Efficient Computation of the Two-Dimensional Periodic Green's Function

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***Abstract*—An efficient scheme is introduced for computing the two-dimensional periodic Green's function. By using Kummer's method to accelerate the Hankel function series, accurate results can be rapidly obtained when the source and field points coincide in the vertical direction. Unlike with the integral acceleration form, convergence of the series is maintained when the source and field points differ horizontally by a complete period.**

***Index Terms*—Green's functions, periodic structures.**

I. Introduction and Formulation

The computation of the electromagnetic field scattered by a periodic two-dimensional object under plane wave illumination requires use of the "periodic Green's function" (PGF) given by [1]

$$\begin{aligned} G(x-x', y-y') &= \sum_{n=-\infty}^{\infty} e^{-jn\beta D} H_0^{(2)}\left(k\sqrt{(\Delta_x - nD)^2 + \Delta_y^2}\right) \\ &= \frac{2}{D}\sum_{n=-\infty}^{\infty} \frac{e^{-j\beta_n \Delta_x}\, e^{-jq_n|\Delta_y|}}{q_n}. \end{aligned} \tag{1}$$

Here, D is the period length of the object, $\beta = -k\cos\phi_0$, $\beta_n = \beta + 2n\pi/D$, $k = \omega\sqrt{\mu_0\varepsilon_0}$, ϕ_0 is the angle of plane wave incidence measured from the x axis, $\Delta_x = x - x'$, $\Delta_y = y - y'$, and $q_n = -j\sqrt{\beta_n^2 - k^2}$. Note that the branch cut is chosen so that $\text{Im}\{q_n\} < 0$.

The convergence of the Hankel function series in (1) is quite poor for most values of Δ_x and Δ_y. In contrast, the exponential series converges rapidly after n exceeds values satisfying $\sqrt{\beta_n^2 - k^2}\,\Delta_y > 1$ due to exponential decay. However, when Δ_y is small, the sum may require many terms before this condition is met. In particular, when $\Delta_y = 0$ (especially important in the case of a flat surface) the exponential series is very poorly convergent.

Various techniques to accelerate the convergence of the PGF have been developed. Often the series is rewritten in the form of a rapidly convergent integral such as [2]

$$G(x-x', y-y') = G_+ + G_- + H_0^{(2)}\left(k\sqrt{\Delta_x^2 + \Delta_y^2}\right) \tag{2}$$

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where

$$\begin{aligned} G_\pm(x-x', y-y') &= -4\,\frac{e^{-j(kDb_\pm - t_\pm)}}{j\pi} \\ &\cdot \int_0^\infty \frac{e^{-kD(b_\pm+1)u^2}\,\cos\left(k\Delta y\sqrt{u^2+2j}\right)}{(1 - e^{-kDu^2}\,e^{jt_\pm})\sqrt{u^2+2j}}\,du \end{aligned} \tag{3}$$

with $b_\pm = \mp\Delta_x/D$ and $t_\pm = -D(k \pm \beta)$. Note that the integrand in G_+ decays rapidly only when $k(D - \Delta_x)u^2 > 1$. Thus, when $\Delta_x \approx D$ the integral converges slowly.

In this paper, we choose to use Kummer's method [3] to accelerate the series. This involves subtracting off an asymptotic form, thus producing a series which is more rapidly convergent. The technique is efficient only when the asymptotic series can be summed efficiently (preferably in closed form). Applying this to the Hankel function series in (1) gives

$$\begin{aligned} G(x-x', y-y') &= H_0^{(2)}\left(k\sqrt{\Delta_x^2+\Delta_y^2}\right) \\ &+ \sum_{n=1}^{\infty}\Big\{\Big[e^{-jn\beta D}\,H_0^{(2)}\left(k\sqrt{(nD-\Delta_x)^2+\Delta_y^2}\right) \\ &\qquad - A_n^-(\Delta_x, \Delta_y)\Big] \\ &\qquad + \Big[e^{jn\beta D}\,H_0^{(2)}\left(k\sqrt{(nD+\Delta_x)^2+\Delta_y^2}\right) \\ &\qquad - A_n^+(\Delta_x, \Delta_y)\Big]\Big\} \\ &\qquad + \sum_{n=1}^{\infty} A_n^-(\Delta_x, \Delta_y) + \sum_{n=1}^{\infty} A_n^+(\Delta_x, \Delta_y). \end{aligned} \tag{4}$$

Since we seek an expression for $A_n^\pm$ valid for large n, we assume that $nD \gg \Delta_x, \Delta_y$ and expand the square root using the binomial theorem as

$$\begin{aligned} \sqrt{(nD\pm\Delta_x)^2+\Delta_y^2} &= nD \pm \Delta_x + \frac{\Delta_y^2}{2nD} \mp \frac{\Delta_x\Delta_y^2}{2(nD)^2} \\ &\quad + \frac{\Delta_x^2\Delta_y^2}{2(nD)^3} - \frac{\Delta_y^4}{8(nD)^3} + \cdots. \end{aligned} \tag{5}$$

Substituting (5) into the asymptotic expansion of the Hankel function [4]

$$H_0^{(2)}(z) = \sqrt{\frac{2j}{\pi z}}\, e^{-jz}\left[1 + j\,\frac{1}{8z} - \frac{9}{128z^2} - j\,\frac{225}{3072z^3} + \frac{11\,025}{98\,304z^4} + \cdots\right] \quad (6)$$

gives the general expression

$$A_n^{\pm} = \sqrt{\frac{2j}{\pi kD}}\, e^{\pm jk\Delta_x}(Z_0^{\pm})^n \sum_{m=0}^{M-1} \frac{\xi_{m+1/2}(\Delta_x, \Delta_y)}{n^{m+1/2}} \quad (7)$$

where $Z_0^{\pm} = \exp\{-jkD(1 \pm \cos\phi_0)\}$.

The convergence rate of (4) depends on the number of terms M we keep in the expansion (7). The formulas for $\xi_{m+1/2}^{\pm}$ for the first four terms in the expansion are given by

$$\begin{aligned}
\xi_{1/2}^{\pm} &= 1 \\
\xi_{3/2}^{\pm} &= \left[\frac{j}{4k} \mp \Delta_x - jk\Delta_y^2\right](2D)^{-1} \\
\xi_{5/2}^{\pm} &= \left[-\frac{9}{8k^2} \mp \frac{3j}{k}\,\Delta_x + 6\Delta_x^2 - 3\Delta_y^2 \right. \\
&\quad \left. \pm\, 12jk\Delta_x\Delta_y^2 - 2k^2\Delta_y^4\right](16D^2)^{-1} \\
\xi_{7/2}^{\pm} &= \left[-\frac{225j}{12k^3} \pm \frac{45}{k^2}\,\Delta_x + \frac{60j}{k}\,\Delta_x^2 - \frac{15j}{k}\,\Delta_y^2 \right. \\
&\quad \mp 80\Delta_x^3 \pm 120\Delta_x\Delta_y^2 - -304jk\Delta_x^2\Delta_y^2 + 60jk\Delta_y^4 \\
&\quad \left. \pm\, 96k^2\Delta_x\Delta_y^4 + \frac{16jk^3}{3}\,\Delta_y^6\right](256D^3)^{-1}.
\end{aligned} \quad (8)$$

The last step is to compute $\sum A_n^{\pm}$. The infinite series is in the form of the "Lerch transcendent" $\Phi(z, s, \nu)$ which also has an integral representation [5]

$$\Phi(z, s, \nu) = \sum_{n=0}^{\infty} (\nu + n)^{-s} z^n = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}\, e^{-(\nu-1)t}}{e^t - z}\, dt. \quad (9)$$

Using this notation, the series can be written as

$$\sum_{n=1}^{\infty} A_n^{\pm}(\Delta_x, \Delta_y) = \sqrt{\frac{2j}{\pi kD}}\, e^{\pm jk\Delta_x} Z_0^{\pm} \cdot \sum_{m=0}^{M-1} \xi_{m+1/2}^{\pm}(\Delta_x, \Delta_y)\Phi(Z_0^{\pm}, m+1/2, 1) \quad (10)$$

where M is the number of terms kept in the representation of $A_n^{\pm}$.

There is a very significant benefit to using the series-accelerated PGF (4) in place of either the original exponential-series PGF (1) or the integral-accelerated PGF (3). Both the exponential-series PGF (PGFES) and the integral-accelerated PGF (PGFIA) are dependent on the source point-field point-differences Δ_x and Δ_y. When these formulas are used in moment method-based applications, the series or integral must be recomputed as Δ_x and Δ_y change. In contrast, the Lerch transcendents involved in the series-accelerated PGF (PGFSA) are independent of Δ_x and Δ_y, and thus need only be computed once (for each value of frequency, incidence angle, and period width). This can result in a very large savings in computation time.

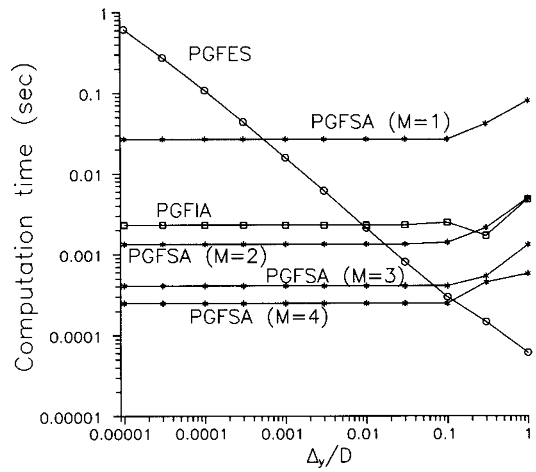

Fig. 1. Dependence of computation time on vertical separation of source and field points. $D/\lambda = 1$, $\Delta_x/D = 0.5$, $\phi_o = 45°$.

Computation of the Lerch transcendent $\Phi(z, s, \nu)$ is fairly straightforward. For $s \geq 3/2$ the series converges rapidly, and there is no need to use the integral representation. For $s = 1/2$ the integral form is much more efficient to compute. By using a change of variables, the integral can be put into the more computer-friendly form

$$\Phi(z, 1/2, 1) = \frac{1}{1-z} + \frac{1}{\sqrt{\pi}}\,\frac{z}{1-z}\int_0^1 \frac{u-1}{1-zu}\,\frac{du}{\sqrt{-lnu}}. \quad (11)$$

II. Numerical Implementation

Examples of the time required to compute the PGF to five significant digits using a 200-MHz Pentium-Pro microcomputer are shown in Figs. 1 and 2. Obviously, the time needed to compute the PGFIA integral (3) depends upon the integration method; in this work, we used the Romberg integration routine from [6]. Also, since the Lerch transcendent is computed once per moment-method solution, its computation time is not included in the PGFSA results shown in the figures.

Fig. 1 shows that as the vertical separation between the source and field point is reduced, the original exponential-series PGF becomes more time consuming to compute. Neither the series-accelerated nor integral-accelerated techniques have this problem, but the series-accelerated technique is significantly faster when the number of terms chosen in the asymptotic expansion is greater than two. In the series-accelerated technique, the vertical distance can be safely set to zero. It is important to note that when the vertical separation is greater than the period D, the series-accelerated technique becomes unreliable due to roundoff error. However, for values of vertical separation greater than about $0.1D$ it is more efficient to use the original exponential series PGF.

In Fig. 2 it is seen that the integral-accelerated technique becomes very time consuming as the horizontal separation

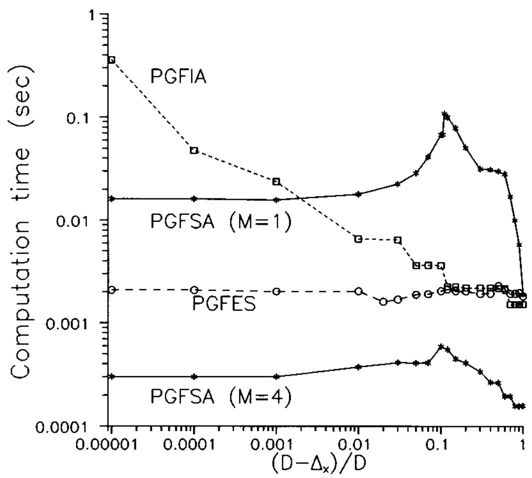

Fig. 2. Dependence of computation time on horizontal separation of source and field points. $D/\lambda = 1$, $\Delta_y/D = 0.01$, $\phi_o = 45°$.

approaches the period D. The series-accelerated solution does not have this problem.

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E. J. Rothwell (S'84–M'85–SM'92), for a photograph and biography, see p. 1278 of the September 1998 issue of this TRANSACTIONS.

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