A Multilevel Formulation of the Finite-Element Method for Electromagnetic Scattering

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***Abstract*— Multigrid techniques for three-dimensional (3-D) electromagnetic scattering problems are presented. The numerical representation of the physical problem is accomplished via a finite-element discretization, with nodal basis functions. A total magnetic field formulation with a vector absorbing boundary condition (ABC) is used. The principal features of the multilevel technique are outlined. The basic multigrid algorithms are described and estimations of their computational requirements are derived. The multilevel code is tested with several scattering problems for which analytical solutions exist. The obtained results clearly indicate the stability, accuracy, and efficiency of the multigrid method.**

***Index Terms*— Electromagnetic scattering, finite-element method, multigrid, multilevel numerical techniques.**

I. Introduction

THE finite-element method (FEM), has been successfully applied in electromagnetism for the solution of problems with arbitrary conductor geometries as well as dielectric inhomogeneities. The most computationally intensive part of the method is the solution of the resulting system of equations. In this context, direct methods are the most robust; but iterative techniques like the preconditioned conjugate gradient method and its variants exhibit higher efficiency because of the sparsity of the finite-element matrices. However, even with the use of advanced iterative methods, the operation count for a finite-element program is extremely high for electrically large three-dimensional (3-D) bodies. A solution to this problem lies in the utilization of multigrid formulations.

Multigrid should not be considered as a single method or even a family of methods, but rather as an entire approach to computational engineering a collection of ideas and attitudes. Multigrid is a prime source of important algorithmic efficiency and its popularity is rapidly increasing. This is because unlike other known methods, multigrid offers the possibility of solving problems with N unknowns with O(N) work and storage, not just for special cases but for large classes of problems. As their name implies, multigrid techniques use a sequence of increasingly finer nested grids. The essential multigrid principle is the approximation of the smooth (long wavelength) part of the error on the coarser grids. The nonsmooth or rough part is reduced with a small number of iterations (independent of N) of a basic iterative method on a fine grid.

The first "true" multigrid publication was by Fedorenko [1], but it was Brandt who clearly outlined the main principles and the practical utility of the multigrid methods [2], [3]. Equally important is the contribution of Hackbush, who laid firm mathematical foundations and provided reliable methods [4], [5]. In the domain of electromagnetism, Kalbasi and Demarest applied the multilevel concepts to the method of moments with impressive results [6]. Ye *et al*. used a multilevel finite-difference method for the Helmholtz wave equation [7]. Herring and Christopoulos introduced the use of multigrid techniques in the transmission line matrix (TLM) method [8]. Costiner *et al.* [9] presented a multilevel formulation of the finite-difference method for the computation of the modes and eigenvalues of resonant cavities. Wang and Fang proposed a multilevel implementation of the diakoptic method [10]. Goverdahanam *et al.* derived a novel multigrid scheme for the finite-difference time-domain (FDTD) technique, by applying the principles of multiresolution analysis [11]. For the efficient finite-element solution of two-dimensional magnetostatic problems, Tsukerman developed a family of multilevel preconditioners [12].

The central theme of this publication, is the incorporation of the multigrid principles to the formulation of a 3-D FEM for electromagnetic scattering. To this end the next section starts with the presentation of the corresponding single-level formulation. Then a short introduction to the fundamental ideas behind multigrid development is given. Section IV provides the description of the most basic multigrid algorithms. Section V deals with the important subject of intergrid transfer operators. In Section VI, implementation details and computational cost estimations are presented. The last two sections contain numerical results and conclusions.

In the following an $e^{j\omega t}$ time dependence is assumed for the field quantities and it is suppressed throughout the remaining sections.

II. Single-Level Formulation

The partial-differential equation, that is to be solved using the FEM, is the well known curl–curl equation for the total

Manuscript received June 2, 1997; revised November 18, 1998.

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Publisher Item Identifier S 0018-926X(99)05820-2.

magnetic field

$$\nabla \times \left(\frac{1}{\varepsilon_r} \nabla \times \overline{\mathbf{H}} \right) - k_o^2 \mu_r \overline{\mathbf{H}} = 0. \tag{1}$$

For the application of the FEM for unbounded problems like those of electromagnetic scattering, the truncation of the computational domain with an artificial boundary S_o is necessary. On this boundary the enforcement of suitable absorbing boundary conditions (ABC's) takes place. These boundary conditions suppress nonphysical reflections from that boundary, thus ensuring the outgoing nature of the waves. The ABC's used in this formulation are the Sommerfeld radiation condition

$$\hat{\mathbf{n}} \times \nabla \times \overline{\mathbf{H}}^S = -jk_o \hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \overline{\mathbf{H}}^S \tag{2}$$

and the second-order ABC [13]

$$\hat{\mathbf{n}} \times \nabla \times \overline{\mathbf{H}}^S = \alpha \overline{\mathbf{H}}_t^S + \beta \nabla \times [\hat{\mathbf{n}} (\nabla \times \overline{\mathbf{H}}^S)_n] + \beta \nabla_t (\nabla \cdot \overline{\mathbf{H}}_t^S) \tag{3}$$

where $\alpha = jk_o$, $\beta = 1/(2jk_o + 2/r)$, $\overline{\mathbf{H}}^S$ represents the scattered magnetic field, $\hat{\mathbf{n}}$ is the unit normal to the surface S_o, and the subscripts t and n denote the transverse and normal component to S_o, respectively.

It is possible to combine (2) and (3) in the more concise expression

$$\hat{\mathbf{n}} \times \nabla \times \overline{\mathbf{H}}^S = P(\overline{\mathbf{H}}^S). \tag{4}$$

For the total field formulation where the unknown is the total magnetic field, the above relation takes the form

$$\hat{\mathbf{n}} \times \nabla \times \overline{\mathbf{H}} = P(\overline{\mathbf{H}}) + \hat{\mathbf{n}} \times \nabla \times \overline{\mathbf{H}}^{\text{inc}} - P(\overline{\mathbf{H}}^{\text{inc}}) \tag{5}$$

where $\overline{\mathbf{H}}^{\text{inc}}$ is the incident magnetic field.

Using the curl–curl equation and the above ABC and through the application of a Galerkin weighted residual procedure, the following weak form is obtained:

$$\int_V \left\{ \frac{1}{\varepsilon_r} (\nabla \times \overline{\mathbf{H}} \times \nabla \phi_i) + \frac{1}{\varepsilon_r} (\nabla \cdot \overline{\mathbf{H}}) \nabla \phi_i - k_o^2 \mu_r \overline{\mathbf{H}} \phi_i \right\} dV = -\oint_{S_o} \{ P(\overline{\mathbf{H}}) \phi_i + \hat{\mathbf{n}} \times \nabla \times \overline{\mathbf{H}}^{\text{inc}} \phi_i - P(\overline{\mathbf{H}}^{\text{inc}}) \phi_i \} \, dS. \tag{6}$$

This expression has been shown to eliminate spurious modes [14]. Tetrahedral elements, with ϕ_i being the standard volume coordinates that have closed-form exact integration formulas on each tetrahedron, are used. By substituting in (6) the expansion of the total magnetic field in terms of ϕ_i inside each element $\overline{\mathbf{H}} = \sum_{j=1}^{N} \overline{\mathbf{H}}_j \phi_j$, the following linear system of equations is derived:

$$Ax = b. \tag{7}$$

The coefficient matrix of this system is complex symmetric and highly sparse. Only the nonzero elements of its upper triangular part are stored using a compressed row storage format. The elements of vector x are the Cartesian components of the magnetic field corresponding to the nodes of the mesh being used.

Closely related to (7) and of extreme importance to multigrid algorithms is the residual equation of (7)

$$Ae = r \tag{8}$$

where for an approximation $\tilde{x}$ to the exact solution x, $e = x - \tilde{x}$ is the error of $\tilde{x}$ and $r = b - A\tilde{x}$ is the corresponding residual.

III. Nested Iteration and Coarse Grid Correction

One way to check the convergence of a finite-element scheme is to compare the computed results for increasingly denser meshes. It is possible to use this comparison to compute an estimate of the discretization error of the scheme [15]. Usually, one gets a finer grid by uniformly shrinking the mesh size. However, a more efficient option is adaptive refinement with *a posteriori* error estimation [16]. In both cases, the result is a sequence of increasingly finer grids. As the number of unknowns increases, the accuracy of the finite-element approximations improves but the price is a much higher computational cost. The idea behind multigrid is to exploit interactions between the different discretization spaces, thus keeping the computational cost as low as possible.

The spaces of discretized functions corresponding to different grids are called levels—level 1 being the coarsest. At each level, the solution of a sparse linear system is required and iterative methods are the most efficient choice [17]. A simple way to speed up the convergence of these methods is to use a good initial guess. A well-known technique for obtaining a improved initial guess at a given level is to solve the problem at a coarser grid and then project in some way this approximation on the original level. The solution at a coarser level is significantly cheaper since there are fewer unknowns and the convergence rate of the iterative techniques is better. This procedure is known as nested iteration or one way multigrid and its application can be performed recursively until the coarsest level is reached. The additional cost for solving the systems associated to the coarser grids is justified by having quicker convergence for the actually sought one.

The solution at each level is composed of fast and slow spatial variations. The cause of the fast variations are local iterations in geometry, while the slow components arise from global interactions. Many iterative methods tend to damp out the fast components of the error quickly, but their convergence rate slows down considerably in the presence of the remaining smooth components. However, the smooth components of one level seem more oscillatory on a coarser one. These important observations have led people to develop the coarse grid correction strategy.

The basic idea is simple and elegant. At a given level, some steps of an iterative method are performed, in order to smooth out the error. This is called presmoothing. When convergence deteriorates, the current residual equation is restricted to a coarser grid. There the resulting projected problem is solved, to obtain an approximation of the error. Then the error estimate is interpolated back to the original grid. By adding this estimation of the error to the previous approximation of the solution at the fine level, a new improved approximation is obtained. This is the coarse grid correction step.

The new approximation contains slow mode information not easily obtained at the fine level. It also contains fast mode errors introduced by the interpolation process. However, these oscillatory errors can be easily eliminated with a few steps of the iterative method at the fine level. This is called this postsmoothing. The correction step can be thought of as a fancy form of iterative improvement [19] using an approximate smaller system of linear equations. Alternately, the combination of presmoothing and coarse grid correction can be seen as a preconditioner for the postsmoothing fine iterations.

It is important to appreciate the complementarity, which is at work in the process. Fine grid iterations eliminate the oscillatory error components, leaving a relatively smooth error. By solving the residual equation on a coarse level where the projected error appears fast, it is possible to deal with the problematic on the fine level slow modes very efficiently. The three steps of the coarse grid correction scheme, can be repeated many times if it is necessary, but usually only a few cycles are enough to arrive at a satisfactory fine level approximation.

It is evident, that the coarse grid problem is not much different from the original one. Therefore, the coarse grid correction scheme can be also applied to the residual equation on the coarse level, which means moving to an even coarser grid for the new correction step. This process can continue recursively, visiting successively coarser grids until a direct solution of the residual equation is negligible at the coarsest level.

Of course, the actual mathematical theory goes far beyond the above simple interpretation as well as beyond the scope of the present paper. Our main objective is to examine possible applications of the multigrid algorithms to electromagnetic scattering.

IV. Multigrid Algorithms

It is convenient to organize the ideas of the previous section in an algorithmic form. Suppose a hierarchy of M discretization levels. At each level a sparse linear system has to be solved

$$A_k x_k = b_k, \qquad k = 1, 2, \cdots, M. \tag{9}$$

The application of an iterative solver on a given level k is represented by $S_k(x_k, b_k)$. This solver is usually a smoother (like Gauss–Seidel), but it can also be a rougher (symmetric successive overrelaxation (SSOR) or conjugate gradients). Only on the coarsest level is (9) solved exactly. This is often done with a direct method (such as LU decomposition). On the rest of the levels, only a few iterations (or even one) are performed each time the iterative solver is called.

P_k denotes the prolongation operator, that interpolates the approximation of the solution or the error from level k to $k+1$. R_k symbolizes the restriction operator that projects the residual from level k to the previous coarser one $k-1$.

There are two principal variants of multigrid algorithms [18]. The first is composed of correction schemes, which start on some level j and only use the coarser levels $k < j$ for solving residual correction problems. Its most elegant description is by means of a recursive formulation

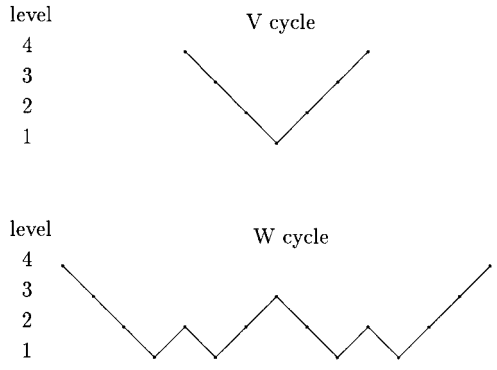

Fig. 1. V ($\mu_k = 1$) and W ($\mu_k = 2$) cycle.

ALGORITHM MGC(k, $\{\mu_\ell\}_{\ell=1}^{k}$, x_k, b_k)
(1) If $k = 1$, then solve $A_1 x_1 = b_1$ exactly.
(2) If $k > 1$, then repeat $i = 1, \cdots, \mu_k$:
(2a) Approximately solve: $x_k \leftarrow S_k^A(x_k, b_k)$.
(2b) Residual Correction:
$b_{k-1} \leftarrow R_k(b_k - A_k x_k)$.
$x_k \leftarrow x_k + P_{k-1}$ MGC($k-1$, $\{\mu_\ell\}_{\ell=1}^{k-1}$, 0, b_{k-1}).
(2c) Approximately solve: $x_k \leftarrow S_k^B(x_k, b_k)$.
(3) Return x_k.

The above definition requires $\mu_1 = 1$. Symmetric multigrid schemes assume that $S_k^A = S_k^B$ and nonsymmetric multigrid schemes usually assume that $S_k^B = I$ where I is the identity operator. However, it is computationally more efficient to assume $S_k^A = I$ since the residual on level $k-1$ is b_{k-1} and does not need to be recomputed.

The second multigrid variant presented utilizes nested iteration schemes, which begin computation on level 1 and work their way to some level j using each level k, $k < j$ both to generate an initial guess for level $k+1$ and for solving residual correction problems

ALGORITHM NIC(j, $\{\mu_k\}_{k=1}^{j}$, x_1, $\{b_k\}_{k=1}^{j}$)
(1) For $k = 1, \cdots, j$, do:
(1a) If $k > 1$, then $x_k \leftarrow P_{k-1} x_{k-1}$.
(1b) $x_k \leftarrow$ MGC(k, $\{\mu_\ell\}_{\ell=1}^{k}$, x_k, b_k).
(2) Return x_j.

The order in which the levels are visited is called the multigrid schedule or multigrid cycle. There are two kinds of schedules—fixed and adaptive. Fixed schedules have their parameters μ_k constant. The term adaptive signifies schedules for which the μ_k parameters depend on intermediate computational results. Fig. 1 shows the order in which MGC visits the levels when $\mu_k = 1$ and $\mu_k = 2$ in the case $M = 4$. A dot represents an approximate solution. Because of the shape of these diagrams, these schedules are called the V and W cycles, respectively. Another commonly used schedule is the sawtooth cycle. This is the name for a special case of the V cycle in which the approximate solution is omitted before the coarse grid correction (presmoothing).

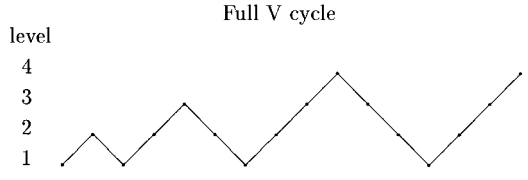

Fig. 2. Full V multigrid.

A very efficient combination is what is called full multigrid [20], which corresponds to algorithm nested iteration correction (NIC) with $\mu_k = 1$. Fig. 2 illustrates full multigrid for $M = 4$. If $\mu_k = 0$, then the NIC algorithm describes what is known as one-way multigrid.

V. Intergrid Transfer Operators

Intergrid transfer operators are mechanisms for communicating information between the levels. They can be thought of as mappings between the neighboring spaces of adjacent levels. For the sake of simplicity in the following discussion, only the case in which each coarse grid has twice the grid spacing of the next finer one is considered. This is known as standard coarsening. The general case is, of course, more complicated, but the basic principles are the same.

When a transition from a coarse level k to the next finer one is necessary, a prolongation operator P_k has to be used. Prolongation operators are interpolation procedures. Many interpolation methods can be used. The degree of interpolation needed depends on the smoothness of the function to be interpolated. Usually linear interpolation is sufficient. In three dimensions trilinear interpolation [exact for $f(x_1, x_2, x_3) = 1, x_1, x_2, x_3, x_1x_2, x_2x_3, x_1x_3, x_1x_2x_3$], approximates fine grid values of a function u using the relations

$$\begin{aligned}(P_k u)_{2i} &= u_i \\ (P_k u)_{2i+e_1} &= \tfrac{1}{2}(u_i + u_{i+e_1}) \\ (P_k u)_{2i+e_1+e_2} &= \tfrac{1}{4}(u_i + u_{i+e_1} + u_{i+e_2} + u_{i+e_1+e_2}) \\ (P_k u)_{2i+e_1+e_2+e_3} &= \tfrac{1}{8}(u_i + \Sigma_{\alpha=1}^{3} u_{i+e_\alpha} + u_{i+e_1+e_2} \\ &\quad + u_{i+e_1+e_3} + u_{i+e_2+e_3} + u_{i+e_1+e_2+e_3})\end{aligned} \tag{10}$$

where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$. In the above expressions, the right-hand sides are coarse level values, while stencil notation is used. Linear interpolation in three dimensions takes place in tetrahedra and it is cheaper because of greater sparsity. The reason trilinear interpolation is preferable in practice is because, unlike linear interpolation, it preserves symmetry exactly.

When information from a fine grid has to be passed to a coarse one, a restriction operator R_k is used. The most obvious restriction operator is injection. During injection each element of the coarse grid residual vector simply takes its value from the corresponding fine grid node

$$(R_k u)_j = u_{2j}. \tag{11}$$

An alternate restriction operator is called full weighting. As its name implies, it approximates a coarse residual value using a weighted average of nearby fine residuals. The full weighting operator is a very popular choice in multigrid literature because for it, the following important relation holds:

$$R_k = cP_{k-1}^T, \qquad c \in \mathbb{R}. \tag{12}$$

The corresponding multilevel procedure is usually referred to as a Galerkin multilevel formulation. For a Galerkin formulation, it is possible to compute the elements of the coefficient matrix of a coarse level k using the fine level matrix A_{k+1} and the intergrid transfer operators P_{k-1} and R_k. This is known as the Galerkin coarse grid approximation

$$A_k = R_{k+1} A_{k+1} P_k. \tag{13}$$

The above identity and the transposition relation between R_k and P_{k-1} are called the variational properties of the multilevel scheme. The grid transfer operators must satisfy the following accuracy requirement [21]:

$$m_P + m_R > 2m \tag{14}$$

where m_R and m_P are the order of the restriction and the prolongation, respectively, and $2m$ is the order of the differential operator.

VI. Implementation Aspects and Computational Costs

This section discusses briefly some practical issues of multigrid implementation. The most basic characteristic of our multigrid program is its high modularity. This means that its various components (for example, iterative or direct solvers, interpolation or restriction subroutines) can be replaced individually. While the use of recursion makes the description of multigrid algorithms easier and more elegant, it is by no means necessary. Our program employs a nonrecursive implementation of the multigrid algorithms, based on the structure diagrams given in [21].

The construction of suitable data structures for multilevel computations is of extreme importance. The solution and the right-hand side vectors of the various grids are stored contiguously in single arrays. The same is done for the mesh data structures and the sparse storage vectors of each level. The storage requirements of a specific level are proportional to the number of its grid nodes. If the finer level M has N nodes, $M-1$ has approximately 2^{-3} times as many (remember standard coarsening was assumed in the previous section), and every lower level has about 2^{-3} times the nodes of its finer predecessor. Adding the storage requirements of all the levels and using the sum of the geometric series as an upper bound gives us that

$$\begin{aligned}\text{Storage} &= cN\{1 + 2^{-3} + 2^{-6} + \cdots + 2^{-3(M-1)}\} \\ &= cN \sum_{k=1}^{M} 2^{-3(M-k)} < cN \frac{1}{1 - 2^{-3}} \\ &= cN \tfrac{8}{7}.\end{aligned} \tag{15}$$

Thus, the storage costs of multigrid algorithms in three dimensions for standard coarsening, is less than 8/7 of the fine grid problem's alone. It is evident that the additional storage requirement posed by multigrid is modest.

Similar reasoning can be used to estimate the computational costs of the multigrid algorithms. It is convenient to measure these costs in terms of work units (WU). One WU is usually defined as the amount of computing work required to evaluate the residual $r_M = b_M - Ax_M$ of (7) on the finest level M. In practice, the work involved in smoothing is by far the dominant part of the total work. Therefore, another definition for the WU is as the cost of performing one iteration of the solver on the finest grid. The two definitions agree with each other more or less, when the iterative solver is simple and cheap. A WU can be thought of as the cost of expressing the fine grid operator. As it is customary in multigrid literature, for simplicity in the following analysis, the cost of intergrid transfer operations is neglected. This amounts to less than 15% of the cost of an entire multigrid cycle.

During a V cycle if only one iteration of the solver is performed, each time the solver is called the cost at the fine level is two WU's. Each of the lower levels is visited twice. Every smoothing iteration on level k adds approximately $2^{3(k-M)}$ WU to the total work. The exact solution by a direct method at the coarsest level is considered negligible and that is the case in practice. Adding the costs of all the levels and again using the geometric series for an upper bound the result is

$$\begin{aligned} V \text{ cycle cost} &= 2\{1 + 2^{-3} + 2^{-6} + \cdots + 2^{-3(M-1)}\} \\ &= 2\sum_{k=1}^{M} 2^{-3(M-k)} < \frac{2}{1-2^{-3}} \\ &= \tfrac{16}{7} \text{ WU}. \end{aligned} \tag{16}$$

In the same way it is found that a single W cycle costs 8/3 WU for a 3-D problem. For a sawtooth cycle the cost is 8/7 WU. With a slight modification, the computational cost of a full multigrid V cycle can be estimated. A V cycle costs 16/7 WU on the fine level M. Each V cycle on every lower level k costs about $2^{3(k-M)}$ as much. Adding all these costs gives us

$$\begin{aligned} \text{Full } V \text{ cost} &= \tfrac{16}{7}\{1 + 2^{-3} + 2^{-6} + \cdots + 2^{-3(M-1)}\} \\ &= \tfrac{16}{7}\sum_{k=1}^{M} 2^{-3(M-k)} < \tfrac{128}{49} \text{ WU}. \end{aligned} \tag{17}$$

It is natural to expect that the fine grid problem cannot be solved at a cost less than a few WU and one should be content if this realized. Multigrid algorithms can achieve this goal because they require only a few cycles to converge. Multigrid convergence analysis is a difficult and open area of computational mathematics and it will not be dealt theoretically in this publication. However, in the next section, a wealth of computational evidence is provided that testifies to the general effectiveness of multigrid methods in electromagnetic scattering.

VII. Computational Experiments

To demonstrate the efficiency of the multilevel algorithms a series of numerical experiments were performed on the silicon graphics power challenge computer. The code was written in FORTRAN and double-precision complex arithmetic was used.

The approximation obtained from a numerical technique was considered sufficiently close to the true solution of the problem if at the finest level the residual r for this approximation satisfied the stopping criterion [22]

$$\|r\| \le e \cdot (\|A\| \cdot \|x\| + \|b\|) \tag{18}$$

where $\|\cdot\|$ denoted matrix or vector norm. Tolerance was set to $e = 10^{-6}$ since lower values of it did not offer significant improvements.

The first problem considered was the scattering of a plane wave by a dielectric sphere. This problem was chosen because an analytical solution exists for it and it can be used to check the accuracy of our code. The sphere modeled had relative electric permittivity $\varepsilon_r = 2$ and radius $a = 0.5$ m. The incident plane wave was x polarized, z traveling, and had frequency $f = 90$ MHz. The Sommerfeld ABC was enforced on a artificial spherical boundary placed at a distance $0.3\lambda_o$ from the surface of the sphere.

On the finest level a total of 15 625 nodes were used, resulting in 46 785 unknowns. The nodal spacing at this level was 12.5 cm. The curvature based criterion derived in [23], was employed for the choice of this sampling rate. Although it led to satisfactory results, it must be admitted that the accuracy of the finite-element approximation can be further increased if a higher discretization density per minimum wavelength like those in [17], [24], and [25] is utilized.

A three-level scheme was composed for the solution of this problem. The number of nodes at the coarsest level was only 343 (1029 unknowns), while at the intermediate level there were 2197 nodes and 6591 unknowns.

It is desirable to keep the problem at level 1 as small as possible, but it was found out by experimentation that if a sampling rate of less than about five nodes per wavelength was used, serious problems with multigrid convergence arose. The reason for this must be the excessive misrepresentation of the finer level oscillatory modes at the lowest level. These problems can be avoided by adopting a discretization density of not less than five nodes at the coarsest grid.

The prolongation operator used was trilinear interpolation. The restriction operator was full weighting. At the coarsest level sparse LU decomposition was employed. The factorization of the system matrix at this level was needed only once at the beginning. After this factorization had taken place, every time the coarsest level was visited, only the very cheap forward and backward substitutions had to be performed.

In Fig. 3 the convergence of the conjugate orthogonal gradient method (COCG) [26] and the one way multigrid with COCG as fine level solver are compared. It is obvious that the latter converged earlier, requiring only 194 fine level iterations while the single level method needed 620. In terms of central processing unit (CPU) time the three-level one way multigrid was more than two times faster from COCG alone.

The COCG was used, because it turned out to be the fastest single-level method that converged, for our problem. The biconjugate gradient stabilized method (BiCGStab) satisfied the stopping criterion in less iterations, but as the COCG required only about half the amount of work per iteration, the CPU time the latter took to converge was clearly shorter. It is interesting that the initial residual of the one way multigrid is larger (almost double) from the corresponding residual of COCG. This is due to errors induced by the interpolation process

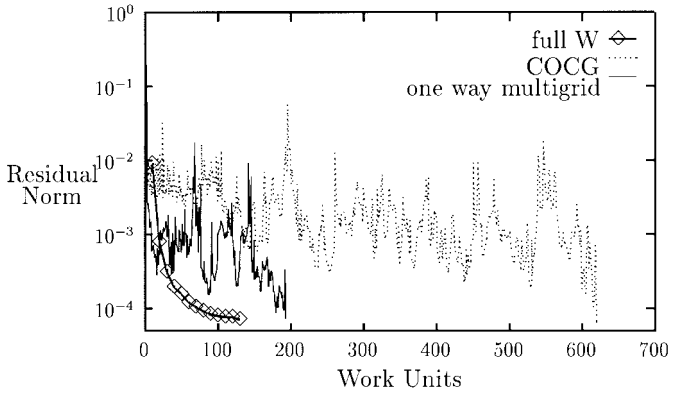

Fig. 3. Convergence comparison of a one-way three-level multigrid the COCG alone and a three level full W scheme.

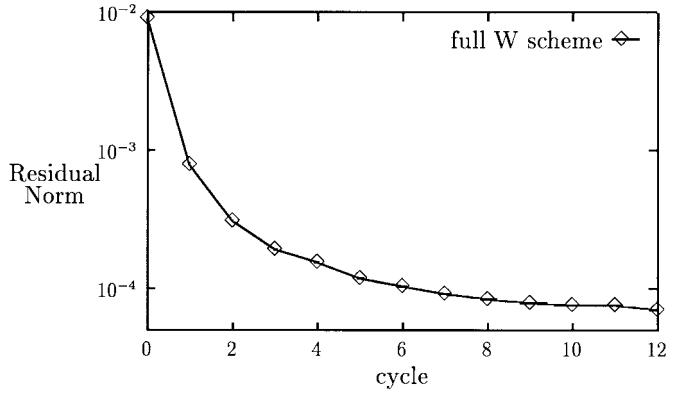

Fig. 4. Convergence history of a three-level full W scheme.

during the nested iteration steps. These errors are fast and thus easily eliminated by the iterative solver at the fine level.

Another noteworthy fact is the highly irregular convergence of COCG, that also characterizes the one way multigrid that uses COCG as its solver. The reason for this is that the COCG implicitly computes and uses the lower diagonal upper (LDU) factorization of an indefinite tridiagonal matrix arising from the underlying Lanczos process. Since no pivoting is used, there is the possibility of encountering small (or even zero) pivots in this factorization. When a small pivot is encountered, typically, the residual norm increases by a large amount on one iteration only to be reduced by a similar amount on the next step, creating a "spike" in the convergence history. Such "spikes" can cause large cancellation errors, and can threaten the numerical stability of COCG. One-way multigrid with COCG as fine level solver also suffers from these "spikes," but their size and especially their number is reduced and as a result the errors they introduce are also reduced.

Even faster than one way multigrid is the "true" multigrid that uses coarse grid corrections. In Fig. 4 the residual norms at the finest level of a "true" multigrid are plotted for every cycle. The cycling scheme used was the full W cycle. In the intermediate and the finer level BiCGStab was used as a solver and two iterations of it were executed each time it was called. The fact that for a problem with 46 875 unknowns only 12 cycles were sufficient for its solution is a clear indication of the high efficiency of the multilevel technique.

The regular and smooth convergence of the full W scheme is clearly obvious. It was further observed that the residual initially (for the first couple of cycles) reduced rapidly, but then its reduction rate slowed down. This happened because as the multigrid process converged to the exact solution on the finer level, the residual error became increasingly smoother. As a result, the remaining extremely slow modes of the error appeared less and less oscillatory on the coarser levels and their elimination there became more difficult with every cycle. However, this situation is not a problem for multigrid, because these slow modes amount to a very small percentage of the total error and can be efficiently eliminated if the multigrid procedure is carefully implemented. This means that the slowing down of the multigrid convergence can be used as an indication that the exact solution is very close. This was verified many times in practice.

Fig. 3 compares the convergence of the full three-level W scheme and the COCG. The performance of the multilevel scheme is evidently superior. The computational cost of it is 131 WU against the 620 WU that are required by COCG. This means that the full W scheme is almost five times faster.

When instead of the Sommerfeld radiation condition the second-order ABC (3) was applied, the full W scheme took 18 cycles to satisfy the stopping criterion. It should be noticed that the solution obtained with this ABC did not appear to be significantly improved in accuracy over the solution of the first-order condition.

The full V multigrid schedule was also tried, for the initial problem (with the Sommerfeld condition). Its performance was also very satisfactory and comparable with that of the full W scheme that proved slightly superior. With the V schedule 16 cycles were needed and 155 WU.

Furthermore, the use of other solvers (apart from BiCGStab) was examined at the various levels. Stationary methods (like Gauss–Seidel) typically led to fast divergence. These methods did not converge for the system of our problem, even on a single level context. Row projection methods such as Kaczmarz iteration [27] had, as a result, very slow reduction of the residual and often stagnation. The method of conjugate gradients on the normal equations (CGNE) showed a better behavior, but it could not compete with BiCGStab as it was characterized by slower convergence and so its use as a solver is not recommended. Finally, the trial of COCG led to divergence the multigrid procedure, probably because of its highly irregular convergence behavior.

One technique that worked very well as a solver was Gauss–Seidel with minimal residual [Orthomin(1)] acceleration (GSMR). Although this method alone stagnated for our problem when it was incorporated in a full W scheme, 27 cycles were sufficient for the satisfaction of the termination criterion. The corresponding computational cost was only 140 WU. The reduction rate of the residual at the initial cycles, was even greater than the one achieved with the use of BiCGStab, which required double computational work per iteration. It appears that it would be a good idea to use the GSMR as a solver for the first few cycles and then switch to BiCGStab for the remaining.

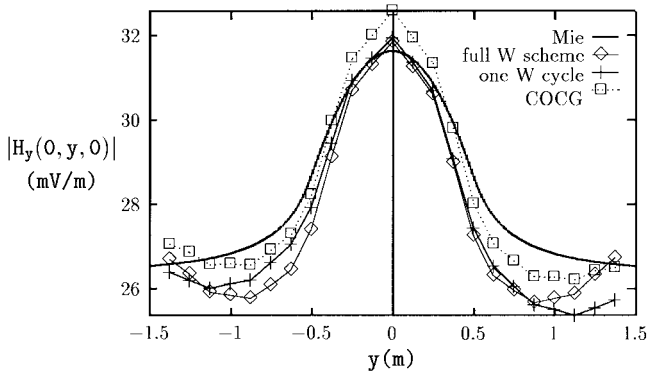

Fig. 5. The y component of the total magnetic field along the y axis for a three-level full W scheme; one W cycle, the COCG alone, and Mie's series solution.

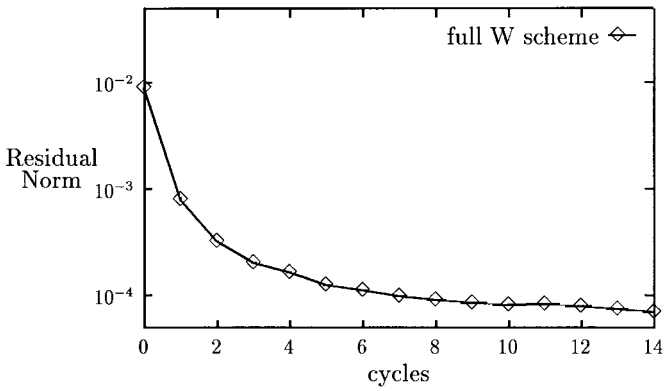

Fig. 6. Convergence behavior of a three-level full W scheme for scattering from a lossy sphere.

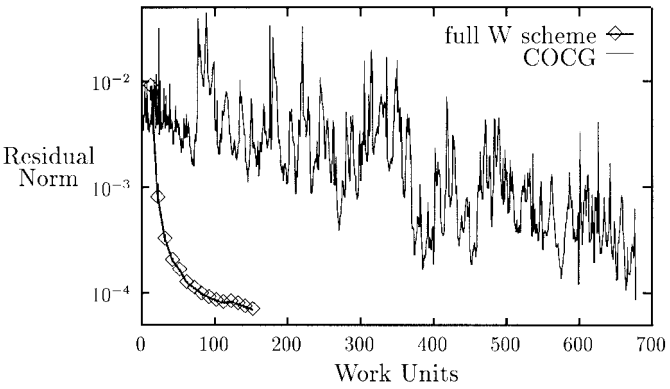

Fig. 7. Convergence comparison between a three-level full W scheme and COCG for the problem of the lossy sphere.

The accuracy of the results of multilevel methods depends on the modeling of the problem at the finest level. The computational price that has to be paid for this accuracy is lower than the cost of solving the finest level system with an ordinary iterative or direct method. Moreover, as with the multigrid schemes, the less number of arithmetic operations are performed, the number of the unavoidable roundoff errors is reduced significantly. Additionally, the smoother convergence history of multigrid makes it more numerically stable from the COCG with its highly irregular convergence behavior. Apart from that, coarse grid corrections are a form of iterative improvement so their application might also serve as a cure to the accumulation of roundoff errors, as is the case for standard iterative improvement [19].

To verify the accuracy of the solutions of our multigrid algorithms, in Fig. 5 a distribution for the total magnetic field that was obtained from the full W scheme and the corresponding analytical curve from Mie's theory are compared. On the same figure we superimposed the field distribution that resulted when the COCG was used alone and the field values provided by the use of only one full W cycle. It is observed that the multigrid results are considerably more accurate than those of the single-level COCG for the same numerical model and equal tolerance $e = 10^{-6}$ for the termination criterion.

The discrepancies between the analytical and the numerical field values are greater close to the surface of the sphere and near the absorbing boundary surface. The errors of the FEM there were higher because of the approximate nature of the ABC and the imperfect geometrical representation of the sphere by a collection of tetrahedra. It should be noted that the results of COCG only appeared to be more accurate at those areas because of an unusual and not exploitable local cancellation of discretization and roundoff errors. From Fig. 5, it is also evident that the use of only one full W cycle (21 WU) gave us a very good approximation of the analytical solution.

The improved accuracy of multigrid has been observed in other comparisons between multilevel and single-level techniques as well. It can allow us to use larger tolerances for the stopping criterion (for example $e = 10^{-5}$), thus arriving at a sufficient approximation at a much shorter time. It might even be possible to safely use single-precision arithmetic and in that way reduce half the memory demands of our programs.

In order to investigate the effect that the electrical properties of the mediums modeled have on the behavior of the multilevel algorithms, the scattering of a plane wave by a lossy sphere was analyzed next. The only difference that the new physical problem had from the previous one examined was that the sphere had a complex relative permittivity of $\varepsilon_r = (2, -1.5)$. For the solution of this problem, again, a three-level scheme was used with the same intergrid operators and solvers as before.

The residual reduction curve at the highest level (3) is drawn in Fig. 6. The full W schedule required 14 cycles to terminate and this cost 153 WU. These values are close to those achieved for the previous lossless problem, but slightly higher. It appears that the presence of electrical loss in the scatterer slightly reduced the multigrid convergence speed. The comparison between the full W scheme and the COCG for the problem of the lossy sphere takes place in Fig. 7.

The difference between the performances of the multigrid schemes and the single-level methods, became more intense as the number of unknowns in the problems solved increased. In Table I, the cycles and the required CPU times for the solution with multigrid of various problems are given. These problems modeled the scattering of plane waves by dielectric spheres. At the same table, the computational requirements for the solution of these problems, with sparse LU decomposition, are included. The latter has extremely large memory demands due to be filled during the factorization process; a fact that makes its practical implementation problematic (or even impossible)

TABLE I
COMPARISON BETWEEN THE COMPUTATIONAL REQUIREMENTS OF A DIRECT METHOD AND VARIOUS MULTIGRID SCHEMES

Number of Unknowns	Initial Nonzero Elements	Nonzero Elements after LU	Sparse LU Decomposition Time (sec)	Number of Levels	Multilevel Cycles	Multilevel Iteration Time (sec)
2187	50943	$5 \cdot 10^5$	204	2	64	118
6591	139227	$5 \cdot 10^6$	2016	2	42	212
14739	289431	10^7	10313	3	24	319
46785	835503	10^8	105606	3	12	476
107811	1808967	10^9	560434	4	3	596

TABLE II
PERFORMANCE OF A THREE-LEVEL MULTIGRID SCHEME FOR VARIOUS PERMITTIVITIES AND FREQUENCIES

ε_r	f(MHz)	Cycles	Work Units
(3,-2)	90	19	207
(4,0)	90	18	194
(7,-3.2)	90	20	220
(7,-3.2)	75	9	99
(16,0)	45	6	65

TABLE III
PERFORMANCE OF A THREE-LEVEL MULTIGRID SCHEME FOR VARIOUS SCATTERERS

Scatterer	f(MHz)	Cycles	Work Units
Cube	90	12	131
Cylinder	55	22	235
Prolate Spheroid	90	23	240
Multilayered Sphere	90	13	141

for large N. It is clear from Table I that multilevel techniques are by far more computationally efficient than direct methods and this performance gap grows rapidly as the number of unknowns increases. The same is also true if multilevel and single-level iterative solvers are compared. For example, the four-level scheme of Table I was almost 16 times faster than COCG (it cost 32 WU, while the corresponding cost of COCG was 511 WU).

In Table II, one can see the effects that the values of relative permittivity and frequency have on the behavior of multigrid. With the exception of ε_r and f, the remaining parameters of the problems investigated were the same with those of this section's first problem. This is also true for the numerical models and the multigrid schemes with which the problems were analyzed.

It is noticed that when f remained constant and ε_r increased, more cycles and subsequently more WU's were required for a multilevel solution. This may be because the increased difference between the free-space permittivity and that of the scatterer introduced more oscillatory (fast) error components that were immune to coarse grid corrections and needed more high-level iterations to be eliminated. As fixed schedules were used with a constant number of iterations at each cycle, this means that more cycles had to be performed. An adaptive cycling scheme would be more efficient in these cases.

Another reason for the slower multigrid convergence for higher ε_r could be the reduced number of nodes per minimum wavelength at the lowest level. As ε_r increased, the minimum wavelength shrunk in size. In order to test this hypothesis quickly and easily, the frequency of some problems was reduced, keeping the number of nodes per minimum wavelength at level 1 constant as ε_r increased. It was found out that when this happened, multigrid convergence speeded up again and was even faster than before. It is evident that an adaptive refinement mesh generator that allocated a sufficient number of nodes inside regions with high ε_r at every grid would take effectively care of this problem.

Finally, Table III presents the performance of the full W three-level scheme for various dielectric scatterers. All the scatterers were enclosed by a conforming absorbing boundary on the surface of which the Sommerfeld radiation condition was enforced. The cube had an edge length of 1 m. The cylinder had 1 m height and 0.5 m radius. The major axis of the spheroid was 1.5 m long while the ratio of its major and minor axes was 1.27. Except from the last case, all the other scatterers were homogenous and had $\varepsilon_r = 2$. The multigrid technique worked very well with all of them.

The last case examined was a double-layered sphere. The radius of the sphere was 0.5 m. The inner layer had a relative permittivity of $\varepsilon_{r1} = 4$ and radius 0.25 m. The permittivity of the outer layer was $\varepsilon_{r2} = 2$. Despite of the inhomogeneity (that is referred as a serious problem in multigrid literature) and the fact that on the coarsest level we were forced to model a homogenous sphere with $\varepsilon_r = 4$, the multilevel method worked surprisingly well and its performance was almost identical to that achieved for a homogenous problem. It is suspected that this promising and impressive behavior was in part owed to the magnetic field formulation of the FEM because H remained continuous even when ε_r changed.

VIII. CONCLUSIONS

In this paper, a multilevel implementation of the 3-D FEM for electromagnetic scattering has been presented. The main features and the characteristic behavior of the multigrid algorithms have been described in detail. It has been shown that the multilevel FEM allows the rapid solution of problems with tens of thousands of unknowns, in only about a dozen multigrid cycles and in a small fraction of the single-level computational cost. It has also been demonstrated that the multilevel approximations exhibit improved accuracy over the corresponding single-level solutions. The multigrid technique has been applied for various problems with different number of levels and unknowns and every time very promising and encouraging results were obtained. These results show that the multilevel method proposed is feasible, numerically stable, reliable, and very computationally efficient.

The efficiency of the method can be further increased by using various preconditioners, more advanced transfer operators (like bicubic interpolation), more sophisticated solvers (like

QMR and GMRES), and adaptive cycling schemes. Another very interesting possibility, seems to be the combination of the multilevel method with an adaptive refinement technique for the mesh generation at the various levels. We are currently in the process of parallelizing the multilevel code in a distributed memory environment.

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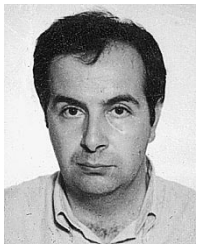

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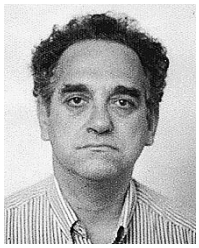

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