Higher Order Interpolatory Vector Bases on Pyramidal Elements

Roberto D. Graglia, *Fellow, IEEE*, and Ioan-L. Gheorma

***Abstract*—In the numerical solution of three-dimensional (3-D) electromagnetic field problems, the regions of interest can be discretized by elements having tetrahedral, brick or prismatic shape. However, such different shape elements cannot be linked to form a conformal mesh; to this purpose pyramidal elements are required. In this paper, we define interpolatory higher order curl- and divergence-conforming vector basis functions on pyramidal elements, with extension to curved pyramids, and discuss their completeness properties. A general procedure to obtain vector bases of arbitrary polynomial order is given and bases up to second order are explicitly reported. These new elements ensure the continuity of the proper vector components across adjacent elements of equal order but different shape. Results to confirm the faster convergence of higher order functions on pyramids are presented.**

***Index Terms*— Electromagnetic fields, finite-element methods, higher order vector elements, method of moments, numerical analysis, pyramidal elements.**

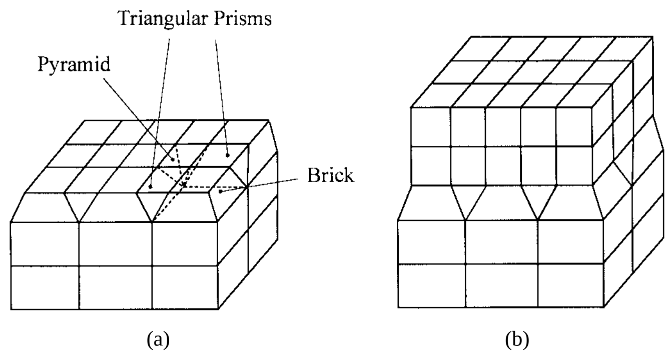

Fig. 1. (a) One *buffer* layer of mixed-shape elements (bricks, tetrahedrons, prisms, and pyramids) is used to link two brick meshes (b) having different mesh-size. In the buffer region, pyramidal elements are required to obtain a conformal mesh.

I. Introduction

INTERPOLATORY higher order vector basis functions of the Nedelec variety [1] have been recently defined in a unified and consistent manner for the most common element shapes [2], [3]. Three-dimensional (3-D) structures can be discretized by elements having tetrahedral, brick, or prismatic shape. However, in general, elements of different shape cannot be used together to form a 3-D conformal mesh without introducing pyramidal elements. Hence, the new element at issue, which has the shape of a pyramid with quadrilateral base, most times is useful as a *filler*; for example, it is required when one has to link a coarse to a dense mesh of bricks, as schematically depicted in Fig. 1. Few previous works considered pyramidal elements [4], [5] and did not addressed the issue of general construction of higher order forms on pyramids or the issue of the existence of spurious modes when using pyramidal elements. In this work, for pyramidal elements, we consider both curl- and divergence-conforming bases, which have continuous tangential or normal components, respectively, across adjacent elements. The basis functions we present here are of interpolatory kind and ensure the continuity of the proper components across adjacent elements of equal order but different shape (tetrahedrons, bricks, prisms). Curl-conforming basis functions are appropriate for discretizing the vector Helmholtz operator, while divergence conforming functions are appropriate for integral operators such as the electric field integral equation. These bases avoid the spurious modes usually encountered when scalar representations are used with one of the foregoing equations and simplify the enforcement of boundary conditions on current or fields in a numerical approach.

Curved pyramids are obtained by parametrically distorting a *parent* pyramidal element. This process requires the introduction of *shape* functions; the shape functions presented here ensure mesh conformity when element of different shape are used in the same mesh. Part of the results reported here were presented in [6].

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The authors are with the Dipartimento di Elettronica, Politecnico di Torino, Torino, 10129 Italy.

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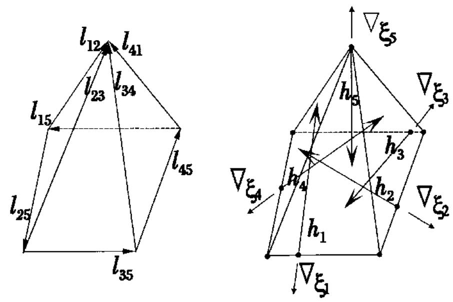
II. Element Geometry Representation

In this section, we define normalized parametric coordinates and related geometrical quantities by assuming rectilinear pyramidal elements; extension to curvilinear elements is easily obtained by use of the results of [2, Appendix] together with (1)–(3) reported below. The geometrical parameters for pyramidal elements are shown in Fig. 2. The faces are numbered to correspond to the indexing of the associated parametric coordinates; that is, the ith face of the pyramid is the zero-coordinate surface for the normalized coordinate ξ_i, which varies linearly across the element, attaining a value of unity at the node or face opposite the zero-coordinate surface. For the pyramidal element we choose as *independent* coordinates ξ_1, ξ_2 and ξ_5 so that $\underline{\nabla}\xi_5 \cdot (\underline{\nabla}\xi_1 \times \underline{\nabla}\xi_2)$ is strictly positive, while ξ_3 and ξ_4 are *dependent* coordinates. In this case, the



(a) (b)

Fig. 2. Pyramidal element. (a) Edge vectors. (b) Height and gradient vectors.

dependency relations are

$$\begin{aligned}\xi_1 + \xi_3 + \xi_5 &= 1\\ \xi_2 + \xi_4 + \xi_5 &= 1.\end{aligned} \tag{1}$$

The coordinates appearing in each dependency relation form a *group* of dependent coordinates. Similarly to what has been done in [2] and [3], we list the coordinates as $\{\xi_1, \xi_3; \xi_2, \xi_4; \xi_5\}$ to put in evidence that ξ_1, ξ_2 and ξ_5 are the independent coordinates while ξ_3 and ξ_4 are dependent; the indexes corresponding to sampled values of the coordinates are listed as $\{i, k; j, \ell; m\}$ to put in evidence that k and ℓ are dependent indices. The element edges are formed by intersection of pairs of zero-coordinates surfaces, and the edge vectors are directed along the cross product of the associated coordinate gradients. The edges are given a two-index label deriving from the two coordinate indexes appearing in this cross product [see Fig. 2(a)]. The *unitary basis vectors* $\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^5$ are derivatives of the element position vector $\boldsymbol{r}$ with respect to the independent coordinates [2] and determine the following edge-vectors:

$$\begin{aligned}\boldsymbol{\ell}_{25} &= -\boldsymbol{\ell}_{45} = \boldsymbol{\ell}^1\\ \boldsymbol{\ell}_{35} &= -\boldsymbol{\ell}_{15} = \boldsymbol{\ell}^2\\ \boldsymbol{\ell}_{12} &= \boldsymbol{\ell}^5\\ \boldsymbol{\ell}_{23} &= \boldsymbol{\ell}^5 - \boldsymbol{\ell}^1\\ \boldsymbol{\ell}_{34} &= \boldsymbol{\ell}^5 - \boldsymbol{\ell}^1 - \boldsymbol{\ell}^2\\ \boldsymbol{\ell}_{41} &= \boldsymbol{\ell}^5 - \boldsymbol{\ell}^2.\end{aligned} \tag{2}$$

In the special case where $\boldsymbol{\ell}^1, \boldsymbol{\ell}^2$, and $\boldsymbol{\ell}^5$ are constant vectors, the pyramid is a right pyramid.

The independent gradient vectors $\underline{\nabla}\xi_1$, $\underline{\nabla}\xi_2$, $\underline{\nabla}\xi_5$ are derived from [2, eq. (41)]; the remaining coordinate gradients are determined by applying the gradient operator to (1) as $\underline{\nabla}\xi_3 = -\underline{\nabla}\xi_1 - \underline{\nabla}\xi_5, \underline{\nabla}\xi_4 = -\underline{\nabla}\xi_2 - \underline{\nabla}\xi_5$. In turn, the gradient vectors yield five height vectors $\boldsymbol{h}_i = -h_i^2 \underline{\nabla}\xi_i$ $(i = 1, 2, 3, 4, 5)$, with $h_i = 1/|\underline{\nabla}\xi_i|$ [2, Appendix].

For curvilinear elements, all these geometrical quantities, including the Jacobian $\mathcal{J} = \boldsymbol{\ell}^1 \cdot \boldsymbol{\ell}^2 \times \boldsymbol{\ell}^5$, vary with position.

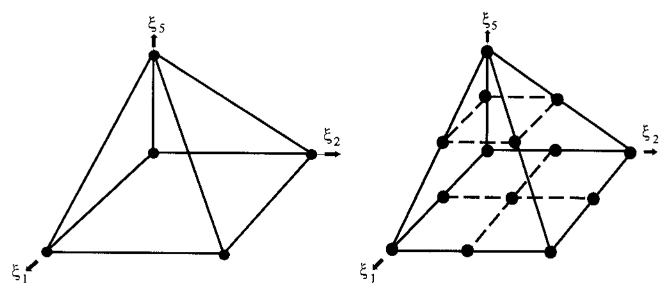
A parametrization of order q for a curvilinear pyramid can be expressed as

$$\boldsymbol{r} = \sum_{\substack{i,j,k,\ell,m=0\\ i+k+m=j+\ell+m=q}}^{q} \boldsymbol{r}_{ik;j\ell;m}\, S_{ik;j\ell;m}(q, \boldsymbol{\xi}) \tag{3}$$

TABLE I
PYRAMID SHAPE FUNCTIONS

Interpolation Point		First Order Shape Functions
base corner	(4)	$\frac{\xi_i \xi_{i+1}}{1-\xi_5}$
tip	(1)	ξ_5
		Second Order Shape Functions
base corner	(4)	$\xi_i \xi_{i+1} \left[\left(\frac{2\xi_i}{1-\xi_5} - 1\right)\left(\frac{2\xi_{i+1}}{1-\xi_5} - 1\right) - \frac{\xi_5}{1-\xi_5}\right]$
base edge midpoint	(4)	$4\,\frac{\xi_{i+1}\xi_{i-1}}{1-\xi_5}\,\xi_i \left(\frac{2\xi_i}{1-\xi_5} - 1\right)$
base middle node	(1)	$16\,\frac{\xi_1\xi_2\xi_3\xi_4}{(1-\xi_5)^2}$
lateral edge midpoint	(4)	$4\,\frac{\xi_i\xi_{i+1}\xi_5}{1-\xi_5}$
tip	(1)	$\xi_5(2\xi_5 - 1)$

Subscripts are counted modulo 4, for $i = 1, 4$



(a) (b)

Fig. 3. Parent domain parametrization for a curvilinear pyramid. The pyramid shape functions interpolate 5 and 14 nodes for (a) first- and (b) second-order parametrization, respectively.

where a quintuplet indexing scheme is used to label the position vector $\boldsymbol{r}_{ik;j\ell;m}$ interpolating the point with normalized coordinates $\boldsymbol{\xi}_{ik;j\ell;m} = (\xi_1, \xi_3; \xi_2, \xi_4; \xi_5) = (i/q, k/q; j/q, \ell/q; m/q)$. Hence, the shape function $S_{ik;j\ell;m}(q, \boldsymbol{\xi})$ is defined to be unity at $\boldsymbol{\xi}_{ik;j\ell;m}$ and with zeros at the other interpolation points.

In applications, second order parametrizations usually suffice. Table I reports the first- and second-order shape functions, while the arrangement of the interpolation points in the parent domain is shown in Fig. 3; from this figure one infers that the number of shape functions for a qth order pyramid is $\Sigma_{n=1}^{q+1}\ n^2 = ((q+1)(q+2)(2q+3)/6)$ [see also (3)]. A very important property of the shape functions regards their completeness, which requires that all *inhomogeneous* monomials of the form $\xi_1^r \xi_2^s \xi_5^t$ with $0 \le r+s+t \le q$ can be expressed as a linear combination of the shape functions of order q; it is readily proved that the shape functions of Table I are complete in this sense (for example, the sum of all the first and second-order shape functions of Table I is equal to unity).

Also, the shape functions of Table I and their first derivatives are always bounded in the element domain, even at the tip of the pyramid where $\xi_5 = 1$, because there one has $\xi_1 = \xi_2 = \xi_3 = \xi_4 = 0$. The point $\xi_5 = 1$ is a point of discontinuity for the Jacobian of the parametrized element although the Jacobian is bounded at that point.

The factors $(1-\xi_5)^{-1}$ appearing in the expressions of the shape functions of Table I are necessary in order to guarantee element conformity when the pyramidal element of order q is connected to a brick, to a tetrahedron or to a prism element of the same order. In fact, in the limit for $\xi_1 = 0$ (or $\xi_2, \xi_3, \xi_4 = 0$) the shape functions of Table I yield the triangular shape functions, while they yield the quadrilateral shape functions in the limit for $\xi_5 = 0$ [2].

III. Curl-Conforming Bases on Pyramids

A. Zeroth-Order Bases

Curl-conforming bases of zeroth order on a pyramid are defined as

$$\begin{aligned}
\boldsymbol{\Omega}_{12}(\boldsymbol{r}) &= \frac{\xi_3\xi_4\underline{\nabla}\xi_5 - \xi_4\xi_5\underline{\nabla}\xi_3 - \xi_3\xi_5\underline{\nabla}\xi_4}{1-\xi_5} - \frac{\xi_3\xi_4\xi_5}{(1-\xi_5)^2}\underline{\nabla}\xi_5\\
\boldsymbol{\Omega}_{23}(\boldsymbol{r}) &= \frac{\xi_4\xi_1\underline{\nabla}\xi_5 - \xi_1\xi_5\underline{\nabla}\xi_4 - \xi_4\xi_5\underline{\nabla}\xi_1}{1-\xi_5} - \frac{\xi_4\xi_1\xi_5}{(1-\xi_5)^2}\underline{\nabla}\xi_5\\
\boldsymbol{\Omega}_{34}(\boldsymbol{r}) &= \frac{\xi_1\xi_2\underline{\nabla}\xi_5 - \xi_2\xi_5\underline{\nabla}\xi_1 - \xi_1\xi_5\underline{\nabla}\xi_2}{1-\xi_5} - \frac{\xi_1\xi_2\xi_5}{(1-\xi_5)^2}\underline{\nabla}\xi_5\\
\boldsymbol{\Omega}_{41}(\boldsymbol{r}) &= \frac{\xi_2\xi_3\underline{\nabla}\xi_5 - \xi_3\xi_5\underline{\nabla}\xi_2 - \xi_2\xi_5\underline{\nabla}\xi_3}{1-\xi_5} - \frac{\xi_2\xi_3\xi_5}{(1-\xi_5)^2}\underline{\nabla}\xi_5\\
\boldsymbol{\Omega}_{15}(\boldsymbol{r}) &= \frac{\xi_2\xi_3\underline{\nabla}\xi_4 - \xi_3\xi_4\underline{\nabla}\xi_2}{1-\xi_5}\\
\boldsymbol{\Omega}_{25}(\boldsymbol{r}) &= \frac{\xi_3\xi_4\underline{\nabla}\xi_1 - \xi_4\xi_1\underline{\nabla}\xi_3}{1-\xi_5}\\
\boldsymbol{\Omega}_{35}(\boldsymbol{r}) &= \frac{\xi_4\xi_1\underline{\nabla}\xi_2 - \xi_1\xi_2\underline{\nabla}\xi_4}{1-\xi_5}\\
\boldsymbol{\Omega}_{45}(\boldsymbol{r}) &= \frac{\xi_1\xi_2\underline{\nabla}\xi_3 - \xi_2\xi_3\underline{\nabla}\xi_1}{1-\xi_5}.
\end{aligned} \tag{4}$$

Notice that the expression of the function $\boldsymbol{\Omega}_{23}, \boldsymbol{\Omega}_{34}$, and $\boldsymbol{\Omega}_{41}$ can be obtained from that of $\boldsymbol{\Omega}_{12}$ by exploiting the symmetry of the pyramidal element. In fact by substituting $(\xi_1, \xi_2, \xi_3, \xi_4)$ with $(\xi_i, \xi_{i+1}, \xi_{i+2}, \xi_{i+3})$ in the expression of $\boldsymbol{\Omega}_{12}$ and by counting the subscripts modulo 4, one immediately obtains the expression of $\boldsymbol{\Omega}_{i,i+1}$ for $i = 2, 3, 4$. $\boldsymbol{\Omega}_{25}, \boldsymbol{\Omega}_{35}$ and $\boldsymbol{\Omega}_{45}$ can be similarly obtained by *rotating* the parent variables (or the subscripts from 1 to 4) in the expression of $\boldsymbol{\Omega}_{15}$. The same symmetry property has also been used to compactly express the shape functions of Table I.

$\boldsymbol{\Omega}_{12}(\boldsymbol{r})$ simplifies to $\xi_4\underline{\nabla}\xi_5 - \xi_5\underline{\nabla}\xi_4 + \xi_4\xi_5(1-\xi_5)^{-1}\underline{\nabla}\xi_1$ on face $\xi_1 = 0$, while $\boldsymbol{\Omega}_{12}(\boldsymbol{r}) = \xi_3\underline{\nabla}\xi_5 - \xi_5\underline{\nabla}\xi_3 + \xi_3\xi_5(1-\xi_5)^{-1}\underline{\nabla}\xi_2$ on face $\xi_2 = 0$. On face $\xi_1 = 0$ one gets $\boldsymbol{\Omega}_{15}(\boldsymbol{r}) = \xi_2\underline{\nabla}\xi_4 - \xi_4\underline{\nabla}\xi_2$, while $\boldsymbol{\Omega}_{15}(\boldsymbol{r}) = \xi_3\underline{\nabla}\xi_4 + \xi_3\xi_4\underline{\nabla}\xi_5$ on face $\xi_5 = 0$. Therefore, by keeping in mind the symmetry of the curl-conforming functions just discussed, it is apparent that the bases have a constant tangential (CT) component along each element edge, which matches with that of the curl-conforming bases of the adjacent element having brick, tetrahedral, or prismatic shape (see [2], [3]). The curl of (4) are compactly written as

$$\begin{aligned}
\nabla \times \boldsymbol{\Omega}_{i,i+1}(\boldsymbol{r}) &= \frac{2}{1-\xi_5}\,\frac{\xi_{i+2}\boldsymbol{\ell}_{i-1,5} + \xi_{i-1}\boldsymbol{\ell}_{i+2,5}}{\mathcal{J}}\\
\nabla \times \boldsymbol{\Omega}_{i5}(\boldsymbol{r}) &= \frac{\nabla \times \boldsymbol{\Omega}_{i,i+1}(\boldsymbol{r})}{2} + \frac{\boldsymbol{\ell}_{i+2,i-1}}{\mathcal{J}}
\end{aligned} \tag{5}$$

for $i = 1, 2, 3, 4$ and by counting the subscripts modulo 4.

The above bases are normalized by ensuring a unit component of $\boldsymbol{\Omega}_{\gamma\beta}(\boldsymbol{r})$ along $\boldsymbol{\ell}_{\gamma\beta}$ at the midpoint of edge $\gamma\beta$ (interpolation point). Hence, the normalized form of the zeroth order bases (4) is $\ell_{\gamma\beta}\boldsymbol{\Omega}_{\gamma\beta}(\boldsymbol{r})$, where $\ell_{\gamma\beta} = |\boldsymbol{\ell}_{\gamma\beta}|$ at the corresponding interpolation point.

B. Completeness of Zeroth-Order Bases

Although the bases (4) contain rational terms, their completeness to zeroth order follows from forming linear combinations that yield three independent constant vectors

$$\begin{aligned}
&\boldsymbol{\Omega}_{25}(\boldsymbol{r}) - \boldsymbol{\Omega}_{45}(\boldsymbol{r}) - \boldsymbol{\Omega}_{23}(\boldsymbol{r}) - \boldsymbol{\Omega}_{34}(\boldsymbol{r}) = \nabla\xi_1\\
&\boldsymbol{\Omega}_{35}(\boldsymbol{r}) - \boldsymbol{\Omega}_{15}(\boldsymbol{r}) - \boldsymbol{\Omega}_{34}(\boldsymbol{r}) - \boldsymbol{\Omega}_{41}(\boldsymbol{r}) = \nabla\xi_2\\
&\boldsymbol{\Omega}_{12}(\boldsymbol{r}) + \boldsymbol{\Omega}_{23}(\boldsymbol{r}) + \boldsymbol{\Omega}_{34}(\boldsymbol{r}) + \boldsymbol{\Omega}_{41}(\boldsymbol{r}) = \nabla\xi_5.
\end{aligned} \tag{6}$$

The bases are complete only to zeroth order since they cannot represent vectors of the form $\xi_i\nabla\xi_i$ $(i = 1, 2, 3, 4)$. Notice that $\nabla\xi_3 = -\nabla\xi_1 - \nabla\xi_5 = \boldsymbol{\Omega}_{45}(\boldsymbol{r}) - \boldsymbol{\Omega}_{25}(\boldsymbol{r}) - \boldsymbol{\Omega}_{12}(\boldsymbol{r}) - \boldsymbol{\Omega}_{41}(\boldsymbol{r})$, $\nabla\xi_4 = -\nabla\xi_2 - \nabla\xi_5 = \boldsymbol{\Omega}_{15}(\boldsymbol{r}) - \boldsymbol{\Omega}_{35}(\boldsymbol{r}) - \boldsymbol{\Omega}_{23}(\boldsymbol{r}) - \boldsymbol{\Omega}_{12}(\boldsymbol{r})$.

Completeness of the curl of the bases to zeroth order follows from

$$\begin{aligned}
\nabla \times [\boldsymbol{\Omega}_{12}(\boldsymbol{r}) + \boldsymbol{\Omega}_{23}(\boldsymbol{r})] &= \frac{2\boldsymbol{\ell}_{25}}{\mathcal{J}}\left(=\frac{2\boldsymbol{\ell}^1}{\mathcal{J}}\right) = -\frac{2\boldsymbol{\ell}_{45}}{\mathcal{J}}\\
\nabla \times [\boldsymbol{\Omega}_{12}(\boldsymbol{r}) + \boldsymbol{\Omega}_{41}(\boldsymbol{r})] &= \frac{2\boldsymbol{\ell}_{35}}{\mathcal{J}}\left(=\frac{2\boldsymbol{\ell}^2}{\mathcal{J}}\right) = -\frac{2\boldsymbol{\ell}_{15}}{\mathcal{J}}\\
\nabla \times [\boldsymbol{\Omega}_{35}(\boldsymbol{r}) + \boldsymbol{\Omega}_{45}(\boldsymbol{r})] &= \frac{2\boldsymbol{\ell}_{12}}{\mathcal{J}}\left(=\frac{2\boldsymbol{\ell}^5}{\mathcal{J}}\right).
\end{aligned} \tag{7}$$

On curvilinear elements, completeness is with respect to these vectors as weighting factors. By *rotating the subscripts* $(1, 2, 3, 4)$ in the last of (7), as discussed following (4), one also gets

$$\begin{aligned}
\underline{\nabla} \times [\boldsymbol{\Omega}_{45}(\boldsymbol{r}) + \boldsymbol{\Omega}_{15}(\boldsymbol{r})] &= \frac{2\boldsymbol{\ell}_{23}}{\mathcal{J}}\\
\underline{\nabla} \times [\boldsymbol{\Omega}_{15}(\boldsymbol{r}) + \boldsymbol{\Omega}_{25}(\boldsymbol{r})] &= \frac{2\boldsymbol{\ell}_{34}}{\mathcal{J}}\\
\underline{\nabla} \times [\boldsymbol{\Omega}_{25}(\boldsymbol{r}) + \boldsymbol{\Omega}_{35}(\boldsymbol{r})] &= \frac{2\boldsymbol{\ell}_{41}}{\mathcal{J}}.
\end{aligned} \tag{8}$$

TABLE II
INTERPOLATORY POLYNOMIALS FOR FIRST-ORDER CURL-CONFORMING BASES

$N^{12}_{ik,j\ell,m}$	$\alpha^{12}_{ik,j\ell,m}(\boldsymbol{\xi})$	interp. point $\boldsymbol{\xi}_{ik,j\ell,m}$	$N^{15}_{ik,j\ell,m}$	$\alpha^{15}_{ik,j\ell,m}(\boldsymbol{\xi})$	interp. point $\boldsymbol{\xi}_{ik,j\ell,m}$
ℓ_{12}	$3\xi_5 - 1$	$(0, \frac{1}{3}; 0, \frac{1}{3}; \frac{2}{3})$	ℓ_{15}	$(3\xi_4 - 1)(2\xi_3 + 2\xi_5 - 1)$	$(0, 1; \frac{1}{3}, \frac{2}{3}; 0)$
ℓ_{12}	$3\xi_3 - 3\xi_2 - 1$	$(0, \frac{2}{3}; 0, \frac{2}{3}; \frac{1}{3})$	ℓ_{15}	$(3\xi_2 - 1)(2\xi_3 + 2\xi_5 - 1)$	$(0, 1; \frac{2}{3}, \frac{1}{3}; 0)$
$\frac{3}{2}\ell_{12}$	$3\xi_1$	$(\frac{1}{3}, \frac{1}{3}; 0, \frac{2}{3}; \frac{1}{3})$	$2\ell_{15}$	$2\xi_1(3\xi_4 - 1)$	$(\frac{1}{2}, \frac{1}{2}; \frac{1}{3}, \frac{2}{3}; 0)$
$\frac{3}{2}\ell_{12}$	$3\xi_2$	$(0, \frac{2}{3}; \frac{1}{3}, \frac{1}{3}; \frac{1}{3})$	$2\ell_{15}$	$2\xi_1(3\xi_2 - 1)$	$(\frac{1}{2}, \frac{1}{2}; \frac{2}{3}, \frac{1}{3}; 0)$
			$\frac{3}{2}\ell_{15}$	$3\xi_5(1 - 3\xi_1)$	$(0, \frac{2}{3}; \frac{1}{3}, \frac{1}{3}; \frac{1}{3})$
			$3\ell_{15}$	$9\xi_1\xi_5$	$(\frac{1}{3}, \frac{1}{3}; \frac{1}{3}, \frac{1}{3}; \frac{1}{3})$

C. Order p Bases

Curl-conforming bases complete to order p are obtained by forming the product of the zeroth-order curl-conforming bases with polynomial factors complete to pth order. The multiplying polynomials we use are of interpolatory form. To ensure the continuity of the tangent component across adjacent elements of equal order but different shape, the interpolation points on each face and edge of the pyramid match with those already reported in [2], [3] for tetrahedrons, bricks, and triangular prisms. Hence, the polynomial factors associated with the zeroth order curl-conforming functions $\boldsymbol{\Omega}_{\gamma\beta}(\boldsymbol{r})$ become equal for $\xi_\gamma = 0$ (or $\xi_\beta = 0$) to the polynomial factors of the triangular element when γ (or β) $= 1, 2, 3$, or 4 or equal to the polynomial factors of the quadrilateral element for γ $(\beta) = 5$ [2]. As far as interpolation points internal to the element are concerned, we notice that for the pyramid, these cannot be obtained by following a *Lagrangian scheme* as done for tetrahedrons, bricks, and triangular prisms [2], [3]. At any rate, higher order curl-conforming bases on a pyramid can be succinctly written as

$$\boldsymbol{\Omega}^{\gamma\beta}_{ik,j\ell,m}(\boldsymbol{r}) = N^{\gamma\beta}_{ik,j\ell,m}\alpha^{\gamma\beta}_{ik,j\ell,m}(\boldsymbol{\xi})\boldsymbol{\Omega}_{\gamma\beta}(\boldsymbol{r}) \tag{9}$$

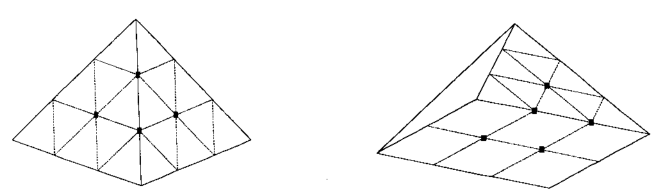
where $\alpha^{\gamma\beta}_{ik,j\ell,m}(\boldsymbol{\xi})$ is an interpolatory polynomial, while $N^{\gamma\beta}_{ik,j\ell,m}$ is a normalization constant chosen to ensure that the component of $\boldsymbol{\Omega}^{\gamma\beta}_{ik,j\ell,m}(\boldsymbol{r})$ along $\boldsymbol{\ell}_{\gamma\beta}$ at the interpolation point is unity.

Because of the symmetry of the pyramidal element, it is sufficient to report the expression of the interpolatory polynomials and normalization constants only for $\boldsymbol{\Omega}^{12}_{ik,j\ell,m}(\boldsymbol{r})$ and $\boldsymbol{\Omega}^{15}_{ik,j\ell,m}(\boldsymbol{r})$. The other polynomials and normalization constants are easily obtained by *rotating the subscripts* $(1, 2, 3, 4)$ as discussed following (4). Tables II and III report the interpolatory multiplying polynomials for first- and second-order bases, respectively; where ℓ_{12} (ℓ_{15}) is the value of $|\boldsymbol{\ell}_{12}|$ $(|\boldsymbol{\ell}_{15}|)$ at the indicated interpolation point $\boldsymbol{\xi}_{ik,j\ell,m}$; in the tables we have omitted the superscript $(ik; j\ell; m)$ to the normalization quantities ℓ_{12}, ℓ_{15} in order to ease the notation.

To clarify the property of our interpolation polynomials notice, for example, how the factors $\alpha^{12}_{ik,j\ell,m}(\boldsymbol{\xi})$ of Table II yield for $\xi_1 = 0$, the first-order multiplying polynomials $3\xi_5 - 1$, $3\xi_4 - 1$, $3\xi_2$ of the triangular element attached to face 1 (see [2]).

TABLE III
INTERPOLATORY POLYNOMIALS FOR SECOND-ORDER CURL-CONFORMING BASES

$N^{12}_{ik,j\ell,m}$	$\alpha^{12}_{ik,j\ell,m}(\boldsymbol{\xi})$	interp. point $\boldsymbol{\xi}_{ik,j\ell,m}$
ℓ_{12}	$(2\xi_5 - 1)(4\xi_5 - 1) + \xi_1\xi_2$	$(0, \frac{1}{4}; 0, \frac{1}{4}; \frac{3}{4})$
ℓ_{12}	$(4\xi_3 - 4\xi_2 - 1)(4\xi_5 - 1) + 3\xi_1\xi_2$	$(0, \frac{1}{2}; 0, \frac{1}{2}, \frac{1}{2})$
ℓ_{12}	$(2\xi_3 - 2\xi_2 - 1)(4\xi_3 - 4\xi_2 - 1) - 9\xi_1\xi_2$	$(0, \frac{3}{4}; 0, \frac{3}{4}, \frac{1}{4})$
$\frac{4}{3}\ell_{12}$	$4\xi_1(4\xi_5 - \xi_2 - 1)$	$(\frac{1}{4}, \frac{1}{4}; 0, \frac{1}{2}; \frac{1}{2})$
$\frac{4}{3}\ell_{12}$	$4\xi_1(4\xi_3 - \xi_2 - 1)$	$(\frac{1}{4}, \frac{1}{2}; 0, \frac{3}{4}; \frac{1}{4})$
$2\ell_{12}$	$2\xi_1(4\xi_1 - \xi_2 - 1)$	$(\frac{1}{2}, \frac{1}{4}; 0, \frac{3}{4}; \frac{1}{4})$
$\frac{4}{3}\ell_{12}$	$4\xi_2(4\xi_5 - \xi_1 - 1)$	$(0, \frac{1}{2}; \frac{1}{4}, \frac{1}{4}; \frac{1}{2})$
$\frac{4}{3}\ell_{12}$	$4\xi_2(4\xi_4 - \xi_1 - 1)$	$(0, \frac{3}{4}; \frac{1}{4}, \frac{1}{2}; \frac{1}{4})$
$2\ell_{12}$	$2\xi_2(4\xi_2 - \xi_1 - 1)$	$(0, \frac{3}{4}; \frac{1}{2}, \frac{1}{4}; \frac{1}{4})$
$\frac{12}{5}\ell_{12}$	$9\xi_1\xi_2$	$(\frac{1}{3}, \frac{1}{3}; \frac{1}{3}, \frac{1}{3}; \frac{1}{3})$
$N^{15}_{ik,j\ell,m}$	$\alpha^{15}_{ik,j\ell,m}(\boldsymbol{\xi})$ with $\chi_2 = \xi_2 + \xi_5$, $\chi_3 = \xi_3 + \xi_5$	interp. point $\boldsymbol{\xi}_{ik,j\ell,m}$
ℓ_{15}	$-\frac{(3\xi_1 - 1)(3\chi_3 - 1)(2\xi_4 - 1)(4\xi_4 - 1)}{2}$	$(0, 1; \frac{1}{4}, \frac{3}{4}; 0)$
ℓ_{15}	$-\frac{(3\xi_1 - 1)(3\chi_3 - 1)(4\chi_2 - 1)(4\xi_4 - 1)}{2} - 4(4\xi_4 - \xi_1 - 1)\xi_5$	$(0, 1; \frac{1}{2}, \frac{1}{2}; 0)$
ℓ_{15}	$\frac{(3\xi_1 - 1)(3\chi_3 - 1)(4\chi_2 - 1)(2\xi_4 - 1)}{2} - 2\xi_5[(4\xi_5 - \xi_1 - 1) + 2(4\xi_2 - \xi_1 - 1)]$	$(0, 1; \frac{3}{4}, \frac{1}{4}; 0)$
$\frac{3}{2}\ell_{15}$	$3\xi_1(3\chi_3 - 1)(2\xi_4 - 1)(4\xi_4 - 1) + \xi_1\xi_5$	$(\frac{1}{3}, \frac{2}{3}; \frac{1}{4}, \frac{3}{4}; 0)$
$\frac{3}{2}\ell_{15}$	$3\xi_1(3\chi_3 - 1)(4\chi_2 - 1)(4\xi_4 - 1) - 5\xi_1\xi_5$	$(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}, \frac{1}{2}; 0)$
$\frac{3}{2}\ell_{15}$	$-3\xi_1(3\chi_3 - 1)(4\chi_2 - 1)(2\xi_4 - 1) - 5\xi_1\xi_5$	$(\frac{1}{3}, \frac{2}{3}; \frac{3}{4}, \frac{1}{4}; 0)$
$3\ell_{15}$	$\frac{3\xi_1(3\xi_1 - 1)(2\xi_4 - 1)(4\xi_4 - 1)}{2}$	$(\frac{2}{3}, \frac{1}{3}; \frac{1}{4}, \frac{3}{4}; 0)$
$3\ell_{15}$	$\frac{3\xi_1(3\xi_1 - 1)(4\chi_2 - 1)(4\xi_4 - 1)}{2}$	$(\frac{2}{3}, \frac{1}{3}; \frac{1}{2}, \frac{1}{2}; 0)$
$3\ell_{15}$	$-\frac{3\xi_1(3\xi_1 - 1)(4\chi_2 - 1)(2\xi_4 - 1)}{2}$	$(\frac{2}{3}, \frac{1}{3}; \frac{3}{4}, \frac{1}{4}; 0)$
$\frac{4}{3}\ell_{15}$	$4\xi_5(4\xi_4 - \xi_1 - 1)$	$(0, \frac{3}{4}; \frac{1}{4}, \frac{1}{2}; \frac{1}{4})$
$\frac{4}{3}\ell_{15}$	$4\xi_5(4\xi_2 - \xi_1 - 1)$	$(0, \frac{3}{4}; \frac{1}{2}, \frac{1}{4}; \frac{1}{4})$
$2\ell_{15}$	$2\xi_5(4\xi_5 - \xi_1 - 1)$	$(0, \frac{1}{2}; \frac{1}{4}, \frac{1}{4}; \frac{1}{2})$
$3\ell_{15}$	$9\xi_1\xi_5$	$(\frac{1}{3}, \frac{1}{3}; \frac{1}{3}, \frac{1}{3}; \frac{1}{3})$



(a) (b)

Fig. 4. Interpolation points for curl-conforming bases of order $p = 1$ on pyramidal elements. (a) Nodes in basis subset $\boldsymbol{\Omega}^{34}_{ik,j\ell,m}(\boldsymbol{r})$. (b) Nodes in basis subset $\boldsymbol{\Omega}^{35}_{ik,j\ell,m}(\boldsymbol{r})$. The interior interpolation point at $\boldsymbol{\xi}_{ik,j\ell,m} = (\frac{1}{3}, \frac{1}{3}; \frac{1}{3}, \frac{1}{3}; \frac{1}{3})$ is omitted for clarity.

Interpolation points for the bases of the form $\boldsymbol{\Omega}^{34}_{ik,j\ell,m}(\boldsymbol{r})$ and $\boldsymbol{\Omega}^{35}_{ik,j\ell,m}(\boldsymbol{r})$ are shown in Fig. 4.

Note that no vertices of the pyramid are interpolated and only a single basis function interpolates components along a given edge. For example, with reference to Fig. 4(a), by considering the node of edge 34 at $\boldsymbol{\xi}_{ik,j\ell,m} = (1/3, 0; 1/3, 0; 2/3)$ one immediately recognizes that there is only one vector function having a nonvanishing (unit) tangent component along edge 34 at this node; this function is one of the functions of the basis subset $\boldsymbol{\Omega}^{34}_{ik,j\ell,m}(\boldsymbol{r})$.

Tangential components at a given interpolation point on a triangular face are interpolated by three bases containing zeroth-order basis factors that interpolate the edges of the triangular face; whereas four basis functions produce tangential components at each interpolation node on the quadrilateral face. But on a face, only two of these tangential components can be independent. Hence, only two basis functions at each interpolation point can be retained; if these contain zeroth-order basis factors associated with edges of the face that share a common vertex, they will be independent. Similarly, on the interior, only three of the eight bases which produce components at each interior node should be retained to provide interpolation of the three independent components. These three should contain zeroth-order basis factors associated with edges having independent edge vectors.

The dependencies for faces and interior nodes arise from linear combinations of the bases which contain one of the following identities as a factor:

$$\begin{aligned} \xi_i\boldsymbol{\Omega}_{i,5}(\boldsymbol{r}) + \xi_{i+2}\boldsymbol{\Omega}_{i+2,5}(\boldsymbol{r}) &= \boldsymbol{0}, \qquad i = 1,2 \\ \xi_1\boldsymbol{\Omega}_{i,1}(\boldsymbol{r}) + \xi_3\boldsymbol{\Omega}_{i,3}(\boldsymbol{r}) + \xi_5\boldsymbol{\Omega}_{i,5}(\boldsymbol{r}) &= \boldsymbol{0}, \qquad i = 2,4 \\ \xi_2\boldsymbol{\Omega}_{i,2}(\boldsymbol{r}) + \xi_4\boldsymbol{\Omega}_{i,4}(\boldsymbol{r}) + \xi_5\boldsymbol{\Omega}_{i,5}(\boldsymbol{r}) &= \boldsymbol{0}, \qquad i = 1,3 \end{aligned} \tag{10}$$

where $\boldsymbol{\Omega}_{i,j}(\boldsymbol{r}) = -\boldsymbol{\Omega}_{j,i}(\boldsymbol{r})$.

D. Completeness to Order p in the Curl

Completeness in the curl to order p is readily proved by following the same procedure reported in [3, Appendix], which requires the use of inhomogeneous multiplying polynomials. Curl-completeness is a consequence of the fact that the zeroth-order curl-conforming functions are able to model the following linear vectors:

$$\begin{aligned} \boldsymbol{\Upsilon}_1(\boldsymbol{r}) &= \xi_1\nabla\xi_5 - \xi_5\nabla\xi_1 = \boldsymbol{\Omega}_{23}(\boldsymbol{r}) + \boldsymbol{\Omega}_{34}(\boldsymbol{r}) \\ \boldsymbol{\Upsilon}_2(\boldsymbol{r}) &= \xi_2\nabla\xi_1 - \xi_1\nabla\xi_2 = -\boldsymbol{\Omega}_{35}(\boldsymbol{r}) - \boldsymbol{\Omega}_{45}(\boldsymbol{r}) \\ \boldsymbol{\Upsilon}_5(\boldsymbol{r}) &= \xi_5\nabla\xi_2 - \xi_2\nabla\xi_5 = -\boldsymbol{\Omega}_{41}(\boldsymbol{r}) - \boldsymbol{\Omega}_{34}(\boldsymbol{r}). \end{aligned} \tag{11}$$

E. Number of Degrees of Freedom

The number of edge and face degrees-of-freedom (DOF) for curl-conforming bases of order p on a pyramid may be determined as follows:

- one component $\times(p+1)$ DOF's $\times$ eight edges $= 8(p+1)$ edge DOF;
- two components $\times(p(p+1))/2$ DOF's $\times$ four triangular faces $= 4p(p+1)$ triangular face DOF;
- two components $\times p(p+1)$ DOF's $\times$ one quadrilateral face $= 2p(p+1)$ quadrilateral face DOF

for a total of $2(p+1)(3p+4)$ DOF on the pyramid boundary.

Interior DOF's are necessary for $p > 0$. Unlike other 3-D elements [2], [3] we have not found a unique way to locate the interior nodes via a Lagrangian scheme so that the position of the interior nodes could be changed by changing the form of the non-Lagrangian interpolating polynomials. The number of interior DOF could also depend on the order of the interpolating polynomials one chooses for interior nodes although, in general, there is no advantage in increasing only the number of interior DOF because the complexity of the code, its memory requirements and computation times increase for increasing number of interior DOF. The interpolating polynomials reported in Tables II and III yield the minimum number of DOF while allowing satisfaction of the continuity condition with adjacent elements of different shape and equal order; that is, the number of DOF's related to the polynomials of Tables II and III cannot be reduced without loosing pth-order completeness. In the case of Tables II and III, the total number of DOF's for the curl-conforming bases on a pyramid is 30 for $p = 1$ and 63 for $p = 2$.

IV. Divergence-Conforming Bases on Pyramids

A. Zeroth-Order Bases

Divergence-conforming bases of zeroth order on a pyramid are defined as

$$\begin{aligned} \boldsymbol{\Lambda}_1(\boldsymbol{r}) &= \frac{1}{2\mathcal{J}}\left(\xi_3\boldsymbol{\ell}_{45} + \xi_5\boldsymbol{\ell}_{23} + \frac{\xi_3}{1-\xi_5}\boldsymbol{\ell}_{45} + \frac{\xi_2\xi_5}{1-\xi_5}\boldsymbol{\ell}_{15}\right) \\ \boldsymbol{\Lambda}_2(\boldsymbol{r}) &= \frac{1}{2\mathcal{J}}\left(\xi_4\boldsymbol{\ell}_{15} + \xi_5\boldsymbol{\ell}_{34} + \frac{\xi_4}{1-\xi_5}\boldsymbol{\ell}_{15} + \frac{\xi_3\xi_5}{1-\xi_5}\boldsymbol{\ell}_{25}\right) \\ \boldsymbol{\Lambda}_3(\boldsymbol{r}) &= \frac{1}{2\mathcal{J}}\left(\xi_1\boldsymbol{\ell}_{25} + \xi_5\boldsymbol{\ell}_{41} + \frac{\xi_1}{1-\xi_5}\boldsymbol{\ell}_{25} + \frac{\xi_4\xi_5}{1-\xi_5}\boldsymbol{\ell}_{35}\right) \\ \boldsymbol{\Lambda}_4(\boldsymbol{r}) &= \frac{1}{2\mathcal{J}}\left(\xi_2\boldsymbol{\ell}_{35} + \xi_5\boldsymbol{\ell}_{12} + \frac{\xi_2}{1-\xi_5}\boldsymbol{\ell}_{35} + \frac{\xi_1\xi_5}{1-\xi_5}\boldsymbol{\ell}_{45}\right) \\ \boldsymbol{\Lambda}_5(\boldsymbol{r}) &= \frac{1}{\mathcal{J}}(-\xi_1\boldsymbol{\ell}_{23} + \xi_2\boldsymbol{\ell}_{35} - \xi_3\boldsymbol{\ell}_{12}). \end{aligned} \tag{12}$$

Notice how the expressions of the first four $\boldsymbol{\Lambda}_i(\boldsymbol{r})$ ($i = 1$, 2, 3, 4) agree with the *rotation of subscripts* rule discussed following (4). By rewriting the bases as

$$\begin{aligned} \boldsymbol{\Lambda}_i(\boldsymbol{r}) = \frac{1}{2\mathcal{J}}\Bigg[&\xi_5\boldsymbol{\ell}_{i,i+1} + \frac{\xi_{i+1}\xi_5}{1-\xi_5}\boldsymbol{\ell}_{i5} \\ &- \left(2 - \xi_i - \frac{\xi_i}{1-\xi_5}\right)\boldsymbol{\ell}_{i+1,5}\Bigg], \\ &\text{for } i = 1,2,3,4 \end{aligned} \tag{13}$$

$$\boldsymbol{\Lambda}_5(\boldsymbol{r}) = \frac{1}{\mathcal{J}}[\xi_1\boldsymbol{\ell}_{25} + \xi_2\boldsymbol{\ell}_{35} - (1-\xi_5)\boldsymbol{\ell}_{12}] \tag{14}$$

it is trivial to prove that the bases have a constant normal (CN) component on each element face which matches with that of the divergence-conforming bases of adjacent elements having brick, tetrahedral, or prismatic shape. In (13) index arithmetic is computed modulo 4 and $\boldsymbol{\ell}_{\alpha\beta} = -\boldsymbol{\ell}_{\beta\alpha}$. The basis function $\boldsymbol{\Lambda}_i(\boldsymbol{r})$ interpolates the vector component normal to the centroid of face i and is readily normalized by ensuring a unit component along $\boldsymbol{h}_i$ at the corresponding interpolation point. The normalized form of the zeroth order bases (12) is $\mathcal{J}\boldsymbol{\Lambda}_i(\boldsymbol{r})/h_i$, where $h_i = 1/|\nabla\xi_i|$ is the magnitude of the height vector $\boldsymbol{h}_i$ at the centroid of face i.

Furthermore, it is interesting to observe that the zeroth order curl-conforming function $\boldsymbol{\Omega}_{\gamma\beta}(\boldsymbol{r})$ cannot be obtained from the vector product $\boldsymbol{\Lambda}_\gamma(\boldsymbol{r}) \times \boldsymbol{\Lambda}_\beta(\boldsymbol{r})$ as occurs for brick, tetrahedral, and prismatic elements whose corners are the intersections of only three element faces. The *geometrical* reason for the exception to the rule "$\boldsymbol{\Omega}_{\gamma\beta}(\boldsymbol{r}) \propto \boldsymbol{\Lambda}_\gamma(\boldsymbol{r}) \times \boldsymbol{\Lambda}_\beta(\boldsymbol{r})$" is due to the presence of the tip of the pyramidal element, which is the point in common to four element faces.

B. Completeness of Zeroth-Order Bases

The bases (12) are complete only to zeroth order. Their completeness follows from the identities

$$\boldsymbol{\Lambda}_3(\boldsymbol{r}) - \boldsymbol{\Lambda}_1(\boldsymbol{r}) = \frac{\boldsymbol{\ell}_{25}}{\mathcal{J}}\left(= \frac{\boldsymbol{\ell}^1}{\mathcal{J}}\right)$$
$$\boldsymbol{\Lambda}_4(\boldsymbol{r}) - \boldsymbol{\Lambda}_2(\boldsymbol{r}) = \frac{\boldsymbol{\ell}_{35}}{\mathcal{J}}\left(= \frac{\boldsymbol{\ell}^2}{\mathcal{J}}\right)$$
$$\boldsymbol{\Lambda}_3(\boldsymbol{r}) + \boldsymbol{\Lambda}_4(\boldsymbol{r}) - \boldsymbol{\Lambda}_5(\boldsymbol{r}) = \frac{\boldsymbol{\ell}_{12}}{\mathcal{J}}\left(= \frac{\boldsymbol{\ell}^5}{\mathcal{J}}\right). \tag{15}$$

On curvilinear pyramids, completeness is with respect to these vectors as weighting factors. Completeness of the divergence to zeroth order with respect to $1/\mathcal{J}$ as a weighting factor (curvilinear elements) follows from

$$\nabla \cdot \boldsymbol{\Lambda}_i(\boldsymbol{r}) = \frac{3}{2\mathcal{J}}, \qquad i = 1, 2, 3, 4$$
$$\nabla \cdot \boldsymbol{\Lambda}_5(\boldsymbol{r}) = \frac{3}{\mathcal{J}}. \tag{16}$$

C. Order p Bases

Divergence-conforming bases complete to pth order and which interpolate a vector on a pyramid can be succinctly written as

$$\boldsymbol{\Lambda}^\gamma_{ik,j\ell,m}(\boldsymbol{r}) = N^\gamma_{ik,j\ell,m}\alpha^\gamma_{ik,j\ell,m}(\boldsymbol{\xi})\boldsymbol{\Lambda}_\gamma(\boldsymbol{r})$$
$$\gamma = 1, 2, 3, 4, 5 \tag{17}$$

where $\alpha^\gamma_{ik,j\ell,m}(\boldsymbol{\xi})$ is an interpolatory polynomial, while $N^\gamma_{ik,j\ell,m}$ is a normalization constant chosen to ensure that the component of $\boldsymbol{\Lambda}^\gamma_{ik,j\ell,m}(\boldsymbol{r})$ along $\boldsymbol{h}_\gamma$ at the interpolation point is unity.

To ensure the continuity of the normal component across adjacent elements of equal order but different shape, the interpolation points on each face of the pyramid match with those of tetrahedrons, bricks, and triangular prisms. That is to say that the polynomial factors associated with the zeroth-order divergence-conforming functions $\boldsymbol{\Lambda}_\gamma(\boldsymbol{r})$ become equal for $\xi_\gamma = 0$ to the polynomial factors of the triangular face of the prism element when $\gamma = 1, 2, 3$, or 4, or equal to the polynomial factors of the quadrilateral face of the prism element for $\gamma = 5$ (see [3, Fig. 3]). As far as interpolation points internal to the element are concerned, once again we notice that for the pyramid, these cannot be obtained by following a *Lagrangian scheme* as possible for tetrahedrons, bricks, and triangular prisms.

Because of the symmetry of the pyramidal element, it is sufficient to report the expression of the interpolatory polynomials and normalization constants only for $\boldsymbol{\Lambda}^1_{ik,j\ell,m}(\boldsymbol{r})$ and $\boldsymbol{\Lambda}^5_{ik,j\ell,m}(\boldsymbol{r})$. The other polynomials and normalization constants are easily obtained by *rotating the subscripts* $(1, 2, 3, 4)$ as discussed following (4). Tables IV and V report the interpolatory multiplying polynomials for first- and second-order bases, respectively. In these tables h_1 (h_5) is the value of $|\boldsymbol{h}_1|$ $(|\boldsymbol{h}_5|)$ at the indicated interpolation point $\boldsymbol{\xi}_{ik,j\ell,m}$; again, the notation has been simplified by omitting the superscript $(ik; j\ell; m)$ to the normalization quantities h_1, h_5.

TABLE IV
INTERPOLATORY POLYNOMIALS FOR
FIRST-ORDER DIVERGENCE-CONFORMING BASES

$N^1_{ik,j\ell,m}$	$\alpha^1_{ik,j\ell,m}(\boldsymbol{\xi})$	interp. point $\boldsymbol{\xi}_{ik,j\ell,m}$	$N^5_{ik,j\ell,m}$	$\alpha^5_{ik,j\ell,m}(\boldsymbol{\xi})$ with $\chi_3 = \xi_3 + \xi_5$, $\chi_4 = \xi_4 + \xi_5$	interp. point $\boldsymbol{\xi}_{ik,j\ell,m}$
$\frac{\mathcal{J}}{h_1}$	$4\xi_2 - \xi_1 - 1$	$(0, \frac{3}{4}; \frac{1}{2}, \frac{1}{4}; \frac{1}{4})$	$\frac{\mathcal{J}}{h_5}$	$(3\chi_3 - 1)(3\chi_4 - 1) - 3\xi_5$	$(\frac{1}{3}, \frac{2}{3}; \frac{1}{3}, \frac{2}{3}; 0)$
$\frac{\mathcal{J}}{h_1}$	$4\xi_4 - \xi_1 - 1$	$(0, \frac{3}{4}; \frac{1}{4}, \frac{1}{2}; \frac{1}{4})$	$\frac{\mathcal{J}}{h_5}$	$(3\xi_2 - 1)(3\chi_3 - 1)$	$(\frac{1}{3}, \frac{2}{3}; \frac{2}{3}, \frac{1}{3}; 0)$
$\frac{\mathcal{J}}{h_1}$	$4\xi_5 - \xi_1 - 1$	$(0, \frac{1}{2}; \frac{1}{4}, \frac{1}{4}; \frac{1}{2})$	$\frac{\mathcal{J}}{h_5}$	$(3\xi_1 - 1)(3\chi_4 - 1)$	$(\frac{2}{3}, \frac{1}{3}; \frac{1}{3}, \frac{2}{3}; 0)$
$\frac{12\mathcal{J}}{7h_1}$	$3\xi_1$	$(\frac{1}{3}, \frac{1}{3}; \frac{1}{3}, \frac{1}{3}; \frac{1}{3})$	$\frac{\mathcal{J}}{h_5}$	$(3\xi_1 - 1)(3\xi_2 - 1)$	$(\frac{2}{3}, \frac{1}{3}; \frac{2}{3}, \frac{1}{3}; 0)$
			$\frac{3\mathcal{J}}{2h_5}$	$3\xi_5$	$(\frac{1}{3}, \frac{1}{3}; \frac{1}{3}, \frac{1}{3}; \frac{1}{3})$

TABLE V
INTERPOLATORY POLYNOMIALS FOR
SECOND-ORDER DIVERGENCE-CONFORMING BASES

$N^1_{ik,j\ell,m}$	$\alpha^1_{ik,j\ell,m}(\boldsymbol{\xi})$	interp. point $\boldsymbol{\xi}_{ik,j\ell,m}$
$\frac{10\mathcal{J}}{7h_1}$	$(5\xi_5 - 1)(5\xi_5 - 2)/2$	$(0, \frac{2}{5}; \frac{1}{5}, \frac{1}{5}; \frac{3}{5})$
$\frac{10\mathcal{J}}{9h_1}$	$(5\xi_2 - 1)(5\xi_2 - 2)/2$	$(0, \frac{4}{5}; \frac{3}{5}, \frac{1}{5}; \frac{1}{5})$
$\frac{10\mathcal{J}}{9h_1}$	$(5\xi_4 - 1)(5\xi_4 - 2)/2$	$(0, \frac{4}{5}; \frac{1}{5}, \frac{3}{5}; \frac{1}{5})$
$\frac{10\mathcal{J}}{9h_1}$	$(5\xi_2 - 1)(5\xi_4 - 1) + 25\xi_1(\xi_5 - \xi_3)/2$	$(0, \frac{4}{5}; \frac{2}{5}, \frac{2}{5}; \frac{1}{5})$
$\frac{5\mathcal{J}}{4h_1}$	$(5\xi_2 - 1)(5\xi_5 - 1) + 5\xi_1(5\xi_4 - 2)$	$(0, \frac{3}{5}; \frac{2}{5}, \frac{1}{5}; \frac{2}{5})$
$\frac{5\mathcal{J}}{4h_1}$	$(5\xi_4 - 1)(5\xi_5 - 1) + 5\xi_1(5\xi_2 - 2)$	$(0, \frac{3}{5}; \frac{1}{5}, \frac{2}{5}; \frac{2}{5})$
$\frac{20\mathcal{J}}{9h_1}$	$5\xi_1(5\xi_1 - 1)/2$	$(\frac{2}{5}, \frac{2}{5}; \frac{2}{5}, \frac{2}{5}; \frac{1}{5})$
$\frac{40\mathcal{J}}{27h_1}$	$5\xi_1(5\xi_3 - 2)$	$(\frac{1}{5}, \frac{3}{5}; \frac{2}{5}, \frac{2}{5}; \frac{1}{5})$
$\frac{15\mathcal{J}}{8h_1}$	$-5\xi_1(5\xi_4 - 2)$	$(\frac{1}{5}, \frac{2}{5}; \frac{2}{5}, \frac{1}{5}; \frac{2}{5})$
$\frac{15\mathcal{J}}{8h_1}$	$-5\xi_1(5\xi_2 - 2)$	$(\frac{1}{5}, \frac{2}{5}; \frac{1}{5}, \frac{2}{5}; \frac{2}{5})$
$N^5_{ik,j\ell,m}$	$\alpha^5_{ik,j\ell,m}(\boldsymbol{\xi})$ with $\chi_3 = \xi_3 + \xi_5$, $\chi_4 = \xi_4 + \xi_5$	interp. point $\boldsymbol{\xi}_{ik,j\ell,m}$
$\frac{\mathcal{J}}{h_5}$	$(2\chi_3 - 1)(2\chi_4 - 1)[(4\chi_3 - 1)(4\chi_4 - 1) - 16\xi_5] + 2\xi_5(4\xi_5 - 1)$	$(\frac{1}{4}, \frac{3}{4}; \frac{1}{4}, \frac{3}{4}; 0)$
$\frac{\mathcal{J}}{h_5}$	$(2\chi_3 - 1)(4\xi_2 - 1)[(4\chi_3 - 1)(4\chi_4 - 1) - 8\xi_5]$	$(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}, \frac{1}{2}; 0)$
$\frac{\mathcal{J}}{h_5}$	$(2\chi_3 - 1)(4\chi_3 - 1)(2\xi_2 - 1)(4\xi_2 - 1)$	$(\frac{1}{4}, \frac{3}{4}; \frac{3}{4}, \frac{1}{4}; 0)$
$\frac{\mathcal{J}}{h_5}$	$(4\xi_1 - 1)(2\chi_4 - 1)[(4\chi_3 - 1)(4\chi_4 - 1) - 8\xi_5]$	$(\frac{1}{2}, \frac{1}{2}; \frac{1}{4}, \frac{3}{4}; 0)$
$\frac{\mathcal{J}}{h_5}$	$(4\xi_1 - 1)(4\xi_2 - 1)[(4\chi_3 - 1)(4\chi_4 - 1) - 4\xi_5]$	$(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}; 0)$
$\frac{\mathcal{J}}{h_5}$	$(4\xi_1 - 1)(4\chi_3 - 1)(2\xi_2 - 1)(4\xi_2 - 1)$	$(\frac{1}{2}, \frac{1}{2}; \frac{3}{4}, \frac{1}{4}; 0)$
$\frac{\mathcal{J}}{h_5}$	$(2\xi_1 - 1)(4\xi_1 - 1)(2\chi_4 - 1)(4\chi_4 - 1)$	$(\frac{3}{4}, \frac{1}{4}; \frac{1}{4}, \frac{3}{4}; 0)$
$\frac{\mathcal{J}}{h_5}$	$(2\xi_1 - 1)(4\xi_1 - 1)(4\xi_2 - 1)(4\chi_4 - 1)$	$(\frac{3}{4}, \frac{1}{4}; \frac{1}{2}, \frac{1}{2}; 0)$
$\frac{\mathcal{J}}{h_5}$	$(2\xi_1 - 1)(4\xi_1 - 1)(2\xi_2 - 1)(4\xi_2 - 1)$	$(\frac{3}{4}, \frac{1}{4}; \frac{3}{4}, \frac{1}{4}; 0)$
$\frac{4\mathcal{J}}{3h_5}$	$16\xi_5(2\chi_3 - 1)(2\chi_4 - 1) - 4\xi_5(4\xi_5 - 1)$	$(\frac{1}{4}, \frac{2}{4}; \frac{1}{4}, \frac{2}{4}; \frac{1}{4})$
$\frac{4\mathcal{J}}{3h_5}$	$8\xi_5(4\xi_2 - 1)(2\chi_3 - 1)$	$(\frac{1}{4}, \frac{2}{4}; \frac{2}{4}, \frac{1}{4}; \frac{1}{4})$
$\frac{4\mathcal{J}}{3h_5}$	$8\xi_5(4\xi_1 - 1)(2\chi_4 - 1)$	$(\frac{2}{4}, \frac{1}{4}; \frac{1}{4}, \frac{2}{4}; \frac{1}{4})$
$\frac{4\mathcal{J}}{3h_5}$	$4\xi_5(4\xi_1 - 1)(4\xi_2 - 1)$	$(\frac{2}{4}, \frac{1}{4}; \frac{2}{4}, \frac{1}{4}; \frac{1}{4})$
$\frac{2\mathcal{J}}{h_5}$	$2\xi_5(4\xi_5 - 1)$	$(\frac{1}{4}, \frac{1}{4}; \frac{1}{4}, \frac{1}{4}; \frac{2}{4})$

Notice that no interpolation points lie at vertices or along edges of the pyramid and that a single basis function interpolates the normal at a point on a given face. At interior points, however, five bases contribute component at each interpolation point. Clearly, only three can be independent; these three should contain zeroth-order basis factors associated with faces having independent gradient vectors. The dependencies for interior nodes arise from linear combinations of the bases, which contain the identities

$$\begin{aligned} \xi_1 \boldsymbol{\Lambda}_1 + \xi_3 \boldsymbol{\Lambda}_3 + \frac{\xi_5}{2}\boldsymbol{\Lambda}_5 &= \mathbf{0} \\ \xi_2 \boldsymbol{\Lambda}_2 + \xi_4 \boldsymbol{\Lambda}_4 + \frac{\xi_5}{2}\boldsymbol{\Lambda}_5 &= \mathbf{0}. \end{aligned} \tag{18}$$

as a factor.

D. Completeness to Order p in the Divergence

Completeness to order p in the divergence is shown using an inhomogeneous multiplying polynomial of order p [2]. In this case, completeness follows from the fact that terms of like order are generated. The divergence of the product of the polynomial and a simple linear combination of the zeroth-order bases is found to be

$$\begin{aligned} \nabla \cdot [\xi_1^\alpha \xi_2^\beta \xi_5^\gamma (\boldsymbol{\Lambda}_3(\boldsymbol{r}) + \boldsymbol{\Lambda}_4(\boldsymbol{r}))] &= \frac{(\alpha+\beta+\gamma+3)\xi_1^\alpha \xi_2^\beta \xi_5^\gamma}{\mathcal{J}}, \\ \nabla \cdot [\xi_2^\alpha \chi_3^\beta \xi_5^\gamma (\boldsymbol{\Lambda}_4(\boldsymbol{r}) + \boldsymbol{\Lambda}_1(\boldsymbol{r}))] &= \frac{(\alpha+\beta+\gamma+3)\xi_2^\alpha \chi_3^\beta \xi_5^\gamma}{\mathcal{J}} \\ \nabla \cdot [\chi_3^\alpha \chi_4^\beta \xi_5^\gamma (\boldsymbol{\Lambda}_1(\boldsymbol{r}) + \boldsymbol{\Lambda}_2(\boldsymbol{r}))] &= \frac{(\alpha+\beta+\gamma+3)\chi_3^\alpha \chi_4^\beta \xi_5^\gamma}{\mathcal{J}} \\ \nabla \cdot [\chi_4^\alpha \xi_1^\beta \xi_5^\gamma (\boldsymbol{\Lambda}_2(\boldsymbol{r}) + \boldsymbol{\Lambda}_3(\boldsymbol{r}))] &= \frac{(\alpha+\beta+\gamma+3)\chi_4^\alpha \xi_1^\beta \xi_5^\gamma}{\mathcal{J}} \\ \nabla \cdot [\xi_1^\alpha \xi_2^\beta \chi_5^\gamma \boldsymbol{\Lambda}_5(\boldsymbol{r})] &= \frac{(\alpha+\beta+\gamma+3)\xi_1^\alpha \xi_2^\beta \chi_5^\gamma}{\mathcal{J}} \end{aligned} \tag{19}$$

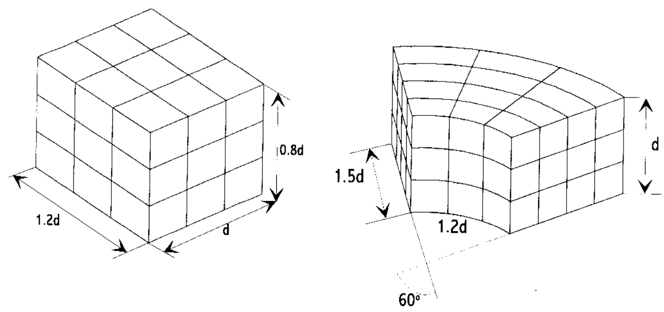
with $\alpha, \beta, \gamma \geq 0$ and $0 \leq \alpha + \beta + \gamma \leq p$, and where $\chi_3 = \xi_3 + \xi_5 = 1 - \xi_1$, $\chi_4 = \xi_4 + \xi_5 = 1 - \xi_2$, and $\chi_5 = 1 - \xi_5$. For curvilinear pyramids for which $\mathcal{J}$ is not a constant, polynomial completeness is with respect to $1/\mathcal{J}$ as a weighting factor. Notice that the first of (19) already suffices to prove completeness in the divergence.

E. Number of Degrees of Freedom (DOF)

The number of *surface* DOF for divergence-conforming bases of order p on a pyramid may be determined as follows.

- One component $\times((p+1)(p+2)/2)$ DOF's $\times$ four triangular faces plus one component $\times(p+1)^2$ DOF's $\times$ one quadrilateral face $= (p+1)(3p+5)$ face DOF.

Interior DOF's are necessary for $p > 0$ and, once again, their number depends on the choice of the interpolating polynomials for interior nodes. By using the interpolating polynomials reported in Tables IV and V, the total number of DOF for the divergence-conforming bases on a pyramid is 19 for $p = 1$ and 46 for $p = 2$.



(a) (b)

Fig. 5. (a) Rectangular cavity discretized with 162 pyramidal elements. (b) Pie-shell cavity discretized with 216 pyramidal elements. The surface of these two cavities is discretized by quadrilaterals, each quadrilateral is the base of one pyramid.

V. Numerical Results

Although 3-D structures should not be discretized by using only pyramidal elements, to illustrate their modeling capabilities we present some results relative to two resonant cavities studied by using only pyramidal elements of zeroth $(p = 0)$ and first $(p = 1)$ order. By showing the modeling capabilities of *structured meshes* of pyramidal elements we implicitly prove the usefulness of pyramids as effective *fillers*, although we do not intend to recommend structured meshes to study general problems; but here we use structured meshes to perform critical test studies.

The results consider the resonant frequencies of the cavities obtained by finding the eigenvalues of the discretized vector Helmholtz equation involving the cavity electric field. As in [2] and [3], a Galerkin form of the finite-element method was used to discretize the Helmholtz equation and curl-conforming bases on pyramids were used to model each cavity; curvilinear pyramids with quadratic distortion were used when necessary. The geometry of the two test cases are discretized by first defining a brick mesh; each brick is then subdivided into six pyramids by joining the eight corners of the brick to its center.

Fig. 5 represents the two geometries at issue. Fig. 5(a) shows a discretization of a rectangular cavity of height $0.8d$ and base with sides of length d and $1.2d$. The mesh of Fig. 5(a) consists of 162 pyramids and the total number of DOF is 360 and 2124 for $p = 0$ and 1, respectively. Since the cavity walls are of perfect electric conducting material, the number of unknowns corresponds to the number of interior DOF, which yield systems of 252 and 1692 unknowns for $p = 0$ and 1, respectively. Fig. 5(b) shows a discretization of the pie-shell cavity already studied in [3, figs. 9, 10], obtained by using 216 curved pyramids, which yield systems of 340 and 2276 unknowns for $p = 0$ and 1, respectively (number of interior DOF).

The error in the computed resonant frequencies versus the number of unknowns is reported in Figs. 6 and 7 for the rectangular and the pie-shell cavity, respectively. The error is averaged over the first eight eigenfrequencies. For sake of comparison, Fig. 7 reports also the results obtained by using zeroth- and first-order prism elements [3, fig. 10]; it can be observed that pyramidal and prism elements of zeroth and first order yield results with similar accuracy for the pie-shell

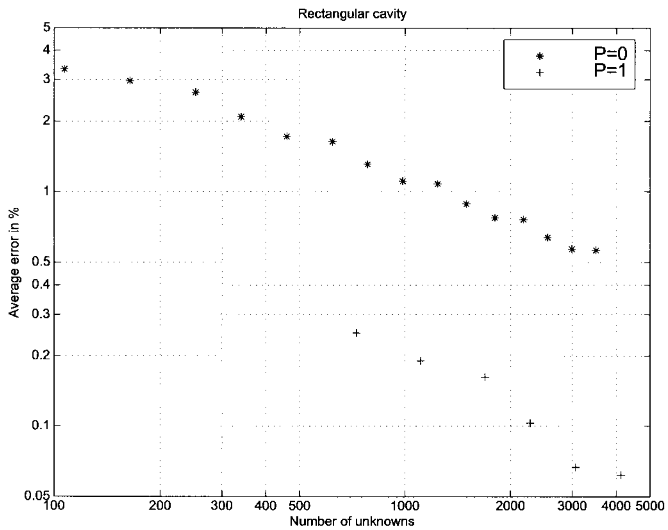


Fig. 6. Average error in computation of first eight resonant frequencies versus number of unknowns for a conducting rectangular cavity.

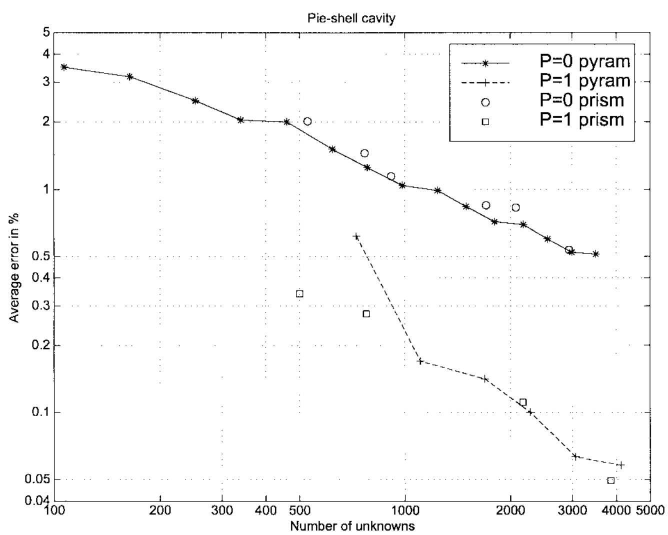


Fig. 7. Average error in computation of first eight resonant frequencies versus number of unknowns for a conducting pie-shell cavity discretized by using pyramids and by using prisms.

cavity of Fig. 5(b). Furthermore, these plots already show the faster convergence of the results for increasing order of the vector bases. Although no spurious nonzero eigenvalues were observed, in the cases of Fig. 5 it is rather difficult to get results with good accuracy using only pyramidal elements for a limited number of unknowns and increasing order of the bases. This is due to the fact that we used structured meshes where the boundary conditions were already constraining the 20% or more of the total number of DOF. As a matter of fact, pyramidal elements should mainly be used as *fillers* for meshes involving elements of different shapes since, usually, it is not convenient to work only with pyramidal elements to discretize a given geometry.

VI. Conclusions

This paper presents a general procedure to obtain higher order interpolatory curl-conforming and divergence-conforming vector basis functions for pyramidal elements. The functions can be consistently used to deal with curvilinear elements and ensure the continuity of the proper vector components across adjacent elements of equal order but different shape. Properties of the vector basis functions are discussed in detail. The reported numerical examples show that higher order functions provide more accurate results than those obtainable with low-order elements.

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Roberto D. Graglia (S'83–M'87–SM'90–F'98), for a photograph and biography, see p. 314 of the March 1997 issue of this Transactions.

Ioan-L. Gheorma, for a biography, see p. 450 of the March 1998 issue of this Transactions.