A New Algorithm for the Complex Exponential Integral in the Method of Moments

Michael S. Kluskens, *Member, IEEE*

***Abstract*—This paper presents a new algorithm for the rapid and accurate calculation of the complex exponential integral associated with the mutual impedance of sinusoidal basis and testing functions in the method of moments. The new algorithm uses Leibniz's theorem to calculate Taylor series expansions of the integral instead of integrating expansions of the integrand as is often done. This results in an algorithm which is twice as fast as and is valid over a wider range than previous algorithms. This technique can be applied to many other integrals encountered in computational electromagnetics as well.**

***Index Terms*—Method of moments.**

I. Introduction

INTEGRALS involving complex exponentials are common in electromagnetics. This paper presents a new algorithm for the rapid and accurate calculation of the exponentially scaled complex exponential integral over a finite interval, i.e.,

$$E(u,\delta)=e^{u}\,⨍_{u-\delta}^{u+\delta}\frac{e^{-z}}{z}\,dz. \tag{1}$$

The primary area of application is the evaluation of a method of moments impedance matrix with sinusoidal basis and testing functions [1]–[14]. Richmond's thin-wire code [2], [12] and the ESP4 method of moments code [7] calculate the required integral as the difference of two evaluations of the complex exponential integral [15]–[21] with a six-region algorithm (EXPJ) using the Taylor series expansion of e^{-z} [16, Sec. 5.1.11], Gauss–Laguerre quadratures [16, Table 25.9], and rational approximations. In addition to the inefficiency of this approach, the error increases as $|\delta|$ decreases due to the cancellation of the most significant digits in the calculation of the two integrals. A more efficient algorithm (EPF) using the Taylor series expansion of e^{-z} for small u and a binomial series of the integrand about an optimal point for large u has been published recently [10]; however, the forward recurrence relation used in the latter is numerically unstable.

In this paper, Leibniz's theorem [16, Sec. 3.3.7] is used to calculate Taylor series expansions of the integral instead of integrating expansions of the integrand. This results in a series which converges uniformly for values of u over the entire complex plane. To achieve high accuracy and efficiency this series is reformulated for $|u|<|\delta|$ and a nearly identical alternative series is used for $|u|>1.79|\delta|$. The efficiency and convergence of these series is improved by the fact that the even-numbered terms are zero. As a result, 0.001% accuracy can be achieved with only the first seven nonzero terms for $|\delta|\le\pi/2$, which corresponds to the practical size limit for sinusoidal basis functions of $\lambda/4$.

Manuscript received June 30, 1998. This work was supported by the Office of Naval Research, Washington, DC. Some computational support was provided by The Joint Services Electronics Program under Grant N00014-89-J-1007 with the Ohio State University Research Foundation.

The author is with the Naval Research Laboratory, Washington, DC 20375 USA.

Publisher Item Identifier S 0018-926X(99)04823-1.

II. Theory

The mutual impedance of the nonplanar-skew sinusoidal monopoles $(0,0,s_1)-(0,0,s_2)$ and $(t_1\sin\psi,d,t_1\cos\psi)-(t_2\sin\psi,d,t_2\cos\psi)$ is given by [8], [14]

$$\begin{aligned}Z_{mn}=\frac{1}{4\pi\omega\epsilon}\Bigg\{&\frac{e^{-jkR_{mn}}}{jR_{mn}}\\&-\frac{k}{4\sin[k(s_2-s_1)]\sin[k(t_2-t_1)]}\\&\cdot\sum_{p,q=\pm1}pq\,e^{jk[ps_{2/m}+qt_{2/n}]}\sum_{i=1}^{2}(-1)^i\\&\cdot\sum_{h=\pm1}e^{hk\beta_{pq}}⨍_{hk\beta_{pq}+jk[R_{i1}+ps_i+qt_1]}^{hk\beta_{pq}+jk[R_{i2}+ps_i+qt_2]}\frac{e^{-z}}{z}\,dz\Bigg\}\end{aligned} \tag{2}$$

for the currents

$$I_s=\frac{\sin k(s_{2/m}-z)}{\sin k(s_{2/m}-s_m)} \tag{3}$$

$$I_t=\frac{\sin k(t_{2/n}-t)}{\sin k(t_{2/n}-t_n)} \tag{4}$$

where

$$\beta_{pq}=\frac{d}{\sin\psi}[p\cos\psi+q] \tag{5}$$

$$R_{mn}=\sqrt{d^2+s_m^2+t_n^2-2s_mt_n\cos\psi} \tag{6}$$

for $m,n=1,2$ and $k=\omega\sqrt{\mu\epsilon}$. To calculate (2) for the widest possible range of d and ψ using single-precision arithmetic it is necessary to include the exponential scale factor $e^{hk\beta_{pq}}$ in the algorithm used for the integrals since it easily exceeds 10^{38} for moderate values of d and ψ; for example, $\psi=5^\circ$ and $d=0.7\lambda$. A Taylor series of $E(u,\delta)$ is an efficient means for calculating (2) since the maximum extent of the domain of integration, $k(|R_{i2}-R_{i1}|+|t_2-t_1|)$, is π for sinusoidal basis functions within the practical size limit of $\lambda/4$. The Taylor series

$$E(u,\delta) \approx E(u,0) + \sum_{n=1}^{N} E^{(n)}(u,0)\frac{\delta^n}{n!} \tag{7}$$

is generated by applying Leibniz's theorem to $E(u,\delta)$ with respect to δ giving $E^{(n)}(u,0) = 2a_n/u$, for odd n and 0 for even n, where $a_1 = 1$ and

$$a_n = 1 + \frac{n-1}{u}\left[1 + \frac{n-2}{u}a_{n-2}\right]. \tag{8}$$

The resulting Taylor series is

$$E(u,\delta) \approx \frac{2}{u} \sum_{n=1,3,\cdots}^{N} a_n \frac{\delta^n}{n!}. \tag{9}$$

Expanding $a_n\delta^n/(un!)$ results in the more efficient form given by

$$E(u,\delta) \approx 2 \sum_{n=1,3,\cdots}^{N} A_n \tag{10}$$

where $A_1 = \delta/u$ and

$$A_n = \frac{\delta^2}{nu^2}\left[\frac{\delta^{n-2}}{(n-1)!}(u+n-1) + (n-2)A_{n-2}\right]. \tag{11}$$

Using Aitken's δ^2-process [16, Sec. 3.9.7] the series is useful down to $|u| = |\delta|$; however, the series diverges for $|u| < |\delta|$.

An expansion for $E(u,\delta)$ which converges uniformly for values of u over the entire complex plane is obtained by separating the logarithmic term from the integral so that the remaining function $f(u,\delta)$ is smooth and continuous everywhere, i.e.,

$$E(u,\delta) = e^u \fint_{u-\delta}^{u+\delta} \frac{1}{z}\, dz + e^u \int_{u-\delta}^{u+\delta} \frac{e^{-z}-1}{z}\, dz \tag{12a}$$

$$= e^u \ln \frac{u+\delta}{u-\delta} + f(u,\delta) \tag{12b}$$

$$\approx e^u \ln \frac{u+\delta}{u-\delta} + f(u,0) + \sum_{n=1}^{N} f^{(n)}(u,0)\frac{\delta^n}{n!}. \tag{12c}$$

Using Leibniz's theorem on $f(u,\delta)$ with respect to δ gives

$$f^{(n)}(u,0) = \frac{2}{u}\left[a_n - \frac{(n-1)!}{u^{n-1}}e^u\right] \tag{13}$$

for odd n, and 0 for even n. However, this can be simplified to $f^{(n)}(u,0) = 2b_n/u$, where $b_1 = 1 - e^u$ and

$$b_n = 1 + \frac{n-1}{u}\left[1 + \frac{n-2}{u}b_{n-2}\right] \tag{14}$$

which differs from a_n only in the starting value of the series. The resulting expansion is

$$E(u,\delta) \approx e^u \ln \frac{u+\delta}{u-\delta} + \frac{2}{u} \sum_{n=1,3,\cdots}^{N} b_n \frac{\delta^n}{n!}. \tag{15}$$

As with a_n, a more efficient form is given by

$$E(u,\delta) \approx e^u \ln \frac{u+\delta}{u-\delta} + 2 \sum_{n=1,3,\cdots}^{N} B_n \tag{16}$$

where $B_1 = \delta(1 - e^u)/u$ and

$$B_n = \frac{\delta^2}{nu^2}\left[\frac{\delta^{n-2}}{(n-1)!}(u+n-1) + (n-2)B_{n-2}\right]. \tag{17}$$

Although mathematically this expansion converges uniformly for all values of u, the numerical noise increases in the right half-plane as $\Re(u)$ increases because of cancellation of the logarithmic term by the series expansion. In addition, the number of terms required also increases as $\Re(u)$ increases. As a result, (10) is more accurate and efficient down to approximately $|u| = 2|\delta|$. In addition, the numerical accuracy of (15) and (16) is reduced as u approaches 0 even though $b_n/u = -1/n$ and $B_n = -\delta^n/(nn!)$ in the limit. However, transforming (14) into a backward recursion and setting $b_n = -uc_n$ gives

$$c_n = \frac{1}{n}\left(1 + \frac{u}{n+1}(1 + uc_{n+2})\right) \tag{18}$$

where the recursion is started using $c_{n+2} = 0$ for a sufficiently large n. The resulting expansion is

$$E(u,\delta) \approx e^u \ln \frac{u+\delta}{u-\delta} - 2 \sum_{n=1,3,\cdots}^{N} c_n \frac{\delta^n}{n!} \tag{19}$$

which converges rapidly for $|u| < |\delta|$.

III. Error Analysis

The error in a Taylor series is approximately given by the first neglected term, which in the case of (9) and (10) for $|u| > N+2$ can be approximated as

$$\frac{2}{u}\, \frac{\delta^{N+2}}{(N+2)!} \tag{20}$$

where N is the highest term calculated. For $|u| < N+2$ the upper limit on the magnitude of the error is approximately

$$\frac{10|u|^{\lfloor u \rfloor}}{(N+2)\lfloor u \rfloor!}\left|\frac{\delta}{u}\right|^{N+2} \tag{21}$$

where $\lfloor u \rfloor$ is the largest integer smaller than the magnitude of u. These approximate error formulas cannot be solved directly for N. However, the behavior of their convergence can be determined from an examination of their parts. In the case of (20) the magnitude of $\delta^n/n!$ is maximum near $n = |\delta|$ and is roughly symmetric about this point, therefore, N is proportional to $|\delta|$. From (21), N is approximately proportional to $1/\ln|\delta/u|$. From this information and actual calculations over the range of $2 \le |u| \le 20$ and $|\delta| \le \pi/2$, the number of terms required for 0.001% accuracy is approximately given by

$$N = 4 + 3.2|\delta| - 3.6/\ln|\delta/u| \tag{22}$$

when Aitken's δ^2-process is used with (9) or (10).

Using that $a_n(u^{n-1})/(n-1)!$ is equal to the first n terms of the Taylor series for e^u in combination with (13), the error in (15) and (16) is given by

$$-2\frac{\delta^{N+2}}{N+2}\sum_{k=N+2}^{\infty}\frac{u^{k-(N+2)}}{k!} \tag{23}$$

which for $|u| < N+2$ is approximately

$$\frac{2}{(N+2)}\,\frac{\delta^{N+2}}{(N+2)!} \tag{24}$$

therefore, N is proportional to $|\delta|$. For $|u| > N+2$ and $|u| > |\delta|$, the error is approximately

$$\frac{2}{u}\,\frac{\delta^{N+2}}{(N+2)!}+\frac{2e^u}{N+2}\left(\frac{\delta}{u}\right)^{N+2} \tag{25}$$

using (13) and (20). Therefore, the convergence of (15) and (16) is dependent on $\Re(u)$; however, the numerical noise when calculating (15) and (16) restricts them to $\Re(u) < 2|\delta|$ where the second term of (25) has little effect. From actual calculations over the range of $0.05 \le |u| \le \pi$ and $|\delta| \le \pi/2$, the number of terms required for 0.001% accuracy is approximately

$$N = 2.8 + 3.7|\delta| \tag{26}$$

when Aitken's δ^2-process is used with (15) or (16).

The error in (19) is the same as that for (15) and (16) except for the additional error in each term from assuming $c_{N+2} = 0$. The total correction for this error is

$$(u\tilde{c}_1 + 1 - e^u)\sum_{n=1,3,\cdots}^{N}\frac{1}{n}\left(\frac{\delta}{u}\right)^n \tag{27}$$

where $\tilde{c}_1$ is the value calculated using (18) with $c_{N+2} = 0$. For $|u| < N+1$ this formula can be approximated by

$$-\frac{u^{N+1}}{(N+1)!}\sum_{n=1,3,\cdots}^{N}\frac{1}{n}\left(\frac{\delta}{u}\right)^n. \tag{28}$$

In tests this error was negligible in the region $|u| < \delta$, as would be expected from (28).

In the final algorithm, referred to below as EXPJ1, (19) is used for $|u| < |\delta|$ with $N = 11$, while (16) is used for $|\delta| \le |u| < 1.79|\delta|$ and (10) is used for $|u| \ge 1.79|\delta|$. The latter two use Aitken's δ^2-process and $N = 13$. Dynamically determining the number of terms using (22) and (26) produced a slower algorithm in actual tests due to the small number of terms and the overhead of a dynamic algorithm.

IV. Numerical Results

The following tables show the accuracy and efficiency of the EXPJ1 algorithm versus the EXPJ [2], [7], [12] and the EPF2 algorithms (see the Appendix). Both the EXPJ1 and EPF2 algorithms were adjusted to a relative accuracy of 10^{-5} for these tests. The EPF [10] algorithm is not shown in the tables because of the large error in several regions as discussed in the Appendix. To minimize the differences between the algorithms and to better reflect the calculation of (2) with real k all three algorithms were internally modified to calculate

$$e^{\Re(u)}\int_{u-\delta}^{u+\delta}\frac{e^{-z}}{z}\,dz. \tag{29}$$

The reference solution for Table I was generated to a relative accuracy of 10^{-9} using multiple double-precision Gauss–Legendre quadratures with Aitken's δ^2-process used to determine convergence. The tables are for 10^7 calculations over a test grid where $u = (mA + jnA)/500$; $m, n = -500, \cdots, 500$ and $\delta = jp\pi/20$; $n = 1, \cdots, 10$ for $A = 5$ and 75[1]. The average error calculations for Table I are for the $A = 5$ test grid only since the error for all the algorithms is concentrated in this region. Table II shows that the EXPJ1 algorithm is 70 to 200% faster than the fastest of other two algorithms and averages 170% faster than the commonly used EXPJ algorithm.

TABLE I
PEAK AND AVERAGE RELATIVE AND ABSOLUTE ERRORS FOR $|\delta| \le \pi/2$

Peak	EXPJ	EPF2	EXPJ1
Relative	4.6×10^{-5}	8.2×10^{-6}	9.9×10^{-6}
Absolute	1.6×10^{-5}	1.6×10^{-5}	4.6×10^{-6}
Average ($A = 5$)			
Relative	1.1×10^{-6}	1.3×10^{-7}	1.3×10^{-7}
Absolute	4.3×10^{-7}	6.7×10^{-8}	5.7×10^{-8}

TABLE II
TIME IN SECONDS FOR 10^7 CALCULATIONS FOR $|\delta| \le \pi/2$

$A = 5$	EXPJ	EPF2	EXPJ1	Speedup
R10K/195	53.9	67.4	21.5	151%
PPC604e/300	110.1	134.7	47.3	133%
A21064A/275	96.5	115.7	51.9	86%
R4400/250	149.8	160.2	56.1	167%
PPC604/120	352.4	325.3	104.9	210%
$A = 75$				
R10K/195	61.4	39.5	20.1	97%
PPC604e/300	135.5	103.8	42.9	142%
A21064A/275	106.5	106.1	46.3	129%
R4400/250	167.5	106.4	52.0	105%
PPC604/120	297.5	233.6	96.0	143%

V. Conclusions

A new algorithm for calculating the complex exponential integral for sinusoidal basis function method of moments has been developed using Leibniz's theorem. This algorithm is more accurate and averages 170% faster than the commonly used EXPJ algorithm. The technique used to develop the algorithm can be applied to many other integrals in computational electromagnetics such as those in [22] and [23].

Appendix

The large argument expansion from [10] in terms of (1) can be written as

$$E(u,\delta) = \frac{1}{c}\left[\sum_{n=0}^{N}\alpha_n(c,\delta) - \sum_{n=0}^{N}\alpha_n(c,-\delta)\right] \tag{30}$$

[1] EXPJ fails catastrophically beyond $A = 75$.

where α_n is calculated in [10] using

$$\alpha_n(c,x) = -\Big(-\frac{x}{c}\Big)e^{-x} - \frac{n}{c}\alpha_{n-1}(c,x) \tag{31}$$

and c is the optimal point for convergence given in [10] which can be reduced to

$$c = \sqrt{u^2-\delta^2}\,\frac{\Im(\delta^*\sqrt{u^2-\delta^2})}{\Im(\delta^* u)} \tag{32}$$

where δ^* is the complex conjugate of δ and $\Im$ indicates the imaginary part of the argument. Equation (31) is a forward recursion relation very similar to that for the Bessel function $J_n(c)$ and both are unstable for $n > |c|$. Numerically, the instability can be avoided if (30) achieves sufficient accuracy before the divergence of (31) destroys that accuracy and the number of terms required can be accurately estimated; however, for $|u| < 5$ this is often not possible. For example, with $u = 2$ and $\delta = j\pi/2$, the single precision calculation of (30) with (31) converges to two digits of accuracy in 11 terms before diverging rapidly. Solving (31) for the backward recursion relation produces a stable algorithm, which can be improved by expanding it into

$$\alpha_n(c,x) = -d_n\Big(-\frac{x}{c}\Big)^n e^{-x} \tag{33}$$

where $d_n = c(1-d_{n-1})/n$ and $d_1 = c$. The EPF2 algorithm in Tables I and II uses the series expansion [16, Sec. 5.1.11] for $|c| \leq 1.3|u+\delta-c|$, (33) with Aitken's δ^2-process [16, Sec. 3.9.7] with $N = 5.0-(4.0+1.1|c|)/\ln|\delta/u|$ for $|u| < 4.5$, and (31) with $N = 1-11.5/\ln|\delta/c|$ for $|u| \geq 4.5$. The instability of (31) precludes the use of Aitken's δ^2-process to accelerate convergence of this series.

Acknowledgment

The author wishes to thank Dr. E. H. Newman of The Ohio State University, Columbus, for suggesting the need for a more efficient algorithm for the method of moments complex exponential integral. The author would also like to thank Dr. F. L. Whetten of Embry-Riddle University, Prescott, AZ, for supplying the computer routine EPF used in [10].

References

[1] J. H. Richmond, "Radiation and scattering by thin-wire structures in the complex frequency domain," Tech. Rep. 2902-10, ElectroScience Lab., Ohio State Univ., Columbus, July 1973, abstracted in [13].

[2] ——, "Radiation and scattering by thin-wire structures in a homogeneous conducting medium," *IEEE Trans. Antennas Propagat.*, vol. AP-22, p. 365, Mar. 1974.

[3] N. N. Wang, J. H. Richmond, and M. C. Gilreath, "Sinusoidal reaction formulation for radiation and scattering from conducting surfaces," *IEEE Trans. Antennas Propagat.*, vol. AP-23, pp. 376–382, May 1975.

[4] J. H. Richmond and N. H. Geary, "Mutual impedance of nonplanar-skew sinusoidal dipoles," *IEEE Trans. Antennas Propagat.*, vol. AP-23, pp. 412–414, May 1975.

[5] E. H. Newman and D. M. Pozar, "Electromagnetic modeling of composite wire and surface geometries," *IEEE Trans. Antennas Propagat.*, vol. AP-26, pp. 784–789, Nov. 1978.

[6] E. H. Newman and P. Tulyathan, "A surface patch model for polygonal plates," *IEEE Trans. Antennas Propagat.*, vol. AP-30, pp. 588–593, July 1982.

[7] E. H. Newman, "A user's manual for the electromagnetic surface patch code: ESP version IV," Tech. Rep. 716199-11, ElectroScience Lab., Ohio State Univ., Columbus, Aug. 1988.

[8] C. W. Chuang, J. H. Richmond, N. Wang, and P. H. Pathak, "New expressions for mutual impedance of nonplanar-skew sinusoidal monopoles," *IEEE Trans. Antennas Propagat.*, vol. 38, pp. 275–276, Feb. 1990.

[9] M. A. Tilston and K. G. Balmain, "On the suppression of asymmetric artifacts arising in an implementation of the thin-wire method of moments," *IEEE Trans. Antennas Propagat.*, vol. 38, pp. 281–285, Feb. 1990.

[10] F. L. Whetten, K. Liu, and C. A. Balanis, "An efficient numerical integral in three-dimensional electromagnetic field computations," *IEEE Trans. Antennas Propagat.*, vol. 38, pp. 1512–1514, Sept. 1990.

[11] K. Liu, C. A. Balanis, and G. C. Barber, "Exact mutual impedance between sinusoidal electric and magnetic dipoles," *IEEE Trans. Antennas Propagat.*, vol. 39, pp. 684–686, May 1991.

[12] J. J. H. Wang, *Generalized Moment Methods in Electromagnetics*. New York: Wiley, 1991.

[13] E. K. Miller, L. Medgyesi-Mitschang, and E. H. Newman, Eds., *Computational Electromagnetics: Frequency-Domain Method of Moments*. New York: IEEE, 1992.

[14] K. E. Schmidt, "Simplified mutual impedance of nonplanar skew dipoles," *IEEE Trans. Antennas Propagat.*, vol. 44, pp. 1298–1299, Sept. 1996.

[15] *Tables of the Exponential Integral for Complex Arguments*, Appl. Math. Ser. 51. Washington, DC: Nat. Bur. Stand., May 1958.

[16] M. Abramowitz and I. A. Stegun, Eds., *Handbook of Mathematical Functions*, Appl. Math. Ser. 55. Washington, DC: Nat. Bur. Stand., 1964.

[17] Y. L. Luke, *The Special Functions and Their Approximations*, vol. II. New York: Academic, 1969.

[18] ——, *Mathematical Functions and Their Approximations*. New York: Academic, 1975.

[19] D. E. Amos, "Computation of exponential integrals of a complex argument," *ACM Trans. Math. Software*, vol. 16, pp. 169–177, June 1990.

[20] ——, "Algorithm 683: A portable FORTRAN subroutine for exponential integrals of a complex argument," *ACM Trans. Math. Software*, vol. 16, pp. 178–182, June 1990.

[21] S. Zhang and J. Jin, *Computation of Special Functions*. New York: Wiley, 1996.

[22] D. H. Werner, P. L. Werner, J. A. Huffman, A. J. Ferraro, and J. K. Breakall, "An exact solution of the generalized exponential integral and its application to moment method formulations," *IEEE Trans. Antennas Propagat.*, vol. 41, pp. 1716–1719, Dec. 1993.

[23] J. H. Whitteker, "Numerical evaluation of one-dimensional diffraction integrals," *IEEE Trans. Antennas Propagat.*, vol. 45, pp. 1058–1061, June 1997.

Michael S. Kluskens (S'81–M'91) received the B.S. and M.S. degrees in electrical engineering from Michigan Technological University, Houghton, in 1984 and 1985, respectively, and the Ph.D. degree in electrical engineering from The Ohio State University, Columbus, in 1991.

From 1986 to 1991, he was a Graduate Research Associate at the ElectroScience Laboratory, Department of Electrical Engineering, The Ohio State University, where he conducted research on method of moments and chiral media. He has been with the Radar Division of the Naval Research Laboratory, Washington, DC, since 1991 and is currently with the Electromagnetics Section in the Analysis Branch. His primary research interest is in computational electromagnetics with emphasis on method of moments, finite-difference time-domain, and radiation and scattering from large complex structures.