Phase-Only Shaped Beam Synthesis via Technique of Approximated Beam Addition

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***Abstract*— A new method for phase-only phased-array beampattern synthesis is derived. The method is appropriate for the synthesis of coverage patterns for satellite communications, where a minimax goal of maximizing the worst-case beamforming gain to a set of service locations is desired. The new approach, called the technique of approximated beam addition, is found to be computationally attractive relative to conventional methods, yet offers optimal performance. Included are a theoretical consideration of optimality and simulation examples comparing the computational complexity and convergence quality to that of proven techniques.**

***Index Terms*— Gradient methods, phased-array antennas, phase-only beamforming, shaped beams.**

I. Introduction

THERE are compelling reasons for applying phased-array technology to commercial satellite communications. First, the array is easily reconfigured to meet the changing needs of customers. Second, satellite energy resources are efficiently used in providing coverage to arbitrarily located ground stations. The ability to easily adjust the radiation pattern can be utilized not only to direct energy to customer stations more efficiently, but also to suppress energy in other directions so as to meet requirements on out-of-region transmission levels. The latter allows for the reuse of frequency bands for spatially displaced customers. Synthesis techniques aimed at solving for the amplitude and phase distribution across the array aperture to realize the beamforming goals have been the subject of many papers [1]–[9].

A real implementation demands efficient use of onboard energy, requiring that the component amplifiers be driven with multiple signals toward device saturation. In line with this, phase-only beamforming applied to transmit is essential, as amplitude weighting represents a real reduction in energy and may require the addition of hardware for the removal of the generated thermal energy.

There is considerably less published material in the area of phase-only synthesis of shaped beams. Straightforward application of standard optimization strategies tend to dominate, as seen in [10]–[12]. An interesting technique based upon a statistical approach for the generation of low sidelobe (SL) patterns appears in [13]. In general, phase-only methods perform surprisingly well in achieving arbitrary beamforming goals relative to those allowing for full-phase and amplitude variation [1], [14] despite the loss of half the number of degrees of freedom.

Phase-only techniques have also been investigated in applications where nulling in the SL region is desired. Utilizing first-order approximations for the beamforming weights $\exp(j\theta) \approx 1 + j\theta$ and the pattern gain, a computationally simple method was derived in [15] to produce nulls in the directions of SL interferers. A closed-form solution for a phase perturbation given in [16] can be used to shift a null in the quiescent pattern to the direction of an interferer. The method assumes the use of arrays with symmetric amplitude tapers. Haupt [17] considered the application of gradient-based optimization methods for adjusting the beamforming phases to induce a common null into existing beams used in conventional monopulse processing. More recently, application of the genetic algorithm to phase-only SL nulling was considered in [18]. This method is well-suited for the common scenario where an analog beamforming network provides only discrete phase steps. Critical cases include those where a considerably high degree of nulling is desired or where only a few phase shifter bits are available so that quantization error is a concern.

In this paper, the synthesis of optimal phase-only beampatterns for arbitrary coverage is discussed. As this coverage pattern is intended to serve multiple customer sites simultaneously where each user is as important as the other, the goal in beampattern synthesis is to maximize the gain at the worst-case station. We allow for SL constraints in the form of minimum rejection levels for the application of frequency reuse or to accommodate a general specification on spatial emissions. We effectively assume that the degree of nulling in the SL region is not so severe as to preclude techniques where the phase settings are computed as the quantized version of optimized continuous values. This is, as stated, a constrained minimax formulation. Although the component gain functions are smooth nonlinear functions, the minimax cost function is poorly behaved. Nevertheless, robust implementations of the standard minimax optimization have been derived [19]–[23]. However, simulations have shown that these approaches converge slowly in applications of interest where the arrays possess a rather *large* number of elements.

An alternative phase-only minimax formulation that proves to be both computationally efficient and offers optimal results is developed and described herein. The technique, termed the approach of approximated beam addition, is based upon iteratively revising the phase-only beamforming weights in order to approximate a linear addition of spot-beam components. As a result, the optimization procedure relies upon

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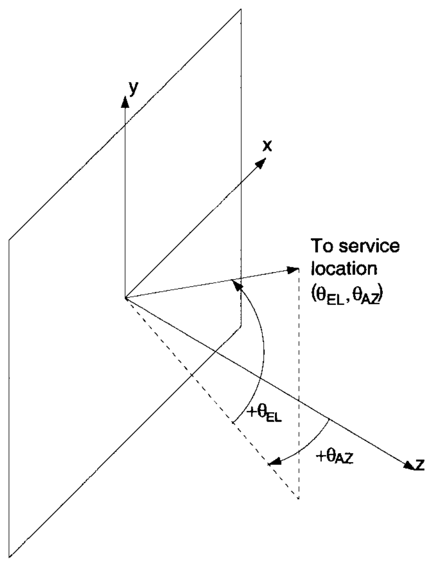

Fig. 1. Coordinate system.

approximations based in a workspace of only M complex independent variables, where M is the combined number of worst-case ground stations and fraction of SL stations meeting their corresponding constraint with equality. This contrasts with the N independent variables (phases) of the conventional approach. Although there may be a large number of ground stations, only a limited percentage are nearly worst case. Thus, the optimization is performed over $2M$ independent real variables where we generally have $2M \ll N$.

Following a description of the signal model with particular regard to the motivational example of coverage from a geosatellite to the U.S. in Section II, a conventional quasi-Newton approach to phase-only minimax optimization is reviewed in Section III. The proposed synthesis technique of approximated beam addition is described in Section IV and evaluated, from a theoretical standpoint, in Section V. Simulations in Section VI address both the computational savings and optimality of the proposed technique relative to that of the conventional method. Finally, concluding remarks are offered in Section VII. With regard to notation, bold lower case letters represent column vectors, bold upper case letters are matrices, and the superscript "H" denotes the conjugate transpose operation.

II. Signal Model and Motivational Example

Appropriate to the intended application, the use of narrow-band signals at wavelength λ is assumed. The reception points on the earth are located in the far field of the satellite array. The array is composed of N identical antenna elements placed on a planar array existing in a local x–y plane as shown in Fig. 1. Note, however, that the identical nature of elements and their arrangement on a planar structure is not necessary for the validity of the results presented herein. As also shown in the figure, $\theta_{\rm EL}$ and $\theta_{\rm AZ}$ are defined as the elevation and azimuth angles, respectively, of a direction of interest relative to the array.

With these definitions, the beamforming gain can now be stated. Let $s(t)$ denote the transmitted signal from a single element and $\alpha s(t-\tau)$ represent that which is received on the ground, where α models the transmission path and τ is a bulk time delay. Through judicious phasing at each of the N elements, the radiated energy can be tailored to the spatial distribution of the customers. Assuming negligible mutual coupling and employing the narrow-band and far-field assumptions, the signal received at a distant location $(\theta_{\rm EL}, \theta_{\rm AZ})$ is increased over that of a single element by

$$g(T_x, T_y) = \sum_{n=1}^{N} \exp\left(-j\,\frac{2\pi}{\lambda}\,[T_x x_n + T_y y_n]\right) \cdot g_e(T_x, T_y)\,\exp(j\theta_n). \tag{1}$$

Here, θ_n is the beamforming phase shift applied to the signal at the nth sensor located at (x_n, y_n). $g_e(T_x, T_y)$ is the element factor for the service direction in T-space coordinates (T_x, T_y), where $T_x = -\cos\theta_{\rm EL}\,\sin\theta_{\rm AZ}$ and $T_y = \sin\theta_{\rm EL}$.

Given that the antenna gain at a specified number N_c of ground stations is of interest, define

$$\mathbf{G}_k \doteq 10\,\log_{10}\left|\mathbf{w}_k^H e^{j\boldsymbol{\theta}}\right|^2, \qquad k = 1, 2, \cdots, N_c \tag{2}$$

where $\mathbf{w}_k$ is the kth column of the $N \times N_c$ matrix of steering vectors $\mathbf{W}$, i.e.,

$$\begin{aligned}[\mathbf{W}]_{nk} &= \mathbf{w}_k(n) \\ &= g_e(T_x(k), T_y(k)) \\ &\quad \cdot \exp\left(-j\,\frac{2\pi}{\lambda}\,[T_x(k)x_n + T_y(k)y_n]\right).\end{aligned} \tag{3}$$

The symbolic notation $e^{j\boldsymbol{\theta}}$ is intended to represent the $N \times 1$ unit-amplitude beamforming vector where the n'th row is $\exp(j\theta_n)$.

As an example, consider a coverage region spanning the United States from a geostationary orbit, as shown in Fig. 2. Here, $N_c = 207$ service locations corresponding to major cities in all the 50 states are shown. As the earth comprises only a small percentage of total space, a six-times spatial oversampling (3-λ element spacing) was selected for use. The on-board phased array is composed of $N = 256$ sensors arranged in a square grid and intended to offer communication service over $K_{\mathbf{u}}$ band. The desired gain profile $\mathbf{G}_k^{\rm DESIRED}$ $k = 1, 2, \cdots, N_c$ is not uniform; rather, the effects of rainfall are factored in to provide uniform coverage under worst-case conditions.

As each customer is equally important, the beamforming strategy is such that the smallest beamforming excess gain be maximized. Mathematically, one wishes to solve

$$\begin{aligned}&\max_{\theta}\,\min_{k} \quad \mathbf{G}_k - \mathbf{G}_k^{\rm DESIRED} && k = 1, 2, \cdots, N_c \\ &\text{subject to:} \quad \mathbf{G}_\ell^{\rm SL} \le \mathbf{G}_\ell^{\rm SL\,MAX} && \ell = 1, 2, \cdots, N_{\rm SL}\end{aligned} \tag{4}$$

where SL constraints have been added for the sake of generality. Here we define an $N \times N_{\rm SL}$ matrix of steering vectors $\mathbf{W}^{\rm SL}$ to the $N_{\rm SL}$ constraint locations. A decibel standard is assumed for proper translation of results. This applies in the intended application where there is a constraint on the signal

Fig. 2. Satellite view of coverage area: $-8/4°$ in azimuth; $\pm 6°$ in elevation.

level at the input to the power amplifiers. Thus, the amplitudes of the multiple customer signals are individually scaled prior to combining at the amplifier input.

III. Conventional Quasi-Newton Synthesis Technique

The most general solution approach to (4) involves the application of readily available tools derived specifically for the minimax structure, where the independent variables are the sensor phases. In this section, the generalities of an accepted method for solving (4) are discussed. Appropriate optimality conditions are derived so that by comparison, the optimality of the technique of Section IV may be addressed.

Although the expression for the excess gain at a ground station is differentiable, the minimax cost function is a composite function that is not differentiable. A proven method of solution [19], [21], [23] is to convert (4) to the nonlinear constrained optimization structure

$$\min_{\{\gamma,\,\theta\}} \gamma$$
$$\text{subject to:}\quad \begin{aligned} &\mathbf{G}_k^{\text{DESIRED}} - \mathbf{G}_k \le \gamma && k = 1, 2, \cdots, N_c \\ &\mathbf{G}_\ell^{\text{SL}} \le \mathbf{G}_\ell^{\text{SL MAX}} && \ell = 1, 2, \cdots, N_{\text{SL}} \end{aligned} \qquad (5)$$

where γ is an introduced dummy variable. Successful gradient-based solution methods employ active-set strategies [20], [21] for iteratively solving locally quadratic functions based upon updated estimates of constraints that are active. The active constraints correspond to those ground stations which, at that stage, share the worst-case descriptor status. This technique forms the basis of the minimax optimization within MATLAB's Optimization Toolbox [23].

A set of phases $\boldsymbol{\theta} = \boldsymbol{\theta}^o$ is known to locally solve a constrained problem by the satisfaction of the Kuhn–Tucker conditions [19], [24], essentially first (for necessity) and second-order (sufficiency) derivative conditions of the Lagrangian. Bandler [25] derived the conditions for the minimax formulation where no SL constraints were defined. Providing for these additional constraints, a local solution $\boldsymbol{\theta}^o$, must satisfy

$$\mathbf{0} = \sum_{k=1}^{M_c} \lambda_k \nabla_\theta \mathbf{G}_k(\boldsymbol{\theta}^o) + \sum_{\ell=1}^{M_{\text{SL}}} \lambda_\ell^{\text{SL}} \nabla_\theta \mathbf{G}_\ell^{\text{SL}}(\boldsymbol{\theta}^o) \qquad (6)$$

$$1 = \sum_{k=1}^{M_c} \lambda_k \qquad (7)$$

$$\lambda_k,\, \lambda_\ell^{\text{SL}} = \begin{cases} \ge 0 & \text{for } k \le M_c,\ \ell \le M_{\text{SL}} \\ = 0 & \text{for } k > M_c,\ \ell > M_{\text{SL}}. \end{cases} \qquad (8)$$

M_c is the number of active converted constraints while M_{SL} is the number of active SL inequality constraints. λ_i and λ_j^{SL} are the Lagrange multipliers associated with the converted and SL constraints, respectively. The above conditions are written for the case where the constraint indices are reordered so that the first M_c ground station gains and the first M_{SL} SL inequality constraints are active. Provided the above conditions are met, $\boldsymbol{\theta}^o$ *is* a local solution if appropriate second-order conditions are satisfied [19], [24].

As needed for comparative purposes later and for use in optimization algorithms, the gradient and Hessian of the kth ground-station excess gain are, respectively

$$\begin{aligned} &\nabla_\theta \{\mathbf{G}_k - \mathbf{G}_k^{\text{DESIRED}}\} \\ &\quad = \nabla_\theta\, 10 \log_{10} \left|\mathbf{w}_k^H e^{j\boldsymbol{\theta}}\right|^2 \\ &\quad = 20(\log_{10} e)\, \text{Im}\Big\{ \big(e^{-j\boldsymbol{\Theta}} \mathbf{w}_k\big) / \big(\mathbf{w}_k^H e^{j\boldsymbol{\theta}}\big)^* \Big\} \end{aligned} \qquad (9)$$

$$e^{j\boldsymbol{\Theta}} \doteq \text{diag}\{e^{j\theta_1},\, e^{j\theta_2},\, \cdots,\, e^{j\theta_N}\} \qquad (10)$$

and

$$\begin{aligned} &\nabla_\theta \nabla_\theta^T \{\mathbf{G}_k - \mathbf{G}_k^{\text{DESIRED}}\} \\ &\quad = 20(\log_{10} e) \nabla_\theta\, \text{Im}^T \Big\{ \big(e^{-j\boldsymbol{\Theta}} \mathbf{w}_k\big) / \big(\mathbf{w}_k^H e^{j\boldsymbol{\theta}}\big)^* \Big\} \\ &\quad = 20(\log_{10} e)\, \text{Re}\Big\{ \big(e^{-j\boldsymbol{\Theta}} \mathbf{w}_k\big)\big(e^{-j\boldsymbol{\Theta}} \mathbf{w}_k\big)^T \Big/ \Big[\big(\mathbf{w}_k^H e^{j\boldsymbol{\theta}}\big)^*\Big]^2 \Big\} \\ &\qquad - 20(\log_{10} e)\, \text{Re}\Big\{ \text{diag}\{e^{-j\boldsymbol{\Theta}} \mathbf{w}_k\} / \big(\mathbf{w}_k^H e^{j\boldsymbol{\theta}}\big)^* \Big\}. \end{aligned} \qquad (11)$$

Note that the gradient and Hessian expressions for a SL constraint have similar forms.

This "conventional" method is applied to a limited set of problems in the simulation section. There are, however, a few general comments that are of note. With regard to the general problem containing nonlinear constraints, computational complexity varies on the order of N_c^3, as found in [19] and [21]. The variation with respect to the N independent variables is not as well defined. Simulations have shown that the general minimax problem converges slowly, relative to, for example, a more well-behaved least-squares cost function. This suggests that a pipelined optimization structure is most effective, where intermediate optimization techniques refine the phases that are used as initial conditions for the final computationally expensive minimax optimization [26].

IV. Technique of Approximated Beam Addition

To motivate the technique of approximated beam addition, suppose that the phase and amplitude of the beamforming weights could be freely varied. Let there be M combined worst-case stations and active SL constraints ($M = M_c + M_{\rm SL}$) with associated steering vectors loaded as columns into the $N \times M$ matrix $\mathbf{W}_M$. The complex voltage gains $\mathbf{g}_k$ at these stations, with initially unit-magnitude beamforming weights, are then

$$\mathbf{g}_k = \left[\mathbf{W}_M^H e^{j\boldsymbol{\theta}}\right]_k, \qquad k = 1, 2, \cdots, M. \tag{12}$$

Now suppose that the beamforming weights are perturbed by an increment $\mathbf{z} \in \mathcal{C}^N$, in general changing the unit-amplitude nature of the weights. Furthermore, consider the increment decomposed to the form

$$\mathbf{z} = \mathbf{W}_M \mathbf{z}_M + \mathbf{W}_\perp \mathbf{z}_\perp \tag{13}$$

where the columns of $\mathbf{W}_\perp$ are any basis for the subspace orthogonal to that spanned by the columns of $\mathbf{W}_M$. The modified station complex gains $\mathbf{g}'_k$ $k = 1, 2, \cdots, M$ become

$$\begin{aligned}\mathbf{g}'_k &= \left[\mathbf{W}_M^H\left(e^{j\boldsymbol{\theta}} + \mathbf{W}_M\mathbf{z}_M + \mathbf{W}_\perp\mathbf{z}_\perp\right)\right]_k \\ &= \mathbf{g}_k + \left[\left(\mathbf{W}_M^H\mathbf{W}_M\right)\mathbf{z}_M\right]_k, \qquad k = 1, 2, \cdots, M.\end{aligned} \tag{14}$$

This implies that to *most efficiently* perturb the gain at a single station k, one should add a component of the steering vector associated with ground station k to the weight vector. When M stations are considered, the weight vector increment should lie in the subspace spanned by the associated M steering vectors. As the station gains and entries of the steering vectors are complex numbers, the optimization involves $2M$ real-valued parameters. In general, the unit-amplitude requirement of the beamforming weights will not allow an increment completely limited to the M-dimensional subspace. As a result, the goal is to find a means of adjusting the beamforming weights to effect a precise change along the desired subspace $\mathcal{R}\{\mathbf{W}_M\}$ with minimal impact along the orthogonal subspace $\mathcal{R}\{\mathbf{W}_\perp\}$. Mathematically stated, the form of the iterated estimate becomes

$$e^{j\boldsymbol{\theta}'} = f\left(e^{j\boldsymbol{\theta}} + \mathbf{D}\boldsymbol{\delta}\right) \tag{15}$$

where the columns of the $N \times M_{\rm eff}$ orthonormal matrix $\mathbf{D}$ form a basis for the *effective* subspace as spanned by the columns of $\mathbf{W}_M$, $\boldsymbol{\delta} = \boldsymbol{\delta}_R + j\boldsymbol{\delta}_I$ are the $2M_{\rm eff}$ independent real variables to determine, and $f(\cdot)$ is a yet unspecified function to approximate the addition of beamforming components.

Given that $\mathbf{USV}^H$ is the singular value decomposition of $\mathbf{W}_M$, $\mathbf{D}$ may be selected as the first $M_{\rm eff}$ columns of $\mathbf{U}$, where

$$\begin{aligned}[\mathbf{S}]_{11} &\geq \cdots \geq [\mathbf{S}]_{M_{\rm eff}M_{\rm eff}} \geq \sigma_T * [\mathbf{S}]_{11} > [\mathbf{S}]_{(M_{\rm eff}+1)(M_{\rm eff}+1)} \\ &\geq \cdots \geq [\mathbf{S}]_{MM}\end{aligned} \tag{16}$$

for some appropriate threshold σ_T, e.g., 1% or $\sigma_T = 0.01$. The orthonormal matrix $\mathbf{D}$ is employed as opposed to the $N \times M$ matrix $\mathbf{W}_M$ for two reasons. First, the dimensionality of the optimization procedure is reduced ($M_{\rm eff} \leq M$), thus reducing the computational complexity. Second, a real/imaginary pair $\boldsymbol{\delta}(\ell) = \boldsymbol{\delta}_R(\ell) + j\boldsymbol{\delta}_I(\ell)$ is decoupled from all other pairs via the orthonormality of $\mathbf{D}$.

Perhaps the simplest choice for $f\{\cdot\}$ is the normalization operation computed as

$$e^{j\theta'_n} = \frac{e^{j\theta_n} + [\mathbf{D}\boldsymbol{\delta}]_n}{\left|e^{j\theta_n} + [\mathbf{D}\boldsymbol{\delta}]_n\right|}. \tag{17}$$

Although other similar techniques were examined, it was determined that this simple normalization operation provided a highly efficient approach yielding satisfactory performance.

An efficient optimization implementation requires an expression for the gradient of the excess gains. Define $\mathbf{d}$ as the real vector of independent variables having the form

$$\mathbf{d} = \begin{bmatrix}\boldsymbol{\delta}_R \\ .. \\ \boldsymbol{\delta}_I\end{bmatrix}. \tag{18}$$

The component gradient with respect to the real-valued $\mathbf{d}$ requires the evaluation of

$$\begin{aligned}&\frac{\partial}{\partial \boldsymbol{\delta}_R(m)}\left\{\mathbf{G}_k - \mathbf{G}_k^{\rm DESIRED}\right\} \\ &\quad = 20(\log_{10} e)\,\mathrm{Re}\left\{\frac{1}{\mathbf{w}_k^H e^{j\boldsymbol{\theta}'}}\,\mathbf{w}_k^H\,\frac{\partial}{\partial \boldsymbol{\delta}_R(m)}\,e^{j\boldsymbol{\theta}'}\right\}\end{aligned} \tag{19}$$

and

$$\begin{aligned}&\frac{\partial}{\partial \boldsymbol{\delta}_I(m)}\left\{\mathbf{G}_k - \mathbf{G}_k^{\rm DESIRED}\right\} \\ &\quad = 20(\log_{10} e)\,\mathrm{Re}\left\{\frac{1}{\mathbf{w}_k^H e^{j\boldsymbol{\theta}'}}\,\mathbf{w}_k^H\,\frac{\partial}{\partial \boldsymbol{\delta}_I(m)}\,e^{j\boldsymbol{\theta}'}\right\}.\end{aligned} \tag{20}$$

One can show

$$\frac{\partial}{\partial \boldsymbol{\delta}_R(m)}\,e^{j\boldsymbol{\theta}'} = je^{j\boldsymbol{\theta}'} \otimes \mathrm{Im}\{\mathbf{D}_m \oslash \boldsymbol{\omega}\} \tag{21}$$

$$\frac{\partial}{\partial \boldsymbol{\delta}_I(m)}\,e^{j\boldsymbol{\theta}'} = je^{j\boldsymbol{\theta}'} \otimes \mathrm{Re}\{\mathbf{D}_m \oslash \boldsymbol{\omega}\} \tag{22}$$

where $\otimes/\oslash$ imply element-wise multiplication/division, $\mathbf{D}_m$ refers to the mth column of $\mathbf{D}$, and $\boldsymbol{\omega}$ are the prenormalized weights

$$\boldsymbol{\omega} = e^{j\boldsymbol{\theta}} + \mathbf{D}\boldsymbol{\delta}. \tag{23}$$

After substituting and repeating for all m, one finds that the gradient with respect to $\mathbf{d}$ can be expressed as

$$\begin{aligned}&\nabla_d\left\{\mathbf{G}_k - \mathbf{G}_k^{\rm DESIRED}\right\} \\ &\quad = 20(\log_{10} e)\begin{bmatrix}\mathrm{Im}\left\{\mathbf{D}\oslash\left(\boldsymbol{\omega}\mathbf{1}^T\right)\right\}^T \\ \ldots\ldots\ldots\ldots\ldots \\ \mathrm{Re}\left\{\mathbf{D}\oslash\left(\boldsymbol{\omega}\mathbf{1}^T\right)\right\}^T\end{bmatrix} \\ &\qquad \cdot \mathrm{Im}\left\{\frac{1}{\left(\mathbf{w}_k^H e^{j\boldsymbol{\theta}}\right)^*}\,e^{-j\boldsymbol{\Theta}}\mathbf{w}_k\right\}\end{aligned} \tag{24}$$

where $\mathbf{1}$ represents an appropriately sized vector of ones. The expression for the gradients of any SL constraint have the same general form.

The algorithm is realized as an iterative scheme with a conventional minimax optimization technique at its core. At the beginning of each stage, the list of perceived active main-lobe/SL stations are updated. Here, stations whose associated beamforming gains that are "sufficiently" close to the worst-case mainlobe station or the stated SL constraint specification are labeled as active. Threshold values that enable/disable a station's activity may also consider their status in the previous stage. As the active list is updated, the matrix of steering vectors $\mathbf{W}_M$ is updated accordingly. The orthonormal matrix $\mathbf{D}$ is computed as the principal components of an eigenvector decomposition of $\mathbf{W}_M$ as described in the paragraph surrounding (16). Usually, a small number (1–3) of steering vectors are added or deleted between stages, so the use of an efficient method of updating $\mathbf{D}$ without the full recomputation of an eigenvector decomposition is recommended. At this point, any standard (minimax) optimization technique is applied to solve for the vector of scaling coefficients $\boldsymbol{\delta}$. The selection of convergence criteria is made to result in rapid overall convergence. The process is chosen to insure that adjustments to the list of active stations to provide accelerated optimization progress is not performed too hastily such that the added complexity of updating $\mathbf{D}$ becomes significant.

General comments relative to the efficiency of this technique can be asserted; verification was addressed through exhaustive simulation. Note that instead of one optimization with N independent variables as obtained in the conventional approach of Section III, the above strategy relies upon the solution of $2M$ working variables, where M is the combined number of worst-case ground stations and active SL constraints. The reduced variable workspace will affect both computation time and memory requirements as the sizes of the component gradients and the Hessian scale with the number of independent variables. M typically increases throughout the procedure so that the most computationally intensive portion of the optimization problem exists at the end. As with the U.S. coverage of Fig. 2, the intended application involves an array with a *large* number of array elements providing service to an area defined by *fewer* discrete locations. Thus, a savings in both computational complexity and working memory requirements is anticipated.

V. Analysis of Approximated Beam Addition Technique

The technique of approximated beam addition as outlined in Section IV provides an efficient means of solving for phase-only beamforming weights as a result of operating in an aptly chosen reduced-dimensional workspace. As there are N phases to evaluate but only $2M \ll N$ working variables at a given stage, the optimality of the iterated estimate is obviously in question. However, simulations presented in Section VI will show that the new technique arrives at a true local solution, as validated by the post application of the conventional technique of Section III. To resolve this issue, a theoretical investigation addressing the subject of optimality is included in this section.

To address the optimality of the beam-addition technique, one must apply the Kuhn–Tucker conditions. The conventional method of Section III is employed as a reference, so that optimality for an estimate generated from the beam-addition technique is validated if the same conditions apply. Recall from Section III that the necessary conditions for optimality include both specifications on the values of the Lagrange multipliers along with a relation involving the gradient of the minimax composite functions. The application of Kuhn–Tucker to the two optimization strategies is done so at a common operating point. The "current" beamforming phases for the competing strategies are $\boldsymbol{\theta} = \boldsymbol{\theta}^o$ so that the techniques have the same set of active worst-case ground stations. Thus, one only needs to consider the functional gradients as in (6). Assume, for simplicity, that there are no additional SL constraints. From (6) and (9), the operating point $e^{j\boldsymbol{\theta}^o}$ satisfies the first-order necessary conditions for a local solution if

$$\sum_{k=1}^{M} \lambda_k \operatorname{Im}\left\{ \frac{1}{\left(\mathbf{w}_k^H e^{j\boldsymbol{\theta}^o}\right)^*} e^{-j\boldsymbol{\Theta}^o} \mathbf{w}_k \right\} = \mathbf{0}. \tag{25}$$

Using (24), the corresponding relation for the approximate beam-addition technique is

$$\begin{bmatrix} \operatorname{Im}\left\{\mathbf{D}\varnothing\left(\boldsymbol{\omega}^o \mathbf{1}^T\right)\right\}^T \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \operatorname{Re}\left\{\mathbf{D}\varnothing\left(\boldsymbol{\omega}^o \mathbf{1}^T\right)\right\}^T \end{bmatrix} \cdot \sum_{k=1}^{M} \lambda_k \operatorname{Im}\left\{ \frac{1}{\left(\mathbf{w}_k^H e^{j\boldsymbol{\theta}^o}\right)^*} e^{-j\boldsymbol{\Theta}^o} \mathbf{w}_k \right\} = \mathbf{0}. \tag{26}$$

By inspection, if an operating point $\boldsymbol{\theta}^o$ is optimal so that (25) is satisfied, then (26) is satisfied. This shows that a true local solution appears as a solution for the approximated beam addition technique. However, it is also desired that no false solution could exist in the proposed technique. This is validated by showing that (26) is only satisfied when (25) is true. Consider the terminal point in the approximated beam-addition optimization where we must have $|\boldsymbol{\delta}| < \epsilon$. Using the definition of the weights $\boldsymbol{\omega}$ in (23), one finds

$$\mathbf{D}\varnothing\left(\boldsymbol{\omega}^o \mathbf{1}^T\right) \approx \mathbf{D}\varnothing\left(e^{j\boldsymbol{\theta}^o} \mathbf{1}^T\right) = e^{-j\boldsymbol{\Theta}^o}\mathbf{D}. \tag{27}$$

Now consider the sum-term in (26) which precisely matches the left-hand side of (25). Observe that in a general sense and not necessarily at $\boldsymbol{\theta} = \boldsymbol{\theta}^o$, it lies in a $2M$-dimensional subspace

$$\begin{aligned} &\sum_{k=1}^{M} \lambda_k \operatorname{Im}\left\{ \frac{1}{\left(\mathbf{w}_k^H e^{j\boldsymbol{\theta}}\right)^*} e^{-j\boldsymbol{\Theta}} \mathbf{w}_k \right\} \\ &\quad \in \mathcal{R}\left\{\left[\operatorname{Re}\left\{e^{-j\boldsymbol{\Theta}}\mathbf{W}_M\right\} \vdots \operatorname{Im}\left\{e^{-j\boldsymbol{\Theta}}\mathbf{W}_M\right\}\right]\right\} \\ &\quad \in \mathcal{R}\left\{\left[\operatorname{Re}\left\{e^{-j\boldsymbol{\Theta}}\mathbf{D}\right\} \vdots \operatorname{Im}\left\{e^{-j\boldsymbol{\Theta}}\mathbf{D}\right\}\right]\right\} \subset R^N \end{aligned} \tag{28}$$

where $\mathcal{R}$ denotes the range of the indicated matrix. Note that $\mathcal{R}\{\mathbf{D}\}$ is only approximately equivalent to $\mathcal{R}\{\mathbf{W}_M\}$ if one follows the suggested method of selecting only the prominent spectral components of $\mathbf{W}_M$ for $\mathbf{D}$. With (27) in mind, the desired result is proven: a set of phases $\boldsymbol{\theta}^o$ satisfies the first-order Kuhn–Tucker necessary conditions for the proposed method if and only if the respective conditions for the conventional technique are satisfied.

The operation of the approximated beam-addition technique is clarified by observing the first-order term in the Taylor series expansion of (17). Incorporating the gradient expressions of (21) and (22), one obtains

$$e^{j\boldsymbol{\theta}'} = e^{j\boldsymbol{\theta}} + \tfrac{1}{2}\mathbf{D}\boldsymbol{\delta} - \tfrac{1}{2}e^{j2\boldsymbol{\Theta}}\mathbf{D}^*\boldsymbol{\delta}^* + o(\delta, \delta^*). \tag{29}$$

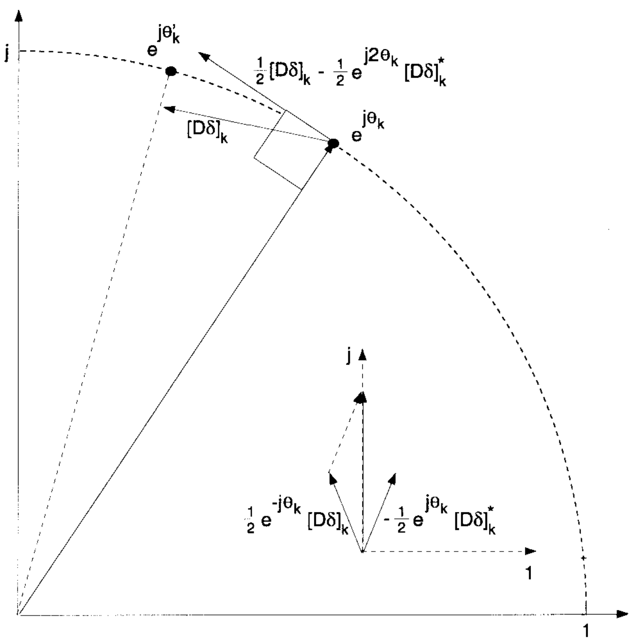

Fig. 3. Geometry of effects of first-order terms.

A geometrical view showing the significance of the two first-order terms for the arbitrary kth sensor weight is shown in Fig. 3. Essentially, the term $\mathbf{D}\boldsymbol{\delta}$ provides the desired movement in the complex plane while the term $-e^{j2\Theta}\mathbf{D}^*\boldsymbol{\delta}^*$ insures, to first order, that the iterated point stays on the unit circle. The desired modification of the worst-case station gain from $\mathbf{D}\boldsymbol{\delta}$ is realized at the expense of an equi-energy perturbation from $e^{j2\Theta}\mathbf{D}^*\boldsymbol{\delta}^*$. This perturbation vector lies in the range space of $e^{j2\Theta}\mathbf{D}^*$. As the optimization proceeds, the phases $\boldsymbol{\theta}$ become scattered over [0, 2π) yielding a perturbation vector not cleanly projected onto the ground-station steering vectors of interest. As a result, the energy of the second term is spread randomly and somewhat uniformly over all space.

VI. Computer Simulations

Two experiments addressing the optimality and computational efficiency of the proposed synthesis technique of approximated beam addition are considered in this section. Experiment 1 represents the intended application to phased-array phase-only beampattern synthesis for satellite communication. Here the computational efficiency of the approximated beam-addition technique, relative to the standard method, is addressed. Experiment 2 represents a more detailed examination of optimality and complexity in the comparison of the two synthesis techniques. Monte Carlo trials are employed in the empirical comparison to develop statistics on the execution times as well as to examine the range and relative quality of local solutions.

Experiment 1: This experiment represents the intended application to satellite communication where $N = 256$ isotropic sensors are placed on a square grid of 3λ spacing. The array is affixed to a satellite in a geostationary orbit. The desired coverage pattern consists of $N_c = 207$ city locations in all the 50 states (refer to Fig. 2). The desired gain profile was such to accommodate worst-case rain levels.

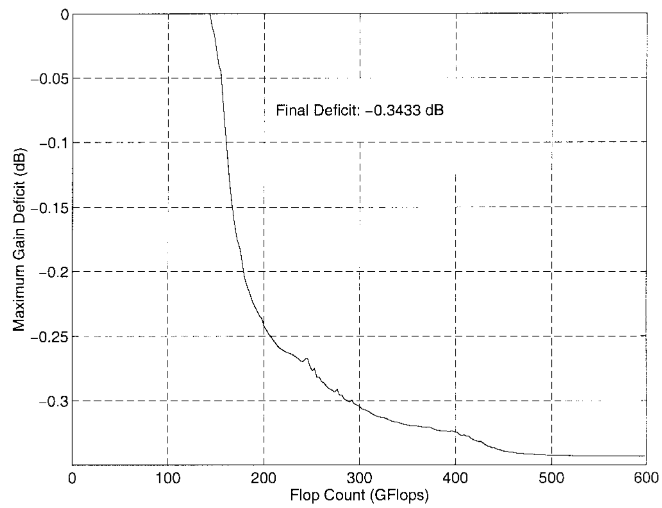

Fig. 4. CONUS minimax optimization timeline—standard method.

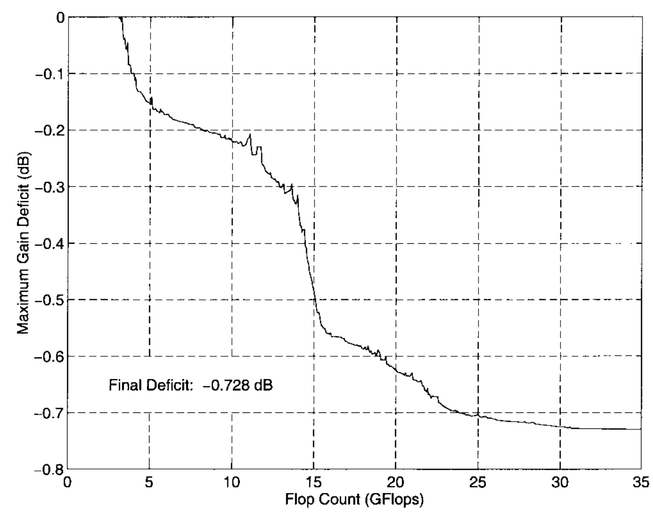

Fig. 5. CONUS optimization timeline—approximated beam-addition method.

MATLAB's minimax algorithm [23] was employed for both optimization methods; applied directly for the standard method of Section III and used as a core utility within the technique of Section IV. The initial element phases input to the optimization were derived using a technique similar to that found in [27] to provide a best-fit rectangular beam encompassing the region of interest. Over the course of the two optimization procedures, the iterated deficit at the worst-case station was tabulated versus the number of floating point operations. For the case of the conventional optimization procedure, the results are presented in Fig. 4. Employing the method of Section IV with the same initial conditions, the optimization performance timeline shown in Fig. 5 was obtained.

Confirming the initial expectations, only 37 of the 207 stations were considered "worst-case" at the end so that the last stage involved an optimization over $2 \times 37 = 74$ variables. This is significantly fewer than the moderately sized 256-element array, which was the number of optimization variables for the conventional method. The replacement of a single large optimization problem with a succession of smaller optimizations ultimately led to a reduction in both computational complexity and memory needs. The considerable reduction in computational time is apparent by comparing the results in Figs. 4 and 5.

The new technique iterated to a better solution—namely 0.728-dB excess gain above specifications as compared to the conventional method where only a 0.343-dB excess was obtained. Note that a 0.4-dB savings is significant to the application of satellite communications. Although this result cannot be expected in every case, one can conclude that the two methods can iterate to different local minima with the same initial conditions. These different solutions may offer widely varying performance levels. Although a detailed examination of the N-dimensional solution space was not attempted, the application of a simulated annealing "wrapper" around the optimization algorithm would be effective in attaining the best operating configuration.

The shape of the optimization timelines in Figs. 4 and 5 are typical of those seen in exhaustive application of the two competing algorithms. Regions of rapid convergence are interlaced with pockets of slowed convergence. To guarantee convergence to a true local solution, it was determined that the stopping criteria had to be set very finely in order to recognize a slight grade in the surface at the expense of an increased computational complexity. In some cases, this resulted in substantial decreases in the functional value.

Experiment 2: In the second experiment, optimality, global convergence, and the relative complexity of the technique of approximated beam addition is addressed through Monte Carlo simulation. Here, 500 trials were performed with random initial beamforming phases. During each trial the two techniques were applied and the computation time to various (gain) milestones were recorded.

To insure that each trial iterated to a true local solution, a scheme with optimization restarts was adopted. Individual optimizations were deemed convergent if the iterative enhancement in the worst-case gain fell below a modest 0.001-dB threshold. The elemental phases were then perturbed about their iterated value over the interval $\pm 10°$ and used as inputs to a restarted optimization. The algorithms were restarted until two successive optimizations yielded a worst-case station gain not significantly better than the current best case. This strategy was found to be effective in guarding against the inefficiencies associated with the algorithm stalling on a plateau. However, the $\pm 10°$ randomization factor was found to be sufficiently small so as not to abort iteration to the local, perhaps nonglobal, solution.

Two array configurations were used to provide a comparison in the complexity of the two optimization methodologies. In the first simulation, an 8 $\times$ 8 square array of λ/2-spaced elements was employed with 25 defined service locations randomly placed in the region $(-0.15 < T_x < -0.15, -0.1 < T_y < 0.1)$. The second scenario involved a 10 $\times$ 10 array with the same elemental spacing. To provide the same perceived distribution of service locations, the region containing the 25 (Tx, Ty) service locations was simply compressed by the factor 8/10 along both dimensions.

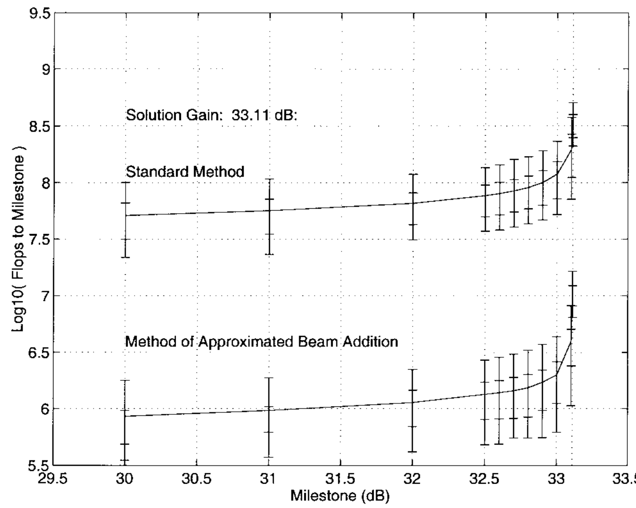

Fig. 6. Computation time to milestone for 8 $\times$ 8 array. Shown are 50 and 90% empirical confidence regions.

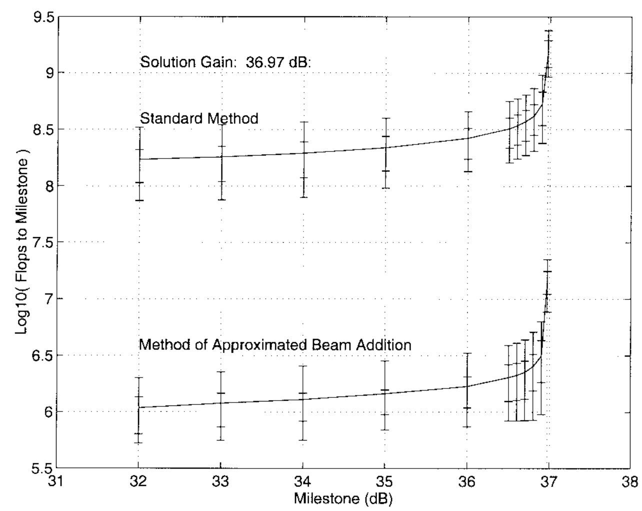

Fig. 7. Computation time to milestone for 10 $\times$ 10 array. Shown are 50 and 90% empirical confidence regions.

Figs. 6 and 7 show the amount of computation to reach the various gain-milestones for the two competing techniques. Observe that the base-ten logarithm of the complexity in MATLAB floating operations (flops) is plotted. The mean time to milestone is shown by the solid line, while the error bars indicate the 50th and 90th percentile confidence regions. Although the shapes of the timelines are similar, there is a dramatic computational improvement using the new method.

Examination of the averaged timelines show that the complexity associated with the technique of approximated beam addition relative to the number of array elements is approximately linear, i.e., by the factor $(100/64)^1$. The complexity factor associated with the conventional method is more dramatic and is approximately $(100/64)^{2.6}$.

The surface containing the multiple local solutions was somewhat probed by storing statistics on the converged solution. Table I shows the distribution of solution gains of the 500 converged trials for the 8 $\times$ 8 and 10 $\times$ 10 array scenarios involving the conventional optimization method. Note that

TABLE I
DISTRIBUTION OF CONVERGED WORST-CASE STATION GAIN FOR EXPERIMENT 2

	8×8 array				10×10 array				
Bin center (dB) (0.1 dB width)	18.8	19.0	19.4	33.1	22.7	22.9	23.3	36.9	37.0
Number of occurrences	4	15	5	476	6	8	10	1	475

application of the first-order Kuhn–Tucker conditions verified that all converged estimates were indeed local solutions. Inspection of the results show that approximately 95% of the independent trials with purely random initial phases iterated to the global solution.

VII. CONCLUSIONS

A method of minimax phase-only beamforming synthesis was presented. It was shown to be more computationally efficient and require less working memory than the standard approach in problems of interest. The optimality of the technique was verified theoretically via application of the Kuhn–Tucker optimality conditions and empirically through Monte Carlo simulation.

The technique allows for iteration within a subspace of $\mathcal{C}^N$ spanned by a set of steering vectors associated with the worst-case service locations. Although the dimensionality of the subspace is typically much less than N, it was shown that there are sufficient degrees of freedom to allow effective adjustment of the beamforming gain at these stations of interest. As the sizes of variables computed within the gradient-based optimization procedure scale with the number of variables, a reduction in both storage and computation are realized with the new technique.

Although all simulations presented here were without defined SL constraints, such problems were analyzed. An example of which is point-to-point communication where optimization is needed to reduce the emission over spatially-adjacent user cells to meet cochannel interference specifications. It was determined that a feasible solution should *first* be obtained with either an auxiliary optimization technique or an alternate version of the beam-addition technique designed solely to accommodate the constraints.

REFERENCES

[1] A. R. Cherrette and D. C. D. Chang, "Phased array contour beam shaping by phase optimization," in *IEEE Antennas Propagat. Symp. Dig.*, 1985, vol. APS-14-4, pp. 475–478.
[2] C. A. Klein, "Design of shaped-beam antennas through minimax gain optimization," *IEEE Trans. Antennas Propagat.*, vol. AP-32, pp. 963–968, Sept. 1984.
[3] B. P. Ng, M. H. Er, and C. Kot, "A flexible array synthesis method using quadratic programming," *IEEE Trans. Antennas Propagat.*, vol. 41, pp. 1541–1550, Nov. 1993.
[4] J. E. Richie and H. N. Kritikos, "Linear program synthesis for direct broadcase satellite phased arrays," *IEEE Trans. Antennas Propagat.*, vol. 36, pp. 345–348, Mar. 1988.
[5] R. S. Elliott and G. J. Stern, "Footprint patterns obtained by planar arrays," *Proc. Inst. Elect. Eng.*, vol. 137, pt. H, no. 2, Apr. 1990.
[6] A. R. Dion and L. J. Ricardi, "A variable-coverage satellite antenna system," *Proc. IEEE*, vol. 59, pp. 252–262, Feb. 1971.
[7] A. I. Zaghloul and D. K. Freeman, "Phased array synthesis for shaped beams using power matrix technique," in *IEEE Antennas Propagat. Soc. Int. Symp. Dig.*, June 1986, pp. 177–180.
[8] J. E. Richie and H. N. Kritikos, "Beamforming for direct broadcast satellite phased array antennas," in *IEEE Antennas Propagat. Soc. Int. Symp. Dig.*, June 1986, pp. 181–184.
[9] S. H. Colodny and R. L. Crane, "Active-array antenna beam shaping for direct broadcast satellites and other applications,," *RCA Rev.*, vol. 46, pp. 376–392, Sept. 1992.
[10] R. C. Voges and J. K. Butler, "Phase optimization of antenna array gain with constrained amplitude excitation," *IEEE Trans. Antennas Propagat.*, vol. AP-20, pp. 432–436, July 1972.
[11] R. Giusto and P. De Vincenti, "Phase-only optimization for the generation of wide deterministic nulls in the radiation pattern of phased arrays," *IEEE Trans. Antennas Propagat.*, vol. AP-31, pp. 814–817, Sept. 1983.
[12] K. Hirasawa, "The application of a biquadratic programming method to phase-only optimization of antenna arrays," *IEEE Trans. Antennas Propagat.*, vol. 36, pp. 1545–1550, Nov. 1988.
[13] J. F. DeFord and O. P. Gandhi, "Phase-only synthesis of minimum peak sidelobe patterns for linear and planar arrays," *IEEE Trans. Antennas Propagat.*, vol. 36, pp. 191–201, Feb. 1988.
[14] R. F. E. Guy, "A synthesis technique for array antennas of general shape with various aperture constraints," in *4th Inst. Elect. Eng. Int. Conf. Antennas Propagat. (ICAP) #248*, Warwick, U.K., 1985, pp. 35–39.
[15] H. Steyskal, "Simple method for pattern nulling by phase perturbation," *IEEE Trans. Antennas Propagat.*, vol. AP-31, pp. 163–166, Jan. 1983.
[16] S. Lundgren and J. Sanford, "A new technique for phase only nulling with equispaced arrays," in *IEEE AP-S Symp. Dig.*, June 1995, vol. 1, pp. 447–450.
[17] R. L. Haupt, "Adaptive nulling in monopulse antennas," *IEEE Trans. Antennas Propagat.*, vol. 36, pp. 202–208, Feb. 1988.
[18] ——, "Phase-only adaptive nulling with a genetic algorithm," *IEEE Trans. Antennas Propagat.*, vol. 45, pp. 1009–1015, June 1997.
[19] P. E. Gill, W. Murray, and M. H. Wright, *Practical Optimization.* New York: Academic, 1981.
[20] P. E. Gill, W. Murray, M. A. Saunders, and M. H. Wright, "Procedures for optimization problems with a mixture of bounds and general linear constraints,," *ACM Trans. Math. Software*, vol. 10, no. 3, pp. 282–298, Sept. 1984.
[21] P. E. Gill, W. Murray, and M. H. Wright, *Numerical Linear Algebra and Optimization.* Reading, MA: Addison-Wesley, 1991, vol. 1.
[22] R. K. Brayton, S. W. Director, G. D. Hachtel, and L. M. Vidigal, "A new algorithm for statistical circuit design based on quasi-Newton methods and function splitting," *IEEE Trans. Circuits Syst.*, vol. 26, pp. 784–794, Sept. 1979.
[23] A. Grace, *Optimization Toolbox User's Guide.* Natick, MA: The MathWorks, 1990.
[24] D. G. Luenberger, *Linear and Nonlinear Programming.* Reading, MA: Addison-Wesley, 1984.
[25] J. W. Bandler, "Conditions for a minimax optimum," *IEEE Trans. Circuit Theory*, vol. 18, pp. 476–479, July 1971.
[26] F. W. Wheeler, "Techniques for phase-only phased array pattern synthesis," Internal Memo, GE Corp. R&D Ctr., 1995.
[27] B. L. Cleaveland and S. R. Carpenter, "A phased array with variable beam width using a simple phase distribution," patent disclosure 35-EL-1838, Mar. 1993.
[28] A. Chakraborty, B. N. Das, and G. S. Sanyal, "Determination of phase functions for a desired one-dimensional pattern," *IEEE Trans. Antennas Propagat.*, vol. AP-29, pp. 502–506, May 1981.
[29] ——, "Beamshaping using nonlinear phase distribution in a uniformly spaced array," *IEEE Trans. Antennas Propagat.*, vol. AP-30, pp. 1031–1034, Sept. 1982.
[30] E. T. Bayliss, "A phase synthesis technique with application to array beam broadening," in *Proc. AP-S Antennas Propagat. Conf.*, 1966, pp. 427–433.

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