An Approximate Generalization of Schelkunoff's Horn-Gain Formulas

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***Abstract*— Schelkunoff's classical horn-gain formulas are applicable only to pyramidal and H-plane sectoral horns, which satisfy the Kirchhoff approximation. In this paper, a general gain formula is proposed that would be applicable also to E-plane sectoral horns and open-ended rectangular waveguides. The new formula is obtained by incorporating the "edge effect" into the classical formulas. Whereas for large apertures the new solution gives the same values as the conventional formulas, for small apertures it gives significant improvement over them and compares well with other improved gain formulas.**

***Index Terms*— Gain, pyramidal horn, rectangular waveguide antennas, Schelkunoff's horn-gain formulas, sectoral horns.**

I. Introduction

Schelkunoff's horn-gain formulas [1] are based on the Kirchoff approximation and, as such, are considered to be applicable only if the aperture dimensions are sufficiently large [2]. Consistent with this notion, these formulas accurately represent the monotonic gain component for the pyramidal horn [3]. There are reasons to expect them to also predict the H-plane sectoral horn gain accurately [4] though no comparison with measurement seems to have been reported as yet.

E-plane sectoral horns and open-ended rectangular waveguides (OEG) do not entirely satisfy the Kirchhoff approximation and Schelkunoff's formulas give inadequate representation of their gains [4], [5]. In [4], Jull presented a new formula for the gain of the E-plane sectoral horn by incorporating the exact on-axis TE_{10} mode field of an open-ended parallel-plate waveguide and obtained an accuracy similar to Schelkunoff's pyramidal horn-gain expression. For open-ended rectangular waveguides, Yaghjian [5] noted that Schelkunoff's formula neglects the "edge effect" or the fringe currents along the shorter edges of the waveguide. Obtaining an estimate of these fringe currents by using a numerical solution to the electric field integral equation (EFIE) applied to the OEG, he derived gain expressions that offer a significant improvement over Schelkunoff's formulas. More recently, Selvan [6] treated the shorter edges of the OEG as isotropic line sources following [7] and derived an alternative representation for the fringe currents and, hence, the gain.

In this paper, we propose a general, approximate formula for the far-field on-axis gain of pyramidal and sectoral horns and open-ended rectangular waveguides. (In subsequent discussions, we will refer to this class of antennas as rectangular waveguide antennas.) To this end, we begin with the recognition that the failure of the Schelkunoff's horn-gain formulas for small apertures is caused primarily by the assumption of insignificant edge effect [8]. By modifying these formulas to include the edge effect, a straightforward representation for which is the fringe current approximation derived in [6] for the open-ended rectangular waveguide, we subsequently arrive at the desired general formula. Comparison with the classical formulas and the improved formulas pertaining to specific apertures shows that the proposed formula gives a reasonably accurate representation of the gain for small as well as large apertures.

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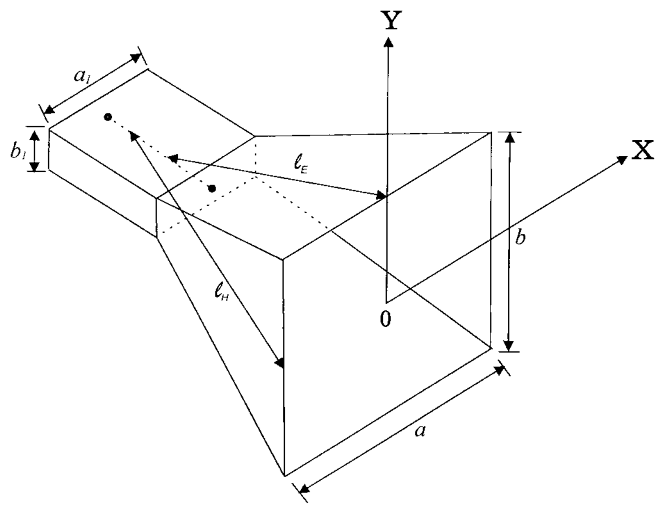

Fig. 1. Geometry of pyramidal horn.

II. A Generalization of Schelkunoff's Horn-Gain Formulas

Schelkunoff's classical formula for the on-axis far-field gain of the pyramidal horn of Fig. 1 is given by [1]

$$G' = \frac{32ab}{\pi\lambda^2} R_E R_H \tag{1}$$

with

$$R_E = \frac{C^2(w) + S^2(w)}{w^2} \tag{2}$$

and

$$R_H = \frac{\pi^2}{4} \frac{\{C(u) - C(v)\}^2 + \{S(u) - S(v)\}^2}{(u - v)^2}. \tag{3}$$

$$C(w) - jS(w) = \int_0^w \exp\left(-j\frac{\pi}{2}t^2\right) dt \tag{4}$$

are the Fresnel integrals and

$$w = \frac{b}{\sqrt{2\lambda l_E}} \tag{5}$$

$$\begin{matrix} u \\ v \end{matrix} = \pm\frac{a}{\sqrt{2\lambda l_H}} + \frac{1}{a}\sqrt{\frac{\lambda l_H}{2}} \tag{6}$$

where λ is the free-space wavelength. R_E and R_H account for the gain reduction due to the E- and H-plane flare of the horn.

Equation (1) makes several approximations, which include insignificant contribution by the fringe currents caused by the discontinuity at the aperture to the gain and negligible mutual interaction between the E- and H-edges at the aperture [8]. Whereas these approximations are valid for large apertures, the one approximation that primarily causes the failure of (1) for small apertures concerns the edge effect, or the fringe currents at the edges, which (1) neglects [5], [9]. Incorporating the effect of these fringe currents into (1), we can propose the following as a general gain formula:

$$G = \frac{32ab}{\pi\lambda^2} K R_E R_H \tag{7}$$

where K is a factor that accounts for the influence of the edge effect on the on-axis gain. A simple representation for this factor obtained from the integral form of the line-source solution [7] is given by [6]

$$K = \left\{1 + \frac{1}{\sqrt{2}}\left(\frac{k}{\beta} - 1\right)\right\}^2 \tag{8}$$

where $\beta/k = \sqrt{1 - (\pi/ka)^2}$ is the normalized propagation constant for the TE_{10} mode. It is particularly interesting to note that the edge effect as represented by (8) does not depend on the mutual interaction between the E and H edges. (This observation has also been made in [8] and [10].) By using (8) in (7), we then get the following modified Schelkunoff's gain formula:

$$G = \frac{32ab}{\pi\lambda^2}\left\{1 + \frac{1}{\sqrt{2}}\left(\frac{k}{\beta} - 1\right)\right\}^2 R_E R_H. \tag{9}$$

We propose (9) to be an approximate general far-field on-axis gain formula for rectangular waveguide antennas with small as well as large apertures. It may be noted that for large apertures, $k/\beta \approx 1$ and (9) reduces to (1)—the classical Schelkunoff's horn-gain formula.

Since $l_H \to \infty$ for the E-plane sectoral horn, to get its gain we must let $R_H \to 1$ in (9). Similarly, for the H-plane sectoral horn, $l_E \to \infty$ and, hence, $R_E \to 1$ in (9). Of course, for the open-ended rectangular waveguide, $l_H = l_E \to \infty$ and, hence, $R_H = R_E \to 1$ in (9).

III. Validity Examination of the New Formula

In Figs. 2–5, we consider the validity of the new formula (9) for pyramidal horn, sectoral horns, and open-ended rectangular waveguide.

For the pyramidal horn, (9) gives values that are scarcely different from those of (1). In the frequency range 5.5–8.5 GHz, an example is illustrated in Fig. 2 for a pyramidal horn with dimensions $a = 28.85$ cm, $b = 21.37$ cm, $l_e = 47.5$ cm, and $l_h = 50.84$ cm. This kind of agreement between (1) and (9) is just as expected.

In Fig. 3 we consider the gain of two E-plane sectoral horns, as computed by the new and the old methods. Showing significant improvement over Schelkunoff's formula (1), the new formula (9) compares very well with Jull's formula [4] over most of the frequency band. The agreement is particularly good at higher frequencies.

For the H-plane sectoral horn, measured data do not seem to be available in the published literature for comparison. However, (9) and (1) as well can be expected to give an accurate

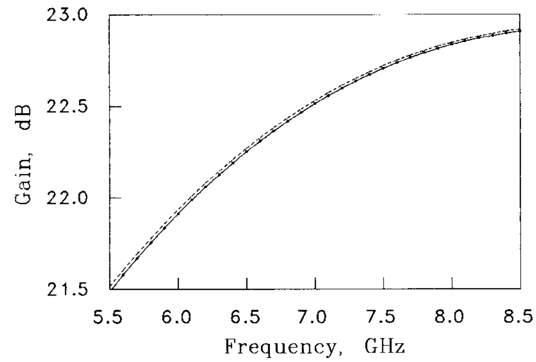

Fig. 2. Gain of pyramidal horn. $a = 28.85$ cm, $b = 21.37$ cm, $l_e = 47.5$ cm, $l_h = 50.84$ cm. -•-•-•- Schelkunoff's formula (1). ----- proposed formula (9).

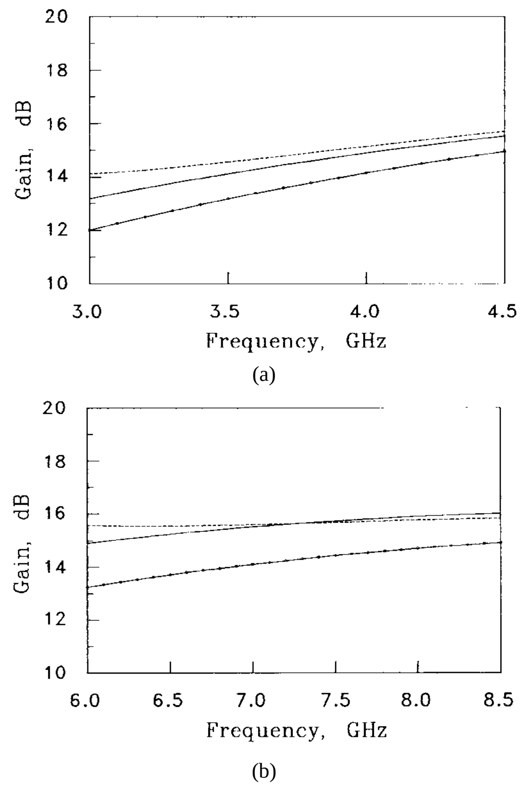

Fig. 3. Gain of E-plane sectoral horn. -•-•-•- Schelkunoff's formula (1). —— Jull's formula [4]. ----- Proposed formula (9). (a) $a_1 = 7.21$ cm, $b = 24$ cm, $l_e = 42.15$ cm. (b) $a_1 = 3.49$ cm, $b = 20$ cm, $l_e = 34.66$ cm.

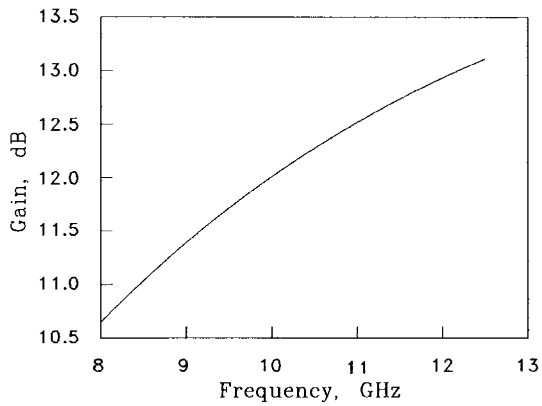

Fig. 4. Gain of H-plane sectoral horn as predicted by (9). $a = 20$ cm, $b_1 = 1.016$ cm, $l_h = 34.66$ cm.

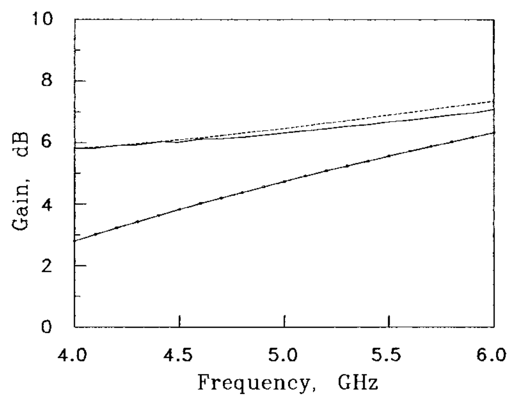

Fig. 5. Gain of open-ended rectangular waveguide, $a_1 = 4.755$ cm, $b_1 = 2.215$ cm. -•-•-•- Schelkunoff's formula (1). —— Yaghjian's formula [5]. - - - - - proposed formula (9).

representation of its gain as all the Kirchoff approximations hold for the H-plane sectoral horn [4]. Gain values predicted by using (9) and (1) are hardly different from each other and are shown in Fig. 4 for an X-band sectoral horn with the dimensions $a = 20$ cm, $b_1 = 1.016$ cm, and $l_h = 34.66$ cm.

For the open-ended rectangular waveguide, $R_E = R_H = 1$ in (9). The resulting formula is of similar accuracy to semi-empirical formulas of Yaghjian [5], [6]. Though a complete discussion is available in [6], for the sake of completeness (9) is compared with Yaghjian's formula [5] and (1) for a standard C-band OEG in Fig. 5.

From the results presented, it can be seen that (9) represents an approximate general on-axis far-field gain formula for rectangular waveguide antennas.

IV. Discussions and Conclusion

An approximate general formula was presented for the far-field on-axis gain of pyramidal and sectoral horns and open-ended rectangular waveguides. The formula was accomplished by incorporating the edge effect into Schelkunoff's classical horn-gain formulas.

The factor K used in the new formula (9) was originally shown in [6] to be a fringe current representation for open-ended rectangular waveguides, similar in accuracy to Yaghjian's equivalent expression [5]. Then, apparently, (9) is the OEG gain equation presented in [6] with gain reduction factors due to the E- and H-plane flares R_E and R_H appearing as multipliers. However, considering that the improved Schelkunoff's formulas derived by Yaghjian [5] have specific pertinence to OEG's unlike the general applicability to rectangular waveguide antennas of (9), we recognize the factor K used in (9) to be an approximate general descriptor of the edge effect of these antennas.

The new formula holds for antennas operating at frequencies well above cutoff. Since the formula assumes negligible wall thickness, it may become less accurate for antennas operating at and beyond Ku-band.

The formula presented in this paper can be generalized for approximate on-axis gain prediction, near and far field, by accounting for finite-range effects in the Fresnel zone in terms of a quadratic phase error in the aperture field [11]. For the cases of open-ended rectangular waveguide and pyramidal horn such generalized solutions have been reported, respectively, in [12] and [13].

In this paper, the gain reduction factors R_E and R_H are calculated by employing the conventional path length approximation. This approximation may be avoided by direct numerical integration of the gain reduction factors [14], [15].

Schelkunoff's formulas, based on the Kirchhoff approximation, do not account for the periodic gain variations with frequency that have been observed for the pyramidal horn [16] and the open-ended rectangular waveguide [17]. Efforts to account for these variations have been made for the case of pyramidal horns in [3] and [18] and for the case of open-ended rectangular waveguides in [19]. Also, our generalization of Schelkunoff's horn-gain formulas in this paper does not account for these oscillations. Investigations on a general gain formula for rectangular waveguide antennas that would also account for gain variations with frequency will be of further significant fundamental interest.

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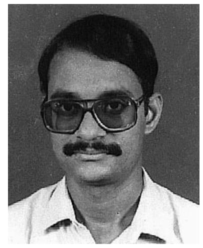

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