1^1} \times \cos\left\{a_1 \operatorname{sgn}\left[m_{p_1^1}^1\right]\left(K_0 V b^{p_1^1} t + \Phi_{p_1^1}\right)\right\}. \tag{23}$$

To find the real part of $\gamma^{(1)}(t)$, we have to consider that the position p_1^1 of the element $\pm a_1$ spans from zero to $N_f - 1$. So (13) is a sum of $2N_f$ terms and the result is

$$\gamma_R^{(1)}(t) = 2\mu(a_1, 1) \sum_{p_1^1=0}^{N_f-1} A_1(p_1^1) b^{a_1(s-2)p_1^1} \times \cos\left\{a_1\left(K_0 V b^{p_1^1} t + \Phi_{p_1^1}\right)\right\} \tag{24}$$

where

$$A_1(p_1^1) = g(\theta_i, \theta_s)\mu_0(p_1^1)\left\{\operatorname{sinc}\left[\left(\upsilon_x + K_0 a_1 b^{p_1^1}\right)L\right] + \operatorname{sinc}\left[\left(\upsilon_x - K_0 a_1 b^{p_1^1}\right)L\right]\right\} \tag{25}$$

is a bounded sequence.

Following Appendix A, we demonstrate that

1) the signal of (24) is a fractal with dimension s for $a_1 = 1$;

2) the signal of (24) is a fractal with dimension less than s for $a_1 > 1$;

3) the same result holds for the imaginary part.

Let us now consider the general case $\alpha_1 > 1$. As a first step we rewrite (13) as follows:

$$\gamma^{(1)}(t) = \sum_{\alpha_1^+=0}^{\alpha_1} \sum_{\underline{k}^1} \gamma^{(1)}(t, \underline{k}^1) \tag{26}$$

where

$$\gamma^{(1)}(t, \underline{k}^1) = \sum_{\underline{p}^1} \gamma^{(1)}(t, \underline{p}^1, \underline{k}^1). \tag{27}$$

By using (21), we get the real part $\gamma_R^{(1)}(t, \underline{k}^1)$ of $\gamma^{(1)}(t)$ in a closed form

$$\gamma_R^{(1)}(t, \underline{k}^1) = \sum_{\underline{p}^1} \mu(a_1, \alpha_1) A(\underline{p}^1, \underline{k}^1) \left\{ \prod_{q_1=1}^{\alpha_1} b^{a_1(s-2)p^1_{q_1}} \right\} \times \cos\left\{ a_1 \sum_{q_1=1}^{\alpha_1} \operatorname{sgn}\left(m^1_{p^1_{q_1}}\right)\left(K_0 V b^{p^1_{q_1}} t + \Phi_{p^1_{q_1}}\right)\right\}. \tag{28}$$

Following Appendix B we get the following conclusions:

4) the real part $\gamma_R^{(1)}(t, \underline{k}^1)$ is a fractal whose dimension is upper bounded by s;

5) the same conclusion holds for the imaginary parts.

From (26) we note that $\gamma^{(1)}(t)$ is the sum of terms $\gamma^{(1)}(t, \underline{k}^1)$. So, by using the theorem of fractal theory relevant to sum of fractal functions and by exploiting the results 4) an 5) we have that:

6) the real and imaginary part of the term $\gamma^{(1)}(t)$ for $\alpha_1 > 1$ is a fractal with dimension upper bounded by s.

For the sake of clarity let us summarize the main results of this section.

R1 Real and imaginary part of the term $\gamma^{(1)}(t)$ is a fractal with dimension equal to s for $\alpha_1 = 1$ and $a_1 = 1$.

R2 Real and imaginary part of the term $\gamma^{(1)}(t)$ is a fractal with dimension upper bonded by s for $\alpha_1 \geq 1$ and $a_1 > 1$.

B. Fractal Analysis of the Term $\gamma^{(2)}(t)$

As just mentioned above the signal $\gamma^{(2)}(t)$ is obtained by summing up terms of sum (5) corresponding with vectors $\underline{m}$ of the class $\underline{m}^2$ selected by fixing $\underline{a} = (a_1, a_2)$ and $\underline{\alpha} = (\alpha_1, \alpha_2)$. To identify a single vector of the class $\underline{m}^2$ we have to specify the order and signum of the elemets of $\underline{m}$ different from zero. To this purpose let us define the following vectors:

•

$$\underline{p}^1 = \{p^1_{q_1}\}_{q_1=1}^{\alpha_1}; \quad 1 \leq p^1_{q_1} \leq N_f - 1; \quad p^1_{q_1-1} < p^1_{q_1}$$

$\underline{p}^1$ is a pointer vector whose the element $p^1_{q_1}$ gives the position of the q_1th element $\pm a_1$ in the vector

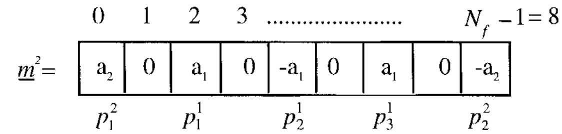

Fig. 3. Example of a vector belonging to class $\underline{m}^2$.

•

$$\underline{k}^1 = \{k^1_{\nu_1}\}_{\nu_1=1}^{\alpha_1^+}; \quad 1 \leq k^1_{\nu_1} \leq \alpha_1; \quad k^1_{\nu_1-1} < k^1_{\nu_1} \quad 0 \leq \alpha_1^+ \leq \alpha_1$$

where α_1^+ is the number of positive elements $+a_1$.

The vector $\underline{k}^1$ is a pointer to vector $\underline{p}^1$ and it locates the positions of positive elements $+a_1$ in the vector $\underline{m}$. Specifically, the value of $k^1_{\nu_1}$ gives the element of the vector $\underline{p}^1$ containing the position of the ν_1th positive element $+a_1$ in the vector. When $\alpha_1^+ = 0$ the elements of $\underline{m}$ different from zero are negative and the vector $\underline{k}^1$ does not exist.

•

$$\underline{p}^2 = \{p^2_{q_2}\}_{q_2=1}^{\alpha_2}; \quad 1 \leq p^2_{q_2} \leq N_f - 1; \quad p^2_{q_2-1} < p^2_{q_2} \quad 1 \leq \alpha_1 + \alpha_2 \leq N_f$$

$\underline{p}^2$ is a pointer vector whose element $p^2_{q_2}$ gives the position of the q_2th element $\pm a_2$ in the vector.

•

$$\underline{k}^2 = \{k^2_{\nu_2}\}_{\nu_2=1}^{\alpha_2^+}; \quad 1 \leq k^2_{\nu_2} \leq \alpha_2^+; \quad k^2_{\nu_2-1} < k^2_{\nu_2}; \quad 0 \leq \alpha_2^+ \leq \alpha_2$$

where α_2^+ is the number of positive elements $+a_2$.

The vector $\underline{k}^2$ is a pointer to vector $\underline{p}^2$ and it locates the positions of positive elements $+a_2$ in the vector $\underline{m}$. Specifically, the value of $k^2_{\nu_2}$ gives the element of vector $\underline{p}^2$ containing the position of the ν_2th positive element $+a_2$ in the vector. When $\alpha_2^+ = 0$ the elements of $\underline{m}$ different from zero are negative and the vector $\underline{k}^2$ does not exist.

To clarify the notation above let us consider the example given in Fig. 3. The vector $\underline{m}$ is obtained by taking $N_f = 9$, $\alpha_1 = 3$, $\alpha_2 = 2$, $\alpha_1^+ = 2$, $\alpha_2^+ = 1$ $\underline{p}^1 = [2, 4, 6]$, $\underline{p}^2 = [0, 8]$, $\underline{k}^1 = [1, 3]$, $\underline{k}^2 = [1]$.

Let us denote as $\underline{m}^2(\underline{p}^1, \underline{p}^2, \underline{k}^1, \underline{k}^2)$ the vector of the first class obtained by fixing $\underline{a}$ and $\underline{\alpha}$ with a suitable choice of vectors $\underline{p}^1, \underline{p}^2, \underline{k}^1, \underline{k}^2$ and let $\gamma^{(2)}(t, \underline{p}^1, \underline{p}^2, \underline{k}^1, \underline{k}^2)$ be the terms of (5) corresponding to $\underline{m}^2(\underline{p}^1, \underline{p}^2 \underline{k}^1, \underline{k}^2)$. The signal $\gamma^{(2)}(t)$ can be rewritten as

$$\gamma^{(2)}(t) = \sum_{\alpha_1^+=0}^{\alpha_1} \sum_{\alpha_2^+=0}^{\alpha_2} \sum_{\underline{k}^1} \sum_{\underline{k}^2} \sum_{\underline{p}^1} \sum_{\underline{p}^2} \gamma^{(2)}(t, \underline{p}^1, \underline{p}^2, \underline{k}^1, \underline{k}^2) \tag{29}$$

where $\sum_{\underline{p}^1} = \sum_{p^1_1} \sum_{p^1_2} \cdots \sum_{p^1_{\alpha_2}}$, $\sum_{\underline{k}^2} = \sum_{k^2_1} \sum_{k^2_2} \cdots \sum_{k^2_{\alpha_2^+}}$. Note that when $\alpha_1^+ = 0$ the sum on $\underline{k}^1$ must be ignored and if $\alpha_2^+ = 0$ the sum on $\underline{k}^2$ must be removed.

As a first step, we determine $\gamma^{(2)}(t, \underline{p}^1, \underline{p}^2, \underline{k}^1, \underline{k}^2)$ in a closed form. Let us rewrite $G_1(\underline{m}^2(\underline{p}^1, \underline{p}^2, \underline{k}^1, \underline{k}^2))$ for the vector $\underline{m}^2(\underline{p}^1,\underline{p}^2,\underline{k}^1,\underline{k}^2)$ as

$$\begin{aligned} G_1[\underline{m}^2(\underline{p}^1,\underline{p}^2,\underline{k}^1,\underline{k}^2)] &= (\pm)\left\{\prod_{q_1=1}^{\alpha_1} J_{a_1}\left[C\nu_z\sigma\cdot b^{(s-2)p^1_{q_1}}\right]\right\} \\ &\cdot\left\{\prod_{q_2=1}^{\alpha_2} J_{a_2}\left[C\nu_z\sigma\cdot b^{(s-2)p^2_{q_2}}\right]\right\} \\ &\times\left\{\prod_{z=0}^{N_f-\alpha_1-\alpha_2} J_0\left[C\nu_z\sigma\cdot b^{(s-2)p^0_z(\underline{p}^1,\underline{p}^2)}\right]\right\} \end{aligned} \quad (30)$$

where $\underline{p}^0(\underline{p}^1,\underline{p}^2)=\{p^0_z(\underline{p}^1,\underline{p}^2)\}_{z=0}^{N_f-\alpha_1-\alpha_2}$ is a pointer vector that gives the positions of the zeros in the vector $\underline{m}$. We assume $p^0_0(\underline{p}^1,\underline{p}^2)=\infty$ so that when $\alpha_1+\alpha_2=N_f$ the product term of (30) involving the $J_0(\bullet)$ Bessel function becomes one. By remembering the Bessel property $J_{-n}(x)=(-1)^n\cdot J_n(x)$ we can argue that the signum minus in (30) only occurs when a_1 and $\alpha_1^-=\alpha_1-\alpha_1^+$ are odd or when a_2 and $\alpha_2^-=\alpha_2-\alpha_2^+$ are odd, i.e. when only one of the two conditions is verified. Note that, apart from a signum, the quantity $G_1[\underline{m}^2(\underline{p}^1,\underline{p}^2\underline{k}^1,\underline{k}^2)]$ only depends on the position of the elements different from zero.

By applying the ascending series expansion of (16) to the Bessel functions in (30) we get

$$\begin{aligned} G_1[\underline{m}^2(\underline{p}^1,\underline{p}^2,\underline{k}^1,\underline{k}^2)] &= \mu(\underline{a},\underline{\alpha})\mu_0(\underline{p}^1,\underline{p}^2)\left\{\prod_{q_1=1}^{\alpha_1} b^{a_1(s-2)p^1_{q_1}}\right\} \\ &\times\left\{\prod_{q_2=1}^{\alpha_2} b^{a_2(s-2)p^2_{q_2}}\right\} \end{aligned} \quad (31)$$

where

$$\mu(\underline{a},\underline{\alpha})=(\pm)\left(\frac{C\nu_z\sigma}{2}\right)^{(a_1\alpha_1+a_2\alpha_2)} \quad (32)$$

$$\begin{aligned} \mu_0(\underline{p}^1,\underline{p}^2) &= \left\{\prod_{z=0}^{N_f-\alpha_1-\alpha_2} J_0\left[C\nu_z\sigma\cdot b^{(s-2)p^0_z(\underline{p}^1,\underline{p}^2)}\right]\right\} \\ &\cdot\left\{\prod_{q_1=1}^{\alpha_1}\rho_{a_1}(q_1)\right\}\cdot\left\{\prod_{q_2=1}^{\alpha_2}\rho_{a_2}(q_2)\right\} \end{aligned} \quad (33)$$

where $\rho_{a_i}(q_i)$ $(i=1,2)$ is given by (17) by substituting x_{q_1} with $x_{q_i}=C\nu_z\sigma b^{(s-2)p^i_{q_i}}$ $(i=1,2)$.

By inserting (31) in (6) and by using (5), we have the expression of the term $\gamma^{(2)}(t,\underline{p}^1,\underline{p}^2\underline{k}^1,\underline{k}^2)$

$$\begin{aligned} &\gamma^{(2)}(t,\underline{p}^1,\underline{p}^2,\underline{k}^1,\underline{k}^2) \\ &= \mu(\underline{a},\underline{\alpha})A(\underline{p}^1,\underline{p}^2,\underline{k}^1,\underline{k}^2) \\ &\times\left\{\prod_{q_1=1}^{\alpha_1} b^{a_1(s-2)p^1_{q_1}}\right\}\left\{\prod_{q_2=1}^{\alpha_2} b^{a_2(s-2)p^2_{q_2}}\right\} \\ &\times\exp\left[j\left\{\sum_{q_1=1}^{\alpha_1}\sum_{q_2=1}^{\alpha_2}\left[\operatorname{sgn}\left(m^2_{p^1_{q_1}}\right)a_1\left(K_0Vb^{p^1_{q_1}}t+\Phi_{p^1_{q_1}}\right)\right.\right.\right. \\ &\left.\left.\left.+\operatorname{sgn}\left(m^2_{p^2_{q_2}}\right)a_2\left(K_0Vb^{p^2_{q_2}}t+\Phi_{p^2_{q_2}}\right)\right]\right\}\right] \end{aligned} \quad (34)$$

where

$$\begin{aligned} &A(\underline{p}^1,\underline{p}^2,\underline{k}^1,\underline{k}^2) \\ &= g(\vartheta_i,\vartheta_s)\mu_0(\underline{p}^1,\underline{p}^2) \\ &\times\operatorname{sinc}\left[\left(v_x+K_0\sum_{q_1=1}^{\alpha_1}\sum_{q_2=1}^{\alpha_2}\left(\operatorname{sgn}\left(m^2_{p^1_{q_1}}\right)a_1b^{p^1_{q_1}}\right.\right.\right. \\ &\left.\left.\left.+\operatorname{sgn}\left(m^2_{p^2_{q_2}}\right)a_2b^{p^2_{q_2}}\right)\right)L\right] \end{aligned} \quad (35)$$

is a bounded function. Let us now rewrite (29) as follows:

$$\gamma^{(2)}(t)=\sum_{\alpha_1^+=0}^{\alpha_1}\sum_{\alpha_2^+=0}^{\alpha_2}\sum_{\underline{k}^1}\sum_{\underline{k}^2}\gamma^{(2)}(t,\underline{k}^1,\underline{k}^2) \quad (36)$$

where

$$\gamma^{(2)}(t,\underline{k}^1,\underline{k}^2)=\sum_{\underline{p}^1}\sum_{\underline{p}^2}\gamma^{(2)}(t,\underline{p}^1,\underline{p}^2\underline{k}^1,\underline{k}^2). \quad (37)$$

By using (34), we get the real part $\gamma_R^{(2)}(t,\underline{k}^1,\underline{k}^2)$ of $\gamma^{(2)}(t,\underline{k}^1,\underline{k}^2)$ in a closed form:

$$\begin{aligned} &\gamma_R^{(2)}(t,\underline{k}^1,\underline{k}^2) \\ &= \sum_{\underline{p}^1}\sum_{\underline{p}^2}\mu(\underline{a},\underline{\alpha})A(\underline{p}^1,\underline{p}^2,\underline{k}^1,\underline{k}^2) \\ &\times\left\{\prod_{q_1=1}^{\alpha_1} b^{a_1(s-2)p^1_{q_1}}\right\}\left\{\prod_{q_2=1}^{\alpha_2} b^{a_2(s-2)p^2_{q_2}}\right\} \\ &\times\cos\left\{\sum_{q_1=1}^{\alpha_1}\sum_{q_2=1}^{\alpha_2}\left[\operatorname{sgn}\left(m^2_{p^1_{q_1}}\right)a_1\left(K_0Vb^{p^1_{q_1}}t+\Phi_{p^1_{q_1}}\right)\right.\right. \\ &\left.\left.+\operatorname{sgn}\left(m^2_{p^2_{q_2}}\right)a_2\left(K_0Vb^{p^2_{q_2}}t+\Phi_{p^2_{q_2}}\right)\right]\right\}. \end{aligned} \quad (38)$$

Following Appendix C, we get the following conclusions.

1) The real part $\gamma_R^{(2)}(t,\underline{k}^1,\underline{k}^2)$ is a fractal whose dimension is upper bounded by s.
2) Same conclusion holds for the imaginary part.

From (36) we note that $\gamma^{(2)}(t)$ is the sum of terms $\gamma^{(2)}(t,\underline{k}^1,\underline{k}^2)$. So, by using the theorem of fractal theory relevant to sum of fractal functions and by exploiting the results 1) an 2) we have:

R3 real and imaginary part of the term $\gamma^{(2)}(t)$ is a fractal with dimension upper bounded by s.

C. Fractal Analysis of the Term $\gamma^{(K)}(t)$

In this section we extend the analysis performed for $K=1,2$ in the general case of $K\in(1,N_f)$.

The identification of a vector $\underline{m}$ belonging to the class $\underline{m}^K$ is obtaining by fixing $\underline{a}=(a_1,a_2,\cdots,a_K)$ and $\underline{\alpha}=(\alpha_1,\alpha_2,\cdots,\alpha_K)$ and providing the following K vectors:

$$\begin{aligned} \underline{p}^n=\{p^n_{q_n}\}_{q_n=1}^{\alpha_n};\quad 1\le n\le K;\quad 1\le p^n_{q_n}\le N_f-1 \\ p^n_{q_n-1}<p^n_{q_n};\quad 1\le\alpha_n\le N_f;\quad \textstyle\sum_{n=1}^K\alpha_n\le N_f \end{aligned}$$

$\underline{p}^n$ is a pointer vector whose element $p^n_{q_n}$ gives the position of the q_nth element $\pm a_n$ in the vector;

$$\underline{k}^n = \{k^n_{\nu_n}\}^{\alpha_n^+}_{\nu_n=1};\quad 1 \le n \le K;\quad 1 \le k^n_{\nu_n} \le \alpha_n;\quad k^n_{\nu_n-1} < k^n_{\nu_n}; 0 \le \alpha_n^+ \le \alpha_n$$

where α_n^+ is the number of positive elements $+a_n$.

Let us use the following notation: $\underline{p}^1\cdots\underline{p}^K = (\underline{p}^1, \underline{p}^2, \cdots, \underline{p}^K)$; $\underline{k}^1\cdots\underline{k}^K = (\underline{k}^1, \underline{k}^2, \cdots, \underline{k}^K)$, denote as $\underline{m}^K(\underline{p}^1\cdots\underline{p}^K, \underline{k}^1\cdots\underline{k}^K)$ the vector of the Kth class obtained by fixing $\underline{a} = (a_1, a_2, \cdots, a_K)$ and $\underline{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_K)$ with a suitable choice of vectors $\underline{p}^1\cdots\underline{p}^K, \underline{k}^1\cdots\underline{k}^K$ and let $\gamma^{(K)}(t, \underline{p}^1\cdots\underline{p}^K, \underline{k}^1\cdots\underline{k}^K)$ be the term of (5) corresponding to $\underline{m}^K(\underline{p}^1\cdots\underline{p}^K, \underline{k}^1\cdots\underline{k}^K)$. The signal $\gamma^{(K)}(t)$ can be rewritten as

$$\gamma^K(t) = \sum_{\underline{\alpha}^+}\sum_{\underline{k}^1\cdots\underline{k}^K}\sum_{\underline{p}^1\cdots\underline{p}^K}\gamma^{(K)}(t, \underline{p}^1\cdots\underline{p}^K, \underline{k}^1\cdots\underline{k}^K) \quad (39)$$

where $\sum_{\underline{\alpha}^+} = \sum_{\alpha_1^+=0}^{\alpha_1}\sum_{\alpha_2^+=0}^{\alpha_2}\cdots\sum_{\alpha_K^+=0}^{\alpha_K}$, $\sum_{\underline{p}^1\cdots\underline{p}^K} = \sum_{\underline{p}^1}\sum_{\underline{p}^2}\cdots\sum_{\underline{p}^K}$, $\sum_{\underline{k}^1\cdots\underline{k}^K} = \sum_{\underline{k}^1}\sum_{\underline{k}^2}\cdots\sum_{\underline{k}^K}$. Note that when $\alpha_n^+ = 0$ the sum involving the vectors $\underline{k}^n$ must be removed.

By extending the result obtained from (30) to (34) we can provide the expression of $\gamma^{(K)}(t, \underline{p}^1\cdots\underline{p}^K, \underline{k}^1\cdots\underline{k}^K)$ in a closed form

$$\begin{aligned}\gamma^{(K)}&(t, \underline{p}^1\cdots\underline{p}^K, \underline{k}^1\cdots\underline{k}^K)\\ &= \mu(\underline{a}, \underline{\alpha})A(\underline{p}^1\cdots\underline{p}^K, \underline{k}^1\cdots\underline{k}^K)\prod_{n=1}^{K}\left\{\prod_{q_n=1}^{\alpha_n} b^{a_n(s-2)p^1_{q_n}}\right\}\\ &\cdot \exp\left[j\left\{\sum_{q_1=1}^{\alpha_1}\sum_{q_2=1}^{\alpha_2}\cdots\sum_{q_K=1}^{\alpha_K}\left[\sum_{n=1}^{K}\operatorname{sgn}\left(m^K_{p^n_{q_n}}\right)\right.\right.\right.\\ &\left.\left.\left.\times a_n\left(K_0 V b^{p^n_{q_n}} t + \Phi_{p^n_{q_n}}\right)\right]\right\}\right]\end{aligned} \quad (40)$$

where

$$\begin{aligned}A&(\underline{p}^1\cdots\underline{p}^K, \underline{k}^1\cdots\underline{k}^K)\\ &= g(\theta_i, \theta_s)\mu_0(\underline{p}^1\cdots\underline{p}^K)\\ &\times \operatorname{sinc}\left[\left(v_x + K_0\sum_{q_1=1}^{\alpha_1}\sum_{q_2=1}^{\alpha_2}\cdots\sum_{q_K=1}^{\alpha_K}\right.\right.\\ &\left.\left.\times\left[\sum_{n=1}^{K}\operatorname{sgn}\left(m^K_{p^n_{q_n}}\right)a_n b^{p^n_{q_n}}\right]\right)L\right]\end{aligned} \quad (41)$$

$$\mu(\underline{a}, \underline{\alpha}) = (\pm)\left(\frac{C\nu_z\sigma}{2}\right)^{\sum_{n=1}^{K} a_n\alpha_n} \quad (42)$$

$$\begin{aligned}\mu_0(\underline{p}^1\cdots\underline{p}^K) &= \left\{\prod_{z=0}^{N_f-\sum_{n=1}^{K}\alpha_n} J_0\left[C\nu_z\sigma\cdot b^{(s-2)p_z^0(\underline{p}^1\cdots\underline{p}^K)}\right]\right\}\\ &\cdot\prod_{n=1}^{K}\left\{\prod_{q_n=1}^{\alpha_n}\rho_{a_n}(q_n)\right\}\end{aligned} \quad (43)$$

with $\rho_{a_n}(q_n)$ $(n = 1, 2, \cdots, K)$ given by (17) by substituting x_{q_1} with $x_{q_n} = C\nu_z\sigma b^{(s-2)p^n_{q_n}}$ $(n = 1, 2, \cdots, K)$.

Let us now rewrite (39) as follows:

$$\gamma^K(t) = \sum_{\underline{\alpha}^+}\sum_{\underline{k}^1\cdots\underline{k}^K}\gamma^{(K)}(t, \underline{k}^1\cdots\underline{k}^K) \quad (44)$$

where

$$\gamma^{(K)}(t, \underline{k}^1\cdots\underline{k}^K) = \sum_{\underline{p}^1\cdots\underline{p}^K}\gamma^{(K)}(t, \underline{p}^1\cdots\underline{p}^K, \underline{k}^1\cdots\underline{k}^K). \quad (45)$$

By using (40), we get the real part $\gamma_R^{(K)}(t, \underline{k}^1\cdots\underline{k}^K)$ of $\gamma^{(K)}(t, \underline{k}^1\cdots\underline{k}^K)$ in a closed form

$$\begin{aligned}\gamma_R^{(K)}&(t, \underline{k}^1\cdots\underline{k}^K)\\ &= \sum_{\underline{p}^1\cdots\underline{p}^K}\mu(\underline{a}, \underline{\alpha})A(\underline{p}^1\cdots\underline{p}^K, \underline{k}^1\cdots\underline{k}^K)\\ &\times\prod_{n=1}^{K}\left\{\prod_{q_n=1}^{\alpha_n} b^{a_n(s-2)p^1_{q_n}}\right\}\\ &\cdot\cos\left\{\sum_{q_1=1}^{\alpha_1}\sum_{q_2=1}^{\alpha_2}\cdots\sum_{q_K=1}^{\alpha_K}\left[\sum_{n=1}^{K}\operatorname{sgn}\left(m^K_{p^n_{q_n}}\right)\right.\right.\\ &\left.\left.\times\ a_n\left(K_0 V b^{p^n_{q_n}} t + \Phi_{p^n_{q_n}}\right)\right]\right\}.\end{aligned} \quad (46)$$

Following Appendix D, we get the following conclusions:

1) the real part $\gamma_R^{(K)}(t, \underline{k}^1\cdots\underline{k}^K)$ is a fractal whose dimension is upper bounded by s;
2) the same conclusion holds for the imaginary part.

From (45), we note that $\gamma^{(K)}(t)$ is the sum of terms $\gamma^{(K)}(t, \underline{k}^1\cdots\underline{k}^K)$. So, by using the theorem of fractal theory relevant to sum of fractal functions and by exploiting the results 1) an 2) we have:

R4 the real and imaginary part of the term $\gamma^{(K)}(t)$ is a fractal with dimension upper bounded by s.

D. Theoretical Result

By exploiting partial results R1, R2, R3, R4, and by using the theorem on sum of fractal functions we get the main result of the paper:

In operating conditions in which Kirchhoff method is applicable, the graphs of the real and imaginary parts of the scattering coefficient (in-phase and quadrature components of the received signal) are fractal curves with box-counting dimension equal to the dimension of the sea profile, i.e. s.

V. Numerical Results

In order to verify the theoretical results obtained in Section IV-D, we have developed a computer program to calculate the scattering coefficient $\gamma(t)$ for the sea-surface model $f(x, t)$ recalled in Section II.

The dimension of the scattering coefficient is estimated by using the morphological covering algorithm proposed by

Maragos [11] and optimized as in [12]. This method calculates the box-counting dimension by linearly fitting the log-log plot of the morphological cover area obtained at different scales. The optimization consists of a suitable selection of the scale interval on which the linear fitting must be performed. The main steps of the algorithm are the following.

1) Determine a raw estimate s_1 of the fractal dimension s by applying the morphological covering technique in the scale interval $\lfloor \varepsilon_m^{(1)} = 1, \varepsilon_M^{(1)} = 10 \rfloor$.
2) Let us calculate the new minimum scale ε_m as
$$\varepsilon_m = \alpha s_1 + \beta$$
and the new maximum scale as
$$\varepsilon_M = \min\left\{ \max\left[\frac{(s_1 - 1.2)N}{1.5}, \left(\varepsilon_m + \varepsilon_M^{(1)}\right) \right], \frac{N}{2} \right\}$$
where N is the number of samples of the signal to be analyzed.
3) Compute the final estimate $\hat{s}$ of the fractal dimension by applying the morphological covering technique in the scale interval $[\varepsilon_m, \varepsilon_M]$.

To reduce computation difficulties and errors in the numerical evaluation of the scattering coefficient we consider a situation in which the dominant terms in (5) are $\gamma^{(1)}(t)$ and $\gamma^{(2)}(t)$. As we see later, this condition also permit us a better interpretation and prediction of the numerical results.

To have $\gamma^{(1)}(t)$ and $\gamma^{(2)}(t)$ as dominant terms, we have to consider a situation in which the term $G_1(\underline{m})$ in (8) and consequently $G(\underline{m})$ in (6) both assume their maximum values in correspondence of vectors $\underline{m}$ belonging to the first and second class. To get this condition we have to select model and geometry parameters so that the argument $x = C\nu_z \sigma b^{(s-2)n}$ in (8) spans in a interval I in which the behavior of the Bessel function can be easily controlled. This occurs when $I = [0, 1.5]$, i.e., when

$$|C\nu_z \sigma| \leq 1.5. \tag{47}$$

In fact for x spanning from 0 to 1.5 in the interval I, the Bessel function $J_0(x)$ decreases and $|J_q(x)|$ with $(q = \pm 1, \pm 2, \pm 3, \cdots, \infty)$ increases. Furthermore, for any $\underline{m}$, $|J_q(x)| > |J_k(x)|$ $(q = 0, \pm 1, \pm 2, \pm 3, \cdots, \infty)$ if $k = q + \mathrm{sgn}(q)$ and $\mathrm{sgn}(\bullet)$ the signum function. In this case, $G_1(\underline{m})$ and $G(\underline{m})$ have their absolute maximum value for $\underline{m} = \underline{0}$ and they assume values comparable with the maximum for vectors $\underline{m}$ that belong to the first classes, namely class 1 and 2. In other words terms of expansion (12) that give a significant contribution to $\gamma(t)$ can be reasonably assumed to be $\gamma^{(1)}(t)$ and $\gamma^{(2)}(t)$. To give a more rigorous justification of this assumption, let us consider the following analysis.

Let us pay attention on term $\gamma_R^{(1)}(t, \underline{k}^1)$ of (28) and note that: 1) $\mu(a_1, \alpha_1) = (\pm)(\frac{C\nu_z\sigma}{2})^{\alpha_1 a_1}$ decreases as a power of $\alpha_1 a_1$ [the argument is less than one because of condition (47)]; 2) due to the term $\{\prod_{q_1=1}^{\alpha_1}(\sum_{r=0}^{\infty} \frac{(-1)^r (x_{q_1}/2)^{2r}}{r!(a_1+r)!})\}$, $\mu_0(\underline{p}^1)$ and, consequently, $A(\underline{p}^1, \underline{k}^1)$ in (22) decreases as $1/(a_1!)^{\alpha_1}$.

From these comments we can draw the following conclusions. The dominant term is obtained for $a_1 = 1$ and $\alpha_1 = 1$. In the worst case, $|C\nu_z\sigma| = 1.5$ the amplitude decay of $\gamma_R^{(1)}(t, \underline{k}^1)$ with respect to the dominant term goes as $\frac{(0.75)^{(a_1\alpha_1 - 1)}}{(a_1!)}$. So, if we assume as significant terms the ones for which $\frac{(0.75)^{(a_1\alpha_1-1)}}{(a_1!)} < 0.1$ we have that $a_1 \leq 3$ for any value of α_1.

Similar remarks can be made for the term $\gamma^{(2)}(t)$. By looking at the expressions of $\mu(\underline{a}, \underline{\alpha})$ and $\mu_0(\underline{p}^1, \underline{p}^2)$ in (32) and (33), respectively, we note that the amplitude of $\gamma^{(2)}(t, \underline{p}^1, \underline{p}^2 \underline{k}^1, \underline{k}^2)$ goes to zero as $(\frac{1}{(a_1!)^{\alpha_1}})(\frac{1}{(a_2!)^{\alpha_2}})(\frac{C\nu_z\sigma}{2})^{\overline{(\alpha_1 a_1 + \alpha_1 a_1)}}$ and it decays faster than $\gamma_R^{(1)}(t, \underline{k}^1)$.

If we repeat the same considerations for terms corresponding to class 3 and more, we note that their amplitudes decay faster than the ones of the classes $K = 1, 2$. So, we can assume that the most important terms are $\gamma^{(1)}(t)$ and $\gamma^{(2)}(t)$ with small values of a_1 and α_1.

This analysis permits us to predict the result we expect from the numerical example carried out under the assumption (47). Since the main term of the scattering coefficient is $\gamma^{(1)}(t)$ with $a_1 = 1$ and $\alpha_1 = 1$ [see (24)], the plots of the real and imaginary part of the scattering coefficient should be similar to the graph of a WM function and its fractal dimension should be equal to the one of the sea-surface profile.

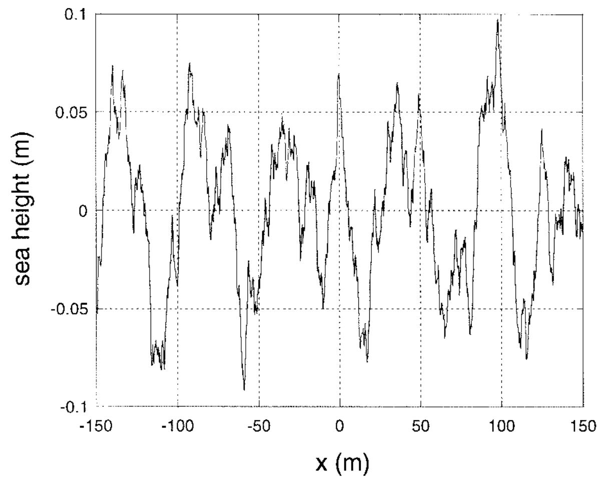

Fig. 4. Sea fractal profile at $t = 0$ and $s = 1.3$.

Let us consider a situation of a radar carried on an aircraft platform flying at a velocity $V = 540$ Km/h and illuminating the sea with an incident angle $\vartheta_i = -\vartheta_s = 80°$. The radar transmits pulses of 2μs at a frequency $f_0 = 5$ GHz with a repetition rate of 1 KHz and it observes the sea for 1 s. By taking one sample per range cell, the number of samples N we acquire in the observation time is 1000, which is a value sufficiently large to successfully apply of the morphological covering algorithm.

To satisfy both the condition of (47) and of (10) and (11), we have considered the case of a sea modeled by assuming in (4) $\Lambda_0 = 60$ m, $N_f = 8$, $b = e/2$ and a significant wave height h_s of about 15 cm. The standard deviation σ is obtained from h_s by $h_s = 4\sigma$ [13].

Figs. 4 and 5 show the sea-surface profiles at the initial time $t = 0$ with fractal dimension $s = 1.3$ and $s = 1.7$, respectively. Figs. 6 and 7 represent the plots of the real part

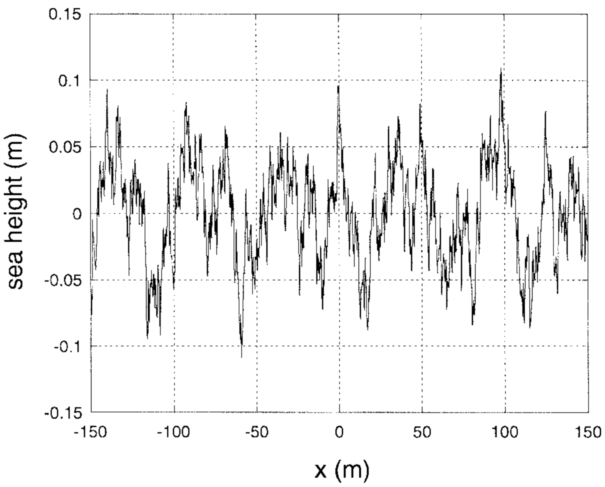

Fig. 5. Sea fractal profile at $t = 0$ and $s = 1.7$.

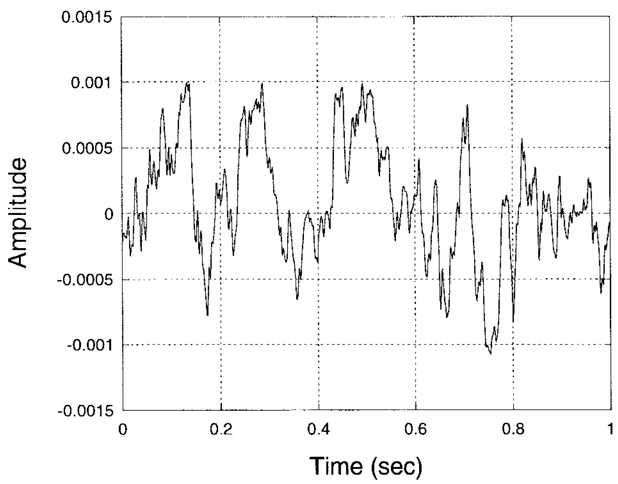

Fig. 6. Real part of the scattering coefficient relevant to the sea profile of Fig. 4.

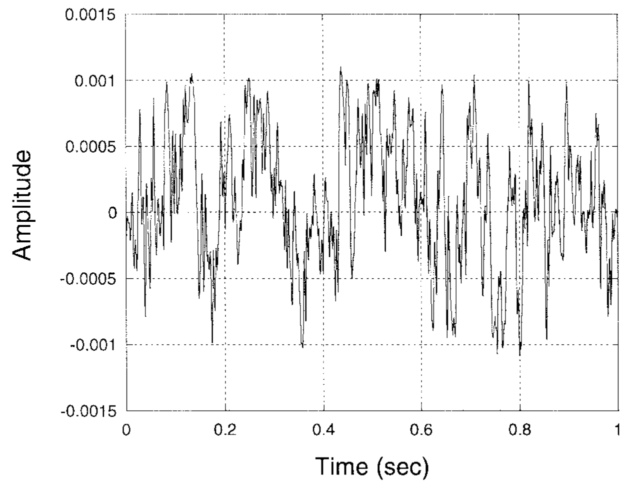

Fig. 7. Real part of the scattering coefficient relevant to the sea profile of Fig. 5.

TABLE I
SIMULATION RESULTS FOR $s = 1.3, 1.5, 1.7$

	$\Re eal$	$\Im maginary$
s=1.3	$\hat{s} = 1.299$ $\varepsilon_{\hat{s}}$ =0.07 %	$\hat{s} = 1.263$ $\varepsilon_{\hat{s}}$ =2.84 %
s=1.5	$\hat{s} = 1.558$ $\varepsilon_{\hat{s}}$ =3.86 %	$\hat{s} = 1.509$ $\varepsilon_{\hat{s}}$ =0.60%
s=1.7	$\hat{s} = 1.741$ $\varepsilon_{\hat{s}}$ =2.41 %	$\hat{s} = 1.713$ $\varepsilon_{\hat{s}}$ =0.76 %

of the scattering coefficient related to the sea of Figs. 4 and 5 respectively. As expected from the theory, this function looks like WM's functions and their shape is quite similar to the one of sea fractal profiles of Figs. 4 and 5.

To validate conclusions of Section IV-D we have estimated the fractal dimension of the scattering coefficient $\gamma(t)$ by applying the morphological cover algorithm procedure previously mentioned. The numerical computation was performed for different values of the sea-surface fractal dimension s. The results are reported in Table I, where $\hat{s}$ is the fractal dimension estimate and $\varepsilon_{\hat{s}}$ denotes the relative error defined as $\varepsilon_{\hat{s}} = |\hat{s} - s|/s$.

Let us discuss the results we obtained.

The mean error is always less than 4%, which is the typical estimation error of the morphological algorithm when applied on WM functions [12]. So, this errors can be reasonably attributed to the estimation algorithm and the theoretical result is confirmed: the received signal and the sea-surface profile has the same fractal dimension.

To remove the limitations that the use of the simplified sea model of (4) could give rise, we have also analyzed a numerical example referred to the most general model of (1) with phases given by (2). The scattering coefficient is calculated by including the effects of the shadowing and the finite conducibility of the sea. The angular frequency Ω_n follows the dispersion relationship [4] and the shadowing effects are taken into account by multiplying the sea profile $f(x,t)$ by a masking function $m(x,t)$ that assumes a value equal to zero in the shadowing zones and one otherwise. The scattering coefficient is calculated by considering the scattering from the modified rough profile $\hat{f}(x,t) = f(x,t)m(x,t)$ and by introducing the finite conducibility of the sea in the reflection coefficient [6].

This example makes use of the same parameters of the previous case. Because of the more general model used, we have to also introduce the following parameters.

1) n_0 =10. This value arises from the common assumption that capillary waves have wavelengths less than 30 cm.
2) $\Phi_n(t)$, $n \geq n_0$ are assumed to be identically distributed white stochastic processes, with pdf of the first order uniform in the interval $[-\pi, \pi]$.
3) The ratio between the sea-surface tension τ_s and water density ρ is $\tau_s/\rho = 74.45$ cm^3 s^{-2}, which is the value in standard conditions.
4) The water conducibility is $\sigma_s = 4$ mho/m [4].
5) The relative dielectric constant is 80 [4].

The results are presented in Table II relevant to $s = 1.3, 1.5, 1.7,$ respectively.

We note that the introduction of physical and geometric effects in the sea-scattering phenomenon do not significantly perturb the final result. The theoretical conclusions are still valid.

TABLE II
SIMULATION RESULTS FOR $s = 1.3, 1.5, 1.7$

	$\Re eal$	$\Im maginary$
s=1.3	$\hat{s} = 1.367$ $\varepsilon_{\hat{s}}$=5.15 %	$\hat{s} = 1.326$ $\varepsilon_{\hat{s}}$=2 %
s=1.5	$\hat{s} = 1.521$ $\varepsilon_{\hat{s}}$=1.40%	$\hat{s} = 1.477$ $\varepsilon_{\hat{s}}$=1.53 %
s=1.7	$\hat{s} = 1.750$ $\varepsilon_{\hat{s}}$=2.94 %	$\hat{s} = 1.759$ $\varepsilon_{\hat{s}}$=3.47 %

The possibility of measuring the fractal dimension of the sea profile by means of a fractal analysis of the received signal could be very interesting in practical applications. In fact, by considering that the fractal dimension of the sea is a measure of the sea roughness and that this geometry characteristic is modified by physical perturbations like surface wind, oil spot, presence of a target, and so on, we can state that the fractal analysis of the sea is an useful tool for sea-surface monitoring.

VI. CONCLUSION

In this paper, we have shown that the scattering coefficient of the sea surface is a fractal function with the same dimension of the model used for the sea surface. This result is very important for two main aspects.

1) A new characterization of the sea-scattered signal is performed. As an example, the possibility of estimate the fractal dimension of the sea surface by a fractal analysis of the received signal permit us to have a measure of the sea-surface roughness. This information could be used for classification purposes or for sea parameters extraction.
2) It represents the basis for a rigorous formulation of a new theory of target fractal detection. The basic idea, proposed by Haykin in [3], consists of comparing the estimated fractal dimension of the received signal with a suitable threshold. Haykin validated this idea by using experimental data without giving any mathematical demonstration. The result of this paper could represent an useful support for a theoretical demonstration of the fractal detection.

The authors are now working on: 1) analysis of the two-dimensional (2-D) sea fractal model proposed in [14]—some results in this direction are reported in [16]; 2) extension of theoretical results to the case of a 2-D sea fractal model; 3) mathematical definition of the fractal detection theory.

APPENDIX A

In this section, we show that function of (24) is a fractal function with dimension less than or equal to s. Let A_0 be a constant such that the sequence $A(p_1^1)$ has an absolute value less than A_0, i.e., $|A(p_1^1)| \leq A_0$. By denoting as $\Phi'_{p_1^1} = a_1\Phi_{p_1^1}|_{2\pi}$, $\mu' = 2\mu(a_1, 1)$, and by defining $B(p_1^1) = A(p_1^1) - 1$, (24) can be rewritten as a sum of two terms

$$\gamma_R^{(1)}(t) = x(t) + y(t) \tag{A.1}$$

where

$$x(t) = \mu' \sum_{p_1^1=0}^{N_f-1} b^{a_1(s-2)p_1^1} \cos\left\{a_1 K_0 V b^{p_1^1} t + \Phi'_{p_1^1}\right\} \tag{A.2}$$

and

$$y(t) = \mu' \sum_{p_1^1=0}^{N_f-1} B_1\left(p_1^1\right) b^{a_1(s-2)p_1^1} \cos\left\{a_1 K_0 V b^{p_1^1} t + \Phi'_{p_1^1}\right\}. \tag{A.3}$$

By using the theorem on sum of fractal functions, the demonstration that the fractal function of $\gamma_R^{(1)}(t)$ is equal to or less than s can be obtained by showing that: 1) $x(t)$ has a dimension equal to or less than s and 2) $y(t)$ has a dimension less than s.

By imposing $b^{a_1(s-2)p_1^1} = b^{(\hat{s}-2)p_1^1}$ we note that the function $x(t)$ is a WM with dimension given by

$$\hat{s} = \begin{cases} s + (2-s)(1-a_1), & \text{if } 0 \leq a_1 \leq 1/(2-s) \\ 1, & \text{otherwise} \end{cases}. \tag{A.4}$$

Note that $\hat{s} \leq s$. In fact, when $a_1 = 1$ the fractal dimension of $x(t)$ is s otherwise it is less than s. To demonstrate that function $y(t)$ has dimension less than or equal to $\hat{s}$ we have to verify the following inequality [15]:

$$\Delta y(t) = |y(t+h) - y(t)| \leq c|h|^{2-\hat{s}} \tag{A.5}$$

where $c > 0$ is independent on h and $|h| < \delta$ for some $\delta > 0$. The proposition of (A.5) is usually demonstrated for function $f(t)$ with $t \in [0, 1]$, however, it can also be applied to our case if we suitably scale the time coordinate of $y(t)$. By substituting (A.3) in (A.5), by reminding that $|B(p_1^1)| \leq B_0 = |A_0 - 1|$ and incorporating the constant B_0 in μ' we have

$$\begin{aligned}
&\Delta y(t) \\
&\quad \leq |\mu'| \sum_{p_1^1=0}^{N} b^{(\hat{s}-2)p_1^1} \Big| \cos\left\{a_1 K_0 V b^{p_1^1}(t+h) + \Phi'_{p_1^1}\right\} \\
&\qquad - \cos\left\{a_1 K_0 V b^{p_1^1} t + \Phi'_{p_1^1}\right\}\Big| \\
&\qquad + |\mu'| \sum_{p_1^1=N+1}^{N_f-1} b^{(\hat{s}-2)p_1^1} \Big| \cos\left\{a_1 K_0 V b^{p_1^1}(t+h) + \Phi'_{p_1^1}\right\} \\
&\qquad - \cos\left\{a_1 K_0 V b^{p_1^1} t + \Phi'_{p_1^1}\right\}\Big|
\end{aligned} \tag{A.6}$$

with N an arbitrary value belonging to the interval $[0, N_f - 2]$.

By maximizing the quantity $|\cos(\alpha) - \cos(\beta)|$ with $|\alpha - \beta|$ in the first sum and with 2 in the second sum of (A.6), we have

$$\Delta y(t) \leq |\mu'| K_0 V a_1 |h| \sum_{p_1^1=0}^{N} b^{(\hat{s}-1)p_1^1} + |2\mu'| \sum_{p_1^1=N+1}^{N_f-1} b^{(\hat{s}-2)p_1^1}. \tag{A.7}$$

By expanding the sums in (A.7) we get

$$\begin{aligned}
\Delta y(t) \leq\ & |\mu'| K_0 V a_1 |h| \frac{b^{(N+1)(\hat{s}-1)} - 1}{b^{(\hat{s}-1)} - 1} \\
& + |2\mu'| \frac{b^{(N+1)(\hat{s}-2)} - b^{N_f(\hat{s}-2)}}{1 - b^{(\hat{s}-2)}}.
\end{aligned} \tag{A.8}$$

By considering that: 1) $b^{(N+1)(\hat{s}-1)}-1\le b^{(N+1)(\hat{s}-1)}$ and 2) $b^{(N+1)(\hat{s}-2)}-b^{N_f(\hat{s}-2)}\le b^{(N+1)(\hat{s}-2)}$ and by assuming that $1/b^{(N+1)}\le|h|\le 1/b^N$, we obtain

$$\Delta y(t)\le c|h|^{2-\hat{s}} \tag{A.9}$$

where

$$c=|\mu'|K_0Va_1\frac{b^{(\hat{s}-1)}}{b^{(\hat{s}-1)}-1}+|2\mu'|\frac{1}{1-b^{(\hat{s}-2)}} \tag{A.10}$$

is a constant greater than zero and independent on h.

Due to the arbitrary choice of N, (A.9) also holds for any value of h such that $|h|\ge 1/b^{N_f}$. In other words, inequality (A.5) is satisfied for any h such that $1/b^{N_f}\le|h|\le\delta$ with $\delta>0$. The latter condition means that $y(t)$ is a fractal up to a scale level corresponding to $1/b^{N_f}$. It is clear that when $N_f\to\infty$ the function $y(t)$ becomes a fractal in a mathematical sense.

For the sake of clarity let us summarize partial and final results.

r1 The function $x(t)$ is a band-limited WM fractal function with dimension less than or equal to s (see A.4)

r2 The function $y(t)$ is a fractal function with dimension upper bounded by s.

RAI) From the theorem on sum of fractal functions and by using the result r1) and r2) we obtain the final conclusion:

a) the signal of $\gamma_R^1(t)$ is a fractal with dimension s for $a_1=1$;
b) the signal of (24) is a fractal with dimension less than s for $a_1>1$;
c) the same result holds for the imaginary part. (The demonstration can be resort to that of the real part by rewriting the sine function as a cosine function and by including a phase $\pi/2$ in $\Phi'_{p_1^1}$.)

Appendix B

In this section, we demonstrate that the signal $\gamma_R^{(1)}(t,\underline{k}^1)$ of (28) is a fractal whose dimension is upper bounded by s. To do that, we show that $\Delta\gamma_R^{(1)}(t,\underline{k}^1)=|\gamma_R^{(1)}(t+h,\underline{k}^1)-\gamma_R^{(1)}(t,\underline{k}^1)|\le c|h|^{2-s}$ for some constant c, independent on h, and for $|h|<\delta$ with $\delta>0$. By means of (28), let us maximize $\Delta\gamma_R^{(1)}(t,\underline{k}^1)$ as

$$\begin{aligned}\Delta\gamma_R^{(1)}&(t,\underline{k}^1)\\ &\le\sum_{\underline{p}^1}|\mu(a_1,\alpha_1)||A(\underline{p}^1,\underline{k}^1)|\left\{\prod_{q_1=1}^{\alpha_1}b^{a_1(s-2)p_{q_1}^1}\right\}\\ &\quad\cdot\left|\cos\left\{a_1\sum_{q_1=1}^{\alpha_1}\operatorname{sgn}\left(m_{p_{q_1}^1}^1\right)\left(K_0Vb^{p_{q_1}^1}(t+h)+\Phi_{p_{q_1}^1}\right)\right\}\right.\\ &\quad\left.-\cos\left\{a_1\sum_{q_1=1}^{\alpha_1}\operatorname{sgn}\left(m_{p_{q_1}^1}^1\right)\left(K_0Vb^{p_{q_1}^1}t+\Phi_{p_{q_1}^1}\right)\right\}\right|\end{aligned} \tag{B.1}$$

Equation (B.1) can be further maximized by means of the following manipulations.

1)
$$|A(\underline{p}^1,\underline{k}^1)|\le A_0.$$
2) Extend each sum of $\sum_{\underline{p}^1}$ with the indexes $p_{q_1}^1$ spanning from 1 to N_f-1 and denote the result as $\sum_{\underline{p}^1=0}^{N_f-1}=\sum_{p_1^1=0}^{N_f-1}\sum_{p_2^1=0}^{N_f-1}\cdots\sum_{p_{\alpha_1}^1=0}^{N_f-1}$. This operation adds positive terms to the second member of (B.1).
3) Decompose the multiple sum of $\sum_{\underline{p}^1}$ as $\sum_{\underline{p}^1=0}^{N_f-1}=\sum_{\underline{p}^1=0}^{N}+\sum_{\underline{p}^1=N+1}^{N_f-1}$ with N an arbitrary value belonging to the interval $[0,N_f-2]$ with $N_f>1$ and where $\sum_{\underline{p}^1=0}^{N}=\sum_{p_1^1=0}^{N}\sum_{p_2^1=0}^{N}\cdots\sum_{p_{\alpha_1}^1=0}^{N}$ and $\sum_{\underline{p}^1=N+1}^{N_f-1}=\sum_{p_1^1=N+1}^{N_f-1}\sum_{p_2^1=N+1}^{N_f-1}\cdots\sum_{p_{\alpha_1}^1=N+1}^{N_f-1}$.
4) Maximize $|\cos\alpha-\cos\beta|$ with $|\alpha-\beta|$ in the multiple sum $\sum_{\underline{p}^1=0}^{N}$and with 2 in the multiple sum $\sum_{\underline{p}^1=N+1}^{N_f-1}$.
5) Consider the following maximization: $b^{p_{q_1}^1}\le b^{a_1p_{q_1}^1}$.

By applying the steps above to (B.1) we have

$$\begin{aligned}\Delta\gamma_R^{(1)}(t,\underline{k}^1)&\le|\mu(a_1,\alpha_1)||A_0|a_1K_0V|h|\\ &\quad\cdot\sum_{\underline{p}^1=0}^{N}\left\{\prod_{q_1=1}^{\alpha_1}b^{a_1(s-2)p_{q_1}^1}\right\}\cdot\left\{\sum_{q_1=1}^{\alpha_1}b^{a_1p_{q_1}^1}\right\}\\ &\quad+2|\mu(a_1,\alpha_1)||A_0|\sum_{\underline{p}^1=N+1}^{N_f-1}\left\{\prod_{q_1=1}^{\alpha_1}b^{a_1(s-2)p_{q_1}^1}\right\}.\end{aligned} \tag{B.2}$$

Let us explicit the first term of (B.2) apart from the constant $|\mu(a_1,\alpha_1)||A_0|a_1K_0V|h|$

$$\begin{aligned}&\sum_{\underline{p}^1=0}^{N}\left\{\prod_{q_1=1}^{\alpha_1}b^{a_1(s-2)p_{q_1}^1}\right\}\cdot\left\{\sum_{q_1=1}^{\alpha_1}b^{a_1p_{q_1}^1}\right\}\\ &=\left\{\begin{array}{l}\sum_{p_1^1=0}^{N}b^{a_1(s-1)p_1^1}\sum_{p_2^1=0}^{N}b^{a_1(s-2)p_2^1}\cdots\sum_{p_{\alpha_1}^1=0}^{N}b^{a_1(s-2)p_{\alpha_1}^1}+\\ \sum_{p_1^1=0}^{N}b^{a_1(s-2)p_1^1}\sum_{p_2^1=0}^{N}b^{a_1(s-1)p_2^1}\cdots\sum_{p_{\alpha_1}^1=0}^{N}b^{a_1(s-2)p_{\alpha_1}^1}+\\ \bullet\\ \bullet\\ \sum_{p_1^1=0}^{N}b^{a_1(s-2)p_1^1}\sum_{p_2^1=0}^{N}b^{a_1(s-2)p_2^1}\cdots\sum_{p_{\alpha_1}^1=0}^{N}b^{a_1(s-1)p_{\alpha_1}^1}\end{array}\right\}\\ &=\alpha_1\left\{\left(\frac{b^{a_1(N+1)(s-1)}-1}{b^{a_1(s-1)}-1}\right)\cdot\left(\frac{1-b^{a_1(N+1)(s-2)}}{1-b^{a_1(s-2)}}\right)^{\alpha_1-1}\right\}.\end{aligned} \tag{B.3}$$

The second term of (B.2) apart from the constant term $2|\mu(a_1,\alpha_1)||A_0|$can be expanded as

$$\begin{aligned}&\sum_{\underline{p}^1=N+1}^{N_f-1}\left\{\prod_{q_1=1}^{\alpha_1}b^{a_1(s-2)p_{q_1}^1}\right\}\\ &\quad=\left(\frac{b^{a_1(N+1)(s-2)}-b^{a_1N_f(s-2)}}{1-b^{a_1(s-2)}}\right)^{\alpha_1}.\end{aligned} \tag{B.4}$$

By inserting (B.4) and (B.3) in (B.2) and considering that: 1) $b^{a_1(N+1)(s-1)} - 1 \le b^{a_1(N+1)(s-1)}$; 2) $b^{a_1(N+1)(s-2)} - b^{a_1 N_f(s-2)} \le b^{a_1(N+1)(s-2)}$; 3) $1 - b^{a_1(N+1)(s-2)} \le 1$; and 4) $b^{a_1(N+1)\alpha_1(s-2)} \le b^{a_1(N+1)(s-2)}$, we obtain

$$\begin{aligned}\Delta\gamma_R^{(1)}(t,\underline{k}^1) &\le |\mu(a_1,\alpha_1)||A_0|a_1K_0V\alpha_1 \\ &\quad\cdot\left(\frac{b^{a_1(s-1)}}{b^{a_1(s-1)}-1}\right)\left(\frac{1}{1-b^{a_1(s-2)}}\right)^{\alpha_1-1}(b^N)^{a_1(s-1)}|h| \\ &\quad+2|\mu(a_1,\alpha_1)||A_0|\left(\frac{1}{1-b^{a_1(s-2)}}\right)^{\alpha_1}b^{a_1(N+1)(s-2)}. \end{aligned} \tag{B.5}$$

By assuming that $1/b^{a_1(N+1)} \le |h| \le 1/b^{a_1 N}$, we have

$$\Delta\gamma_R^{(1)}(t,\underline{k}^1) \le c|h|^{2-s} \tag{B.6}$$

where

$$\begin{aligned}c = |\mu(a_1,\alpha_1)||A_0|\Bigg\{ & a_1K_0V\alpha_1\left(\frac{b^{a_1(s-1)}}{b^{a_1(s-1)}-1}\right) \\ &\cdot\left(\frac{1}{1-b^{a_1(s-2)}}\right)^{\alpha_1-1}+2\left(\frac{1}{1-b^{a_1(s-2)}}\right)^{\alpha_1}\Bigg\}.\end{aligned} \tag{B.7}$$

is a constant greater than zero and independent on h.

Due to the arbitrary choice of N, (B.6) also holds for any value of h such that $|h| \ge 1/b^{a_1 N_f}$. In other words, inequality (B.6) is satisfied for any h such that $1/b^{a_1 N_f} \le |h| \le \delta$ with $\delta > 0$. The latter condition means that $\gamma_R^{(1)}(t,\underline{k}^1)$ is a fractal up to a scale level corresponding to $1/b^{a_1 N_f}$. It is clear that when $N_f \to \infty$ the function $\gamma_R^{(1)}(t,\underline{k}^1)$ becomes a fractal in a mathematical sense.

From (B.6) we can conclude that:

RAII) The function $\gamma_R^{(1)}(t,\underline{k}^1)$ is a fractal whose dimension is upper bounded by s.

The conclusion RAII) is also valid for the imaginary part. We can follow the same demonstration by rewriting the sine function as a cosine function and by including a term $\pi/2$ in the phases $\Phi_{p^1_{q_1}}$.

APPENDIX C

In this section, we demonstrate that the signal $\gamma_R^{(2)}(t,\underline{k}^1,\underline{k}^2)$ of (38) is a fractal whose dimension is upper bounded by s. We show that $\Delta\gamma_R^{(2)}(t,\underline{k}^1,\underline{k}^2) = |\gamma_R^{(2)}(t+h,\underline{k}^1,\underline{k}^2) - \gamma_R^{(2)}(t,\underline{k}^1,\underline{k}^2)| \le c|h|^{2-s}$ for some constant c, independent on h, and for $|h| < \delta$ with $\delta > 0$. By means of (38), we have

$$\begin{aligned}\Delta\gamma_R^{(2)}&(t,\underline{k}^1,\underline{k}^2) \\ &\le \sum_{\underline{p}^1}\sum_{\underline{p}^2}|\mu(\underline{a},\underline{\alpha})||A(\underline{p}^1,\underline{p}^2,\underline{k}^1,\underline{k}^2)| \\ &\quad\cdot\left\{\prod_{q_1=1}^{\alpha_1} b^{a_1(s-2)p^1_{q_1}}\right\}\left\{\prod_{q_2=1}^{\alpha_2} b^{a_2(s-2)p^2_{q_2}}\right\} \\ &\quad\cdot\Bigg|\cos\Bigg\{\sum_{q_1=1}^{\alpha_1}\sum_{q_2=1}^{\alpha_2}\left[\text{sgn}\left(m^2_{p^1_{q_1}}\right)a_1\left(K_0Vb^{p^1_{q_1}}(t+h)+\Phi_{p^1_{q_1}}\right)\right. \\ &\quad\left.+\,\text{sgn}\left(m^2_{p^2_{q_2}}\right)a_2\left(K_0Vb^{p^2_{q_2}}(t+h)+\Phi_{p^2_{q_2}}\right)\right]\Bigg\} \\ &\quad-\cos\Bigg\{\sum_{q_1=1}^{\alpha_1}\sum_{q_2=1}^{\alpha_2}\left[\text{sgn}\left(m^2_{p^1_{q_1}}\right)a_1\left(K_0Vb^{p^1_{q_1}}t+\Phi_{p^1_{q_1}}\right)\right. \\ &\quad\left.+\,\text{sgn}\left(m^2_{p^2_{q_2}}\right)a_2\left(K_0Vb^{p^2_{q_2}}t+\Phi_{p^2_{q_2}}\right)\right]\Bigg\}.\Bigg|\end{aligned} \tag{C.1}$$

Equation (C.1) can be maximized by using the following conditions.

1) $|A(\underline{p}^1,\underline{p}^2,\underline{k}^1,\underline{k}^2)| \le A_0$.
2) Extend each sum of $\sum_{\underline{p}^n}$ $(n=1,2)$ with the indexes $p^n_{q_n}$ spanning from 1 to N_f-1 and denote the result as $\sum_{\underline{p}^n=0}^{N_f-1} = \sum_{p^n_1=0}^{N_f-1}\sum_{p^n_2=0}^{N_f-1}\cdots\sum_{p^n_{\alpha_n}=0}^{N_f-1}$. This operation adds positive terms to the second member of (C.1).
3) Decompose the multiple sum of $\sum_{\underline{p}^n}$ as $\sum_{\underline{p}^n=0}^{N_f-1} = \sum_{\underline{p}^n=0}^{N} + \sum_{\underline{p}^n=N+1}^{N_f-1}$ with $n=1,2$, and N an arbitrary value belonging to the interval $[0, N_f-2]$ with $N_f > 1$ and $\sum_{\underline{p}^n=0}^{N} = \sum_{p^n_1=0}^{N}\sum_{p^n_2=0}^{N}\cdots\sum_{p^n_{\alpha_n}=0}^{N}$, $\sum_{\underline{p}^n=N+1}^{N_f-1} = \sum_{p^n_1=N+1}^{N_f-1}\sum_{p^n_2=N+1}^{N_f-1}\cdots\sum_{p^n_{\alpha_1}=N+1}^{N_f-1}$.
4) Maximize $|\cos\alpha - \cos\beta|$ with $|\alpha-\beta|$ in the multiple sum $\sum_{\underline{p}^n=0}^{N}$ and with 2 in the multiple sum $\sum_{\underline{p}^n=N+1}^{N_f-1}$.
5) Consider the following maximization $b^{p^n_{q_n}} \le b^{a_n p^n_{q_n}}$ $(n=1,2)$.

By applying the steps above to (C.1) we have

$$\begin{aligned}\Delta\gamma_R^{(2)}&(t,\underline{k}^1,\underline{k}^2) \\ &\le |\mu(\underline{a},\underline{\alpha})||A_0|K_0V|h|\Bigg\{\sum_{\underline{p}^1=0}^{N}\sum_{\underline{p}^2=0}^{N}\left\{\prod_{q_1=1}^{\alpha_1} b^{a_1(s-2)p^1_{q_1}}\right\} \\ &\quad\cdot\left\{\prod_{q_2=1}^{\alpha_2} b^{a_2(s-2)p^2_{q_2}}\right\}\left\{a_1\sum_{q_1=1}^{\alpha_1} b^{a_1p^1_{q_1}} + a_2\sum_{q_2=1}^{\alpha_2} b^{a_2p^2_{q_2}}\right\}\Bigg\} \\ &\quad+2|\mu(\underline{a},\underline{\alpha})||A_0|\sum_{\underline{p}^1=N+1}^{N_f-1}\sum_{\underline{p}^2=N+1}^{N_f-1}\left\{\prod_{q_1=1}^{\alpha_1} b^{a_1(s-2)p^1_{q_1}}\right\} \\ &\quad\times\left\{\prod_{q_2=1}^{\alpha_2} b^{a_2(s-2)p^2_{q_2}}\right\}.\end{aligned} \tag{C.2}$$

Let us explicit the first term of (C.2) apart from the term $|\mu(a_1,\alpha_1)||A_0|K_0V|h|$

$$\begin{aligned}&\sum_{\underline{p}^1=0}^{N}\sum_{\underline{p}^2=0}^{N}\left\{\prod_{q_1=1}^{\alpha_1} b^{a_1(s-2)p^1_{q_1}}\right\}\cdot\left\{\prod_{q_2=1}^{\alpha_2} b^{a_2(s-2)p^2_{q_2}}\right\} \\ &\times\left\{a_1\sum_{q_1=1}^{\alpha_1} b^{a_1p^1_{q_1}}\right\} + \sum_{\underline{p}^1=0}^{N}\sum_{\underline{p}^2=0}^{N}\left\{\prod_{q_1=1}^{\alpha_1} b^{a_1(s-2)p^1_{q_1}}\right\} \\ &\cdot\left\{\prod_{q_2=1}^{\alpha_2} b^{a_2(s-2)p^2_{q_2}}\right\}\left\{a_2\sum_{q_2=1}^{\alpha_2} b^{a_2p^2_{q_2}}\right\} \\ &= a_1\Bigg\{\sum_{p^1_1=0}^{N} b^{a_1(s-1)p^1_1}\sum_{p^1_2=0}^{N} b^{a_1(s-2)p^1_2}\cdots\sum_{p^1_{\alpha_1}=0}^{N} b^{a_1(s-2)p^1_{\alpha_1}}\end{aligned}$$

$$
\begin{aligned}
&+\sum_{p_1^1=0}^{N} b^{a_1(s-2)p_1^1}\sum_{p_2^1=0}^{N} b^{a_1(s-1)p_2^1}\cdots\sum_{p_{\alpha_1}^1=0}^{N} b^{a_1(s-2)p_{\alpha_1}^1}\\
&+\cdots+\sum_{p_1^1=0}^{N} b^{a_1(s-2)p_1^1}\sum_{p_2^1=0}^{N} b^{a_1(s-2)p_2^1}\cdots\\
&\left.\times\sum_{p_{\alpha_1}^1=0}^{N} b^{a_1(s-1)p_{\alpha_1}^1}\right\}\\
&\cdot\left\{\sum_{p_1^2=0}^{N} b^{a_2(s-2)p_1^2}\sum_{p_2^2=0}^{N} b^{a_2(s-2)p_2^2}\sum_{p_{\alpha_2}^2=0}^{N} b^{a_2(s-2)p_{\alpha_2}^2}\right\}\\
&+a_2\left\{\sum_{p_1^2=0}^{N} b^{a_2(s-1)p_1^2}\sum_{p_2^2=0}^{N} b^{a_2(s-2)p_2^2}\cdots\right.\\
&\times\sum_{p_{\alpha_2}^2=0}^{N} b^{a_2(s-2)p_{\alpha_2}^1}+\sum_{p_1^2=0}^{N} b^{a_2(s-2)p_1^2}\sum_{p_2^2=0}^{N} b^{a_2(s-1)p_2^2}\cdots\\
&\times\sum_{p_{\alpha_2}^2=0}^{N} b^{a_2(s-2)p_{\alpha_2}^1}+\cdots+\sum_{p_1^2=0}^{N} b^{a_2(s-2)p_1^2}\\
&\left.\times\sum_{p_2^2=0}^{N} b^{a_2(s-2)p_2^2}\cdots\sum_{p_{\alpha_2}^2=0}^{N} b^{a_2(s-1)p_{\alpha_2}^1}\right\}\\
&\cdot\left\{\sum_{p_1^1=0}^{N} b^{a_1(s-2)p_1^1}\sum_{p_2^1=0}^{N} b^{a_1(s-2)p_2^1}\sum_{p_{\alpha_1}^1=0}^{N} b^{a_1(s-2)p_{\alpha_1}^1}\right\}\\
&=a_1\alpha_1\left\{\left(\frac{b^{a_1(N+1)(s-1)}-1}{b^{a_1(s-1)}-1}\right)\right.\\
&\left.\cdot\left(\frac{1-b^{a_1(N+1)(s-2)}}{1-b^{a_1(s-2)}}\right)^{\alpha_1-1}\left(\frac{1-b^{a_2(N+1)(s-2)}}{1-b^{a_2(s-2)}}\right)^{\alpha_2}\right\}\\
&+a_2\alpha_2\left\{\left(\frac{b^{a_2(N+1)(s-1)}-1}{b^{a_2(s-1)}-1}\right)\right.\\
&\left.\cdot\left(\frac{1-b^{a_2(N+1)(s-2)}}{1-b^{a_2(s-2)}}\right)^{\alpha_2-1}\left(\frac{1-b^{a_1(N+1)(s-2)}}{1-b^{a_1(s-2)}}\right)^{\alpha_1}\right\}.
\end{aligned}
\tag{C.3}
$$

The second term of (C.2) apart from the constant term can be expanded as

$$
\begin{aligned}
&\sum_{\underline{p}^1=N+1}^{N_f-1}\left\{\prod_{q_1=1}^{\alpha_1} b^{a_1(s-2)p_{q_1}^1}\right\}\sum_{\underline{p}^2=N+1}^{N_f-1}\left\{\prod_{q_2=1}^{\alpha_2} b^{a_2(s-2)p_{q_2}^2}\right\}\\
&\quad=\left(\frac{b^{a_1(N+1)(s-2)}-b^{a_1N_f(s-2)}}{1-b^{a_1(s-2)}}\right)^{\alpha_1}\\
&\quad\cdot\left(\frac{b^{a_2(N+1)(s-2)}-b^{a_2N_f(s-2)}}{1-b^{a_2(s-2)}}\right)^{\alpha_2}.
\end{aligned}
\tag{C.4}
$$

By inserting (C.3) and (C.4) in (C.2) and considering that: 1) $b^{a_n(N+1)(s-1)}-1\leq b^{a_n(N+1)(s-1)}$; 2) $b^{a_n(N+1)(s-2)}-b^{a_nN_f(s-2)}\leq b^{a_n(N+1)(s-2)}$; 3) $1-b^{a_n(N+1)(s-2)}\leq 1$; and 4) $b^{a_n(N+1)\alpha_n(s-2)}\leq b^{a_n(N+1)(s-2)}$ we obtain

$$
\begin{aligned}
&\Delta\gamma_R^{(2)}(t,\underline{k}^1,\underline{k}^2)\\
&\leq|\mu(\underline{a},\underline{\alpha})||A_0|K_0V|h|\\
&\quad\cdot\left\{a_1\alpha_1\left\{\left(\frac{b^{a_1(s-1)}}{b^{a_1(s-1)}-1}\right)\cdot\left(\frac{1}{1-b^{a_1(s-2)}}\right)^{\alpha_1-1}\right.\right.\\
&\quad\left.\times\left(\frac{1}{1-b^{a_2(s-2)}}\right)^{\alpha_2}b^{a_1N(s-1)}\right\}+a_2\alpha_2\left\{\left(\frac{b^{a_2(s-1)}}{b^{a_2(s-1)}-1}\right)\right.\\
&\quad\left.\left.\cdot\left(\frac{1}{1-b^{a_2(s-2)}}\right)^{\alpha_2-1}\left(\frac{1}{1-b^{a_1(s-2)}}\right)^{\alpha_1}b^{a_2N(s-1)}\right\}\right\}\\
&\quad+2|\mu(\underline{a},\underline{\alpha})||A_0|\left(\frac{1}{1-b^{a_1(s-2)}}\right)^{\alpha_1}\\
&\quad\times\left(\frac{1}{1-b^{a_2(s-2)}}\right)^{\alpha_2}b^{(N+1)(s-2)(a_1+a_2)}.
\end{aligned}
\tag{C.5}
$$

If $a=\max(a_1,a_2)$ we have that: 1) $b^{a_nN(s-1)}\leq b^{aN(s-1)}$ and 2) $b^{(N+1)(s-2)(a_1+a_2)}\leq b^{(N+1)(s-2)a}$. By applying the inequalities 1) and 2) and by assuming that $1/b^{a(N+1)}\leq|h|\leq 1/b^{aN}$ we have

$$\Delta\gamma_R^{(2)}(t,\underline{k}^1,\underline{k}^2)\leq c|h|^{2-s} \tag{C.6}$$

where

$$
\begin{aligned}
c&=|\mu(\underline{a},\underline{\alpha})||A_0|K_0V\\
&\quad\cdot\left\{a_1\alpha_1\left\{\left(\frac{b^{a_1(s-1)}}{b^{a_1(s-1)}-1}\right)\cdot\left(\frac{1}{1-b^{a_1(s-2)}}\right)^{\alpha_1-1}\right.\right.\\
&\quad\left.\times\left(\frac{1}{1-b^{a_2(s-2)}}\right)^{\alpha_2}\right\}+a_2\alpha_2\left\{\left(\frac{b^{a_2(s-1)}}{b^{a_2(s-1)}-1}\right)\right.\\
&\quad\left.\left.\cdot\left(\frac{1}{1-b^{a_2(s-2)}}\right)^{\alpha_2-1}\left(\frac{1}{1-b^{a_1(s-2)}}\right)^{\alpha_1}\right\}\right\}\\
&\quad+2|\mu(\underline{a},\underline{\alpha})||A_0|\left(\frac{1}{1-b^{a_1(s-2)}}\right)^{\alpha_1}\left(\frac{1}{1-b^{a_2(s-2)}}\right)^{\alpha_2}
\end{aligned}
\tag{C.7}
$$

is a constant greater than zero and independent on h.

Due to the arbitrary choice of N, (C.7) also holds for any value of h such that $|h|\geq 1/b^{aN_f}$. In other words, inequality (C.7) is satisfied for any h such that $1/b^{aN_f}\leq|h|\leq\delta$ with $\delta>0$. The latter condition means that $\gamma_R^{(2)}(t,\underline{k}^1,\underline{k}^2)$ is a fractal up to a scale level corresponding to $1/b^{aN_f}$. It is clear that when $N_f\rightarrow\infty$ the function $\gamma_R^{(2)}(t,\underline{k}^1,\underline{k}^2)$ becomes a fractal in a mathematical sense.

From (C.7) we can conclude that:

RAIII) the real part $\gamma_R^{(2)}(t,\underline{k}^1,\underline{k}^2)$ is a fractal whose dimension is upper bounded by s. Same conclusion holds for the imaginary parts (let us follow the same demonstration).

Appendix D

In this section, we demonstrate that the signal $\gamma_R^{(K)}(t,\underline{k}^1\cdots\underline{k}^K)$ of (46) is a fractal whose dimension is upper bounded by s.

We have to show that $\Delta\gamma_R^{(K)}(t,\underline{k}^1\cdots\underline{k}^K) = |\gamma_R^{(K)}(t+h,\underline{k}^1\cdots\underline{k}^K)-\gamma_R^{(K)}(t,\underline{k}^1\cdots\underline{k}^K)| \leq c|h|^{2-s}$ for some constant c, independent on h, and for $|h|<\delta$ with $\delta>0$. By means of (46), we have

$$\begin{aligned}\Delta\gamma_R^{(K)}&(t,\underline{k}^1\cdots\underline{k}^K)\\ &\leq \sum_{\underline{p}^1\cdots\underline{p}^K}|\mu(\underline{a},\underline{\alpha})||A(\underline{p}^1\cdots\underline{p}^K,\underline{k}^1\cdots\underline{k}^K)|\\ &\quad\times\left[\prod_{n=1}^{N_f}\left\{\prod_{q_n=1}^{\alpha_n} b^{a_1(s-2)p_{q_1}^1}\right\}\right]\\ &\quad\cdot\Bigg|\cos\Bigg\{\sum_{q_1=1}^{\alpha_1}\sum_{q_2=1}^{\alpha_2}\cdots\sum_{q_K=1}^{\alpha_K}\\ &\quad\times\left[\sum_{n=1}^{K}\mathrm{sgn}\left(m_{p_{q_n}^n}^n\right)a_n\left(K_0Vb^{p_{q_n}^n}(t+h)+\Phi_{p_{q_n}^n}\right)\right]\Bigg\}\\ &\quad-\cos\Bigg\{\sum_{q_1=1}^{\alpha_1}\sum_{q_2=1}^{\alpha_2}\cdots\sum_{q_K=1}^{\alpha_K}\\ &\quad\times\left[\sum_{n=1}^{K}\mathrm{sgn}\left(m_{p_{q_n}^n}^n\right)a_n\left(K_0Vb^{p_{q_n}^n}t+\Phi_{p_{q_n}^n}\right)\right]\Bigg\}\Bigg|.\end{aligned} \tag{D.1}$$

By applying the steps 1) to 5) of Appendix C with $1\leq n\leq K$, we have

$$\begin{aligned}\Delta\gamma_R^{(K)}&(t,\underline{k}^1\cdots\underline{k}^K)\\ &\leq|\mu(\underline{a},\underline{\alpha})||A_0|K_0V|h|\\ &\quad\cdot\left\{\sum_{\underline{p}^1\cdots\underline{p}^K=0}^{N}\left[\prod_{n=1}^{K}\left\{\prod_{q_n=1}^{\alpha_n}b^{a_n(s-2)p_{q_n}^n}\right\}\right]\right.\\ &\quad\cdot\left.\left\{\sum_{n=1}^{K}\left(a_n\sum_{q_n=1}^{\alpha_n}b^{a_np_{q_n}^n}\right)\right\}\right\}+2|\mu(\underline{a},\underline{\alpha})||A_0|\\ &\quad\cdot\sum_{\underline{p}^1\cdots\underline{p}^K=N+1}^{N_f-1}\left[\prod_{n=1}^{K}\left\{\prod_{q_n=1}^{\alpha_n}b^{a_n(s-2)p_{q_n}^n}\right\}\right].\end{aligned} \tag{D.2}$$

By expanding (D.2) with the same procedure of (C.3) and by considering that: 1) $b^{a_n(N+1)(s-1)}-1\leq b^{a_n(N+1)(s-1)}$; 2) $b^{a_n(N+1)(s-2)}-b^{a_nN_f(s-2)}\leq b^{a_n(N+1)(s-2)}$; 3) $1-b^{a_n(N+1)(s-2)}\leq 1$; and 4) $b^{a_n(N+1)\alpha_n(s-2)}\leq b^{a_n(N+1)(s-2)}$, we obtain

$$\begin{aligned}\Delta\gamma_R^{(K)}&(t,\underline{k}^1\cdots\underline{k}^K)\\ &\leq|\mu(\underline{a},\underline{\alpha})||A_0|K_0V|h|\\ &\quad\cdot\sum_{n=1}^{K}a_n\alpha_n\left\{\left(\frac{b^{a_n(s-1)}}{b^{a_n(s-1)}-1}\right)\cdot\left(\frac{1}{1-b^{a_n(s-2)}}\right)^{\alpha_n-1}\right.\\ &\quad\times\left.\left[\prod_{\substack{j=1\\ j\neq n}}^{K}\left(\frac{1}{1-b^{a_j(s-2)}}\right)^{\alpha_j}\right]b^{a_nN(s-1)}\right\}\\ &\quad+2|\mu(\underline{a},\underline{\alpha})||A_0|\left[\prod_{n=1}^{K}\left(\frac{1}{1-b^{a_n(s-2)}}\right)^{\alpha_n}\right]\\ &\quad\times b^{(N+1)(s-2)\sum_{n=1}^{K}a_n}.\end{aligned} \tag{D.3}$$

If $a=\max_{n=1,2,\cdots,K}(a_n)$ we have that: 1) $b^{a_nN(s-1)}\leq b^{aN(s-1)}$ and 2) $b^{(N+1)(s-2)\sum_{n=1}^{K}a_n}\leq b^{(N+1)(s-2)a}$. By applying the inequalities 1) and 2) and by assuming $1/b^{a(N+1)}\leq|h|\leq 1/b^{aN}$, we have

$$\Delta\gamma_R^{(K)}(t,\underline{k}^1\cdots\underline{k}^K)\leq c|h|^{2-s} \tag{D.4}$$

where

$$c=\sum_{n=1}^{K}c_n+d \tag{D.5}$$

with

$$\begin{aligned}c_n=|\mu(\underline{a},\underline{\alpha})||A_0|K_0Va_n\alpha_n&\left\{\left(\frac{b^{a_n(s-1)}}{b^{a_n(s-1)}-1}\right)\right.\\ &\cdot\left.\left(\frac{1}{1-b^{a_n(s-2)}}\right)^{\alpha_n-1}\left[\prod_{\substack{j=1\\ j\neq n}}^{K}\left(\frac{1}{1-b^{a_j(s-2)}}\right)^{\alpha_j}\right]\right\}\end{aligned} \tag{D.6}$$

and

$$d=2|\mu(\underline{a},\underline{\alpha})||A_0|\left[\prod_{n=1}^{K}\left(\frac{1}{1-b^{a_n(s-2)}}\right)^{\alpha_n}\right]. \tag{D.7}$$

Note that c is a constant greater than zero and independent on h.

Due to the arbitrary choice of N, (D.4) also holds for any value of h such that $|h|\geq 1/b^{aN_f}$. In other words, inequality (D.4) is satisfied for any h such that $1/b^{aN_f}\leq|h|\leq\delta$ with $\delta>0$. The latter condition means that $\gamma_R^{(K)}(t,\underline{k}^1\cdots\underline{k}^K)$ is a fractal up to a scale level corresponding to $1/b^{aN_f}$. It is clear that when $N_f\to\infty$ the function $\gamma_R^{(K)}(t,\underline{k}^1\cdots\underline{k}^K)$ becomes a fractal in a mathematical sense.

From (D.4) we can conclude that:

RAIV) The real part $\gamma_R^{(K)}(t,\underline{k}^1\cdots\underline{k}^K)$ is a fractal whose dimension is upper bounded by s.

The same conclusion holds for the imaginary parts (use the same demonstration).

ACKNOWLEDGMENT

The authors would like to thank the students summarized in the technical reports [7], [12], and [14] for their help and their extensive studies carried out in 1996.

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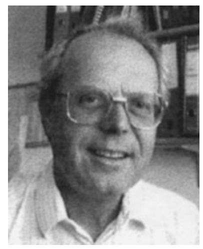

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