Efficient Calculation of Lattice Sums for Free-Space Periodic Green's Function

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***Abstract*— An efficient method to calculate the lattice sums is presented for a one-dimensional (1-D) periodic array of line sources. The method is based on the recurrence relations for Hankel functions and the Fourier integral representation of the zeroth-order Hankel function. The lattice sums of arbitrary high order are then expressed by an integral of elementary functions, which is easily computed using a simple scheme of numerical integration. The calculated lattice sums are used to evaluate the free-space periodic Green's function. The numerical results show that the proposed method provides a highly accurate evaluation of the Green's function with far less computation time, even when the observation point is located near the plane of the array.**

***Index Terms*—Green's function, lattice sums, periodic array.**

I. Introduction

THE calculation of the free-space periodic Green's function is a key to the efficient numerical analysis of electromagnetic scattering by periodic structures. The spatial form of the Green's function for a one-dimensional (1-D) periodic array of line sources is expressed as an infinite sum of Hankel functions multiplied by trigonometric angular dependencies. Its spectral form is represented by an infinite sum of plane waves which satisfy Floquet's theorem. It is well known that both of two forms converge very slowly whenever the observation point lies near the plane of the array. Various analytical or numerical techniques [1]–[8] have been devised to overcome the slow convergence in the periodic Green's function in the spatial or spectral domain.

An alternative to evaluating the periodic Green's function is to use a Neumann series expansion with coefficients given by the lattice sums [9]. This series converges sufficiently rapidly. The lattice sums depend only on the geometrical parameters of the structure and are independent of the location of the observation point. For a given problem of scattering, we need to calculate the lattice sums only once. This decreases remarkably the time of numerical evaluation of the Green's function at any point of the observation. Then the usefulness of the Neumann series expansion is greatly enhanced when combined with an efficient method to calculate the lattice sums. Recently, a hybrid technique [9], based on the recurrence formula and lattice sum identities, has been devised for calculating the lattice sums of arbitrary high order. However, this technique requires much computation time. It should be noted [9] that the overall time to evaluate the Green's function at one observation point is dominated by the time needed to compute the lattice sums. The efficient and accurate evaluation of the lattice sums is of significant importance from another point of view. When the electromagnetic scattering by a periodic structure is formulated using the T-matrix method, one needs inevitably to calculate a large number of the lattice sums [10].

In this paper, we shall present a very efficient numerical method to calculate the lattice sums. The approach is based on the recurrence formula for Hankel functions [11] and the Fourier integral representation of the zeroth-order Hankel function. The lattice sums of arbitrary high order are then represented by an integral form of elementary functions. The integration is easily performed by using a simple trapezoidal formula of numerical integration. The accuracy and efficiency in computation of the proposed method are examined by the numerical tests for certain typical cases. The numerical results show that the method provides a highly accurate evaluation of the Green's function with far less computation time, even when the observation point is located near the plane of the array.

II. Formulation

When the time dependence with $e^{j\omega t}$ is assumed, the Green's function for a 1-D array of line sources spaced d units apart along the x axis is given by

$$G(x, y; k, d, \phi_i) = \frac{1}{4j}\sum_{n=-\infty}^{\infty} H_0^{(2)}\left(k\sqrt{(x-nd)^2+y^2}\right)e^{jnkd\cos\phi_i} \quad (1)$$

where $H_0^{(2)}$ is the zeroth-order Hankel function of the second kind, k is the wavenumber of the medium, ϕ_i is the incidence angle of a hypothetical plane wave with respect to the x axis, and the location of the reference source is assumed at the origin. The infinite series in the spatial domain Green's function (1) converges very slowly. The spectral domain form of (1) is represented by

$$G(x, y; k, d, \phi_i) = \frac{1}{2jd}\sum_{n=-\infty}^{\infty} \frac{1}{\gamma_n} e^{-j(\beta_n x+\gamma_n|y|)} \quad (2)$$

where

$$\beta_n = \frac{2\pi n}{d} - k\cos\phi_i \quad (3)$$

$$\gamma_n = \begin{cases} \sqrt{k^2-\beta_n^2}, & |\beta_n| \le k \\ -j\sqrt{\beta_n^2-k^2}, & |\beta_n| > k. \end{cases} \quad (4)$$

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The infinite series (2) in the spectral domain converges rapidly for the *off plane* case ($y \neq 0$). This occurs because as n increases, the plane wave response changes from propagating waves to exponentially decaying evanescent waves. For the *on plane* case ($y = 0$), however, the series converges very slowly because it no longer has the exponentially decaying factor.

An alternative to evaluating the free-space periodic Green's function is to use a Neumann series representation for (1). Within the domain defined by $-d/2 \leq x \leq d/2$ and $-d/2 \leq y \leq d/2$, Graf's addition theorem for the Hankel functions is used and (1) is expanded in terms of Bessel functions as follows [9]:

$$G(x, y; k, d, \phi_i) = \frac{1}{4j}[H_0^{(2)}(k\rho) + S_0(kd, \phi_i)J_0(k\rho) + 2\sum_{n=1}^{\infty} S_n(kd, \phi_i)J_n(k\rho)\cos(n\phi)] \tag{5}$$

with

$$S_n(kd, \phi_i) = \sum_{m=1}^{\infty} H_n^{(2)}(mkd)e^{jmkd\cos\phi_i} + \sum_{m=1}^{\infty}(-1)^n H_n^{(2)}(mkd)e^{-jmkd\cos\phi_i} \tag{6}$$

where $kd = 2\pi d/\lambda$, J_n is Bessel function of the nth order, and $\rho = \sqrt{x^2+y^2}$ and $\phi = \cos^{-1}(x/\rho)$ are the polar coordinates in the x–y plane. The coefficient $S_n(kd, \phi_i)$ in the expansion is usually referred to as the lattice sum of the nth order.

The lattice sums depend only on the geometrical parameters of the structure and are independent of the location of the observation point (x, y). For a given problem of scattering, we need to calculate the lattice sums only once. This decreases remarkably the time of numerical evaluation of the Green's function at any point of observation. However, we notice again that the lattice sums given by two semi-infinite sums of the Hankel functions are unfortunately slowly convergent. Since the Neumann series itself converges rapidly, the advantage of using (5) is greatly enhanced when combined with an efficient method to calculate the lattice sums. In what follows, we shall discuss a method to evaluate efficiently two semi-infinite sums in (6).

Consider the evaluation of the first sum in (6). The second sum can be treated in the same manner. To do this, we use the recurrence formula [11] for Hankel functions

$$H_n^{(2)}(k\rho)e^{-jn\phi} = (-1)^n\frac{1}{k^n}\left(\frac{\partial}{\partial x} - j\frac{\partial}{\partial y}\right)^n H_0^{(2)}(k\rho) \quad (n \geq 0) \tag{7}$$

and the Fourier integral representation of the zeroth-order Hankel function of the second kind

$$H_0^{(2)}(k\rho) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{\kappa(\xi)}e^{-j[\xi x + \kappa(\xi)|y|]}\,d\xi \tag{8}$$

where

$$\kappa(\xi) = \begin{cases} \sqrt{k^2-\xi^2}, & k \geq |\xi| \\ -j\sqrt{\xi^2-k^2}, & |\xi| > k. \end{cases} \tag{9}$$

Substituting (8) into (7) and assuming that $x = 0$, $y > 0$, and $\phi = \pi/2$, we obtain the following formula for Hankel functions:

$$H_n^{(2)}(ky) = (-1)^n\frac{1}{\pi k^n}\int_{-\infty}^{\infty}\frac{[\xi - j\kappa(\xi)]^n}{\kappa(\xi)}e^{-j\kappa(\xi)y}\,d\xi \quad (y > 0,\ n \geq 0). \tag{10}$$

Using this formula by letting $y = md$, the first sum in (6) is expressed as

$$\sum_{m=1}^{\infty} H_n^{(2)}(mkd)e^{jmkd\cos\phi_i} = (-1)^n\frac{1}{\pi k^n}\lim_{M\to\infty}\int_{-\infty}^{\infty}\frac{[\xi - j\kappa(\xi)]^n}{\kappa(\xi)}T_M(\xi)\,d\xi \tag{11}$$

with

$$T_M(\xi) = \sum_{m=1}^{M} e^{-j[k_x+\kappa(\xi)]md} = e^{-j[k_x+\kappa(\xi)]d}\frac{1-e^{-j[k_x+\kappa(\xi)]Md}}{1-e^{-j[k_x+\kappa(\xi)]d}} \tag{12}$$

where $k_x = -k\cos\phi_i$. When the limit $M \to \infty$ is taken, the function $e^{-j\kappa(\xi)Md}$ oscillates very rapidly for $|\xi| \leq k$ and decays very fast for $|\xi| > k$. Then the second term in the numerator of (12) vanishes after the integration and (11) leads to

$$\sum_{m=1}^{\infty} H_n^{(2)}(mkd)e^{jmkd\cos\phi_i} = (-1)^n\frac{e^{-jk_xd}}{\pi k^n}\int_{-\infty}^{\infty}\frac{[\xi - j\kappa(\xi)]^n}{\kappa(\xi)}\frac{e^{-j\kappa(\xi)d}\,d\xi}{1-e^{-j[k_x+\kappa(\xi)]d}} \quad (n \geq 0). \tag{13}$$

Using the normalization of variables, (13) is rewritten as

$$\sum_{m=1}^{\infty} H_n^{(2)}(mkd)e^{jmkd\cos\phi_i} = (-1)^n\frac{e^{jkd\cos\phi_i}}{\pi k^n}\int_{-\infty}^{\infty} G_n(t)F(t; kd, \cos\phi_i)\,dt \quad (n \geq 0) \tag{14}$$

where

$$G_n(t) = \left(t - j\sqrt{1-t^2}\right)^n \tag{15}$$

$$F(t; kd, \cos\phi_i) = \frac{e^{-jkd\sqrt{1-t^2}}}{\sqrt{1-t^2}[1-e^{-jkd(\sqrt{1-t^2}-\cos\phi_i)}]}. \tag{16}$$

To avoid the difficulties of integration near the singular points of $F(t; kd, \cos\phi_i)$, the integration in (14) is evaluated along the straight line which makes the angle of $\pi/4$ rad with respect to the real axis in the complex ξ plane. After several manipulations, the integration in (14) can be approximated with a high enough accuracy as follows:

$$\sum_{m=1}^{\infty} H_n^{(2)}(mkd)e^{jmkd\cos\phi_i} = (-1)^n e^{j(\pi/4+kd\cos\phi_i)}\frac{\sqrt{2}}{\pi}\int_0^a [G_n(\tau) + G_n(-\tau)] \cdot F(\tau; kd, \cos\phi_i)\,dt \tag{17}$$

where $\tau = (1+j)t$ and $a(\gg 1)$ is a positive real number which is chosen so that the integration converges with a required accuracy.

The presence of exponential function $e^{-jkd\sqrt{1-t^2}}$ in (16), which decays very rapidly for $t \gg 1$, enables us to approximate the infinite integral (14) in terms of the finite integral (17). The choice of the upper limit a in the integration depends on the value of parameter $kd = 2\pi d/\lambda$ and is determined by the numerical testing.

Following the same procedure, the second sum in (6) can be expressed as follows:

$$\sum_{m=1}^{\infty}(-1)^n H_n^{(2)}(mkd)e^{-jmkd\cos\phi_i} = e^{j(\pi/4-kd\cos\phi_i)}\frac{\sqrt{2}}{\pi}\int_0^a [G_n(\tau)+G_n(-\tau)] \cdot F(\tau;\, kd, -\cos\phi_i)\,dt. \quad (18)$$

The integrals in (17) and (18) can be accurately and efficiently calculated using a standard formula of numerical integration for elementary functions. The advantage of using (17) and (18) is that both the real and imaginary parts of the lattice sums of arbitrary high order are simultaneously evaluated by a simple FORTRAN implementation. Although the hybrid technique [9] is an accurate approach, it involves much complicated intermediate formulas using special functions and requires to evaluate separately the real and imaginary parts of the lattice sums. Moreover, the numerical algorithms used in the calculation are different for the lower order lattice sums and for the higher order ones. Those features of the hybrid technique inevitably increases the computation time.

III. Numerical Examples

To confirm the validity of the proposed method, (17) and (18) were used to calculate the lattice sums (6). Although a substantial number of numerical tests could be generated, we will present here only the results related to certain typical cases with $\phi_i = 5\pi/8$ rad and $\lambda/d = 0.23, 0.46$, and 1.77. These cases have been investigated by using the hybrid technique [9] and the numerical results obtained by a MATHEMATICA program run on a SPARC-10 workstation have been given in tabular form. For the comparison of computation time, we also used a SPARC-10 workstation for the calculation.

The integrals in (17) and (18) were evaluated using a trapezoidal formula, which is the simplest scheme in numerical integration. The upper limit a and a small division Δt in the integration were chosen so that the values of the lattice sums are obtained with the accuracy of eight significant figures. Since the integrands $G_n(\pm\tau)$ and $F(\tau;\, kd, \pm\cos\phi_i)$ in (17) and (18) are smooth as functions of t, the results of numerical integration are not so sensitive to the choice of division Δt.

The first 50 values of the lattice sums calculated by the present method are given in Table I for $\lambda/d = 0.23$ and compared with those of the hybrid technique. $\mathrm{Re}[S_n]$ and $\mathrm{Im}[S_n]$ denote the real and imaginary parts of the lattice sum of nth order. The results of the hybrid technique are reproduced from [9, Table I] by changing the signs, since their lattice sums are defined using Hankel functions of the first kind. The results of the imaginary part of odd order and the real part of even order have not been given in [9]. The present results were obtained by choosing $a = 10$ and $\Delta t = 0.01$. We can see that both results are in close agreement for $0 \le n \le 27$ and $42 \le n \le 49$. However, there appears a slight difference between them for $28 \le n \le 41$.

TABLE I
The First 50 Values of Lattice Sums $S_n(kd,\, \cos\phi_i)$ Calculated for $\lambda/d = 0.23$ and $\phi_i = 5\pi/8$ rad. $\mathrm{Re}[S_n]$ and $\mathrm{Im}[S_n]$ Denote the Real and Imaginary Parts of S_n. The Results Termed by "Hybrid Technique" Have Been Reproduced from [9, Table I] (a) $S_0 \sim S_{24}$. (b) $S_{25} \sim S_{49}$

	Present method		Hybrid technique [9]	
n	$\mathrm{Re}[S_n]$	$\mathrm{Im}[S_n]$	$\mathrm{Re}[S_n]$	$\mathrm{Im}[S_n]$
0	0.73329394	-0.16104658		-0.16104658
1	0.15571020	0.94792609	0.15571020	
2	-0.74532753	0.17175436		0.17175436
3	-0.21469466	-0.96391981	-0.21469466	
4	0.78034690	-0.20898202		-0.20898202
5	0.33567720	0.97644809	0.33567720	
6	-0.83144551	0.28765032		0.28765032
7	-0.51161439	-0.94708194	-0.51161439	
8	0.87553498	-0.42758564		-0.42758564
9	0.70236456	0.83201906	0.70236456	
10	-0.86297714	0.63440080		0.63440080
11	-0.82304880	-0.62643188	-0.82304881	
12	0.73043828	-0.86221953		-0.86221953
13	0.79590391	0.42642515	0.79590391	
14	-0.46970407	0.99299669		0.99299669
15	-0.69407376	-0.40323220	-0.69407376	
16	0.22177171	-0.92542628		-0.92542628
17	0.80574556	0.54742971	0.80574556	
18	-0.19868483	0.78977182		0.78977182
19	-0.12340508×10^{1}	0.41845553	-1.23405074	
20	0.28889005	-0.92086219		-0.92086218
21	0.13245685×10^{1}	-0.24580446	1.32456842	
22	-0.99393100×10^{-2}	0.11880266×10^{1}		1.18802661
23	-0.66196266	0.46392024	-0.66196236	
24	-0.36957004	-0.93854258		-0.93854254

(a)

	Present method		Hybrid technique [9]	
n	$\mathrm{Re}[S_n]$	$\mathrm{Im}[S_n]$	$\mathrm{Re}[S_n]$	$\mathrm{Im}[S_n]$
25	0.73723907	0.14123900	0.73723782	
26	0.10726318	0.84071344		0.84071354
27	-0.12529709×10^{1}	0.53186404	-1.25296581	
28	-0.63515270	-0.12523502×10^{1}		-1.25235165
29	0.27216853	-0.25754992	0.27214775	
30	0.62154229	0.21205696		0.21206723
31	-0.23244301×10^{1}	0.75454460	-2.32434610	
32	-0.72175799	-0.19025565×10^{1}		-1.90261541
33	-0.30441925×10^{1}	-0.96010038	-3.04452690	
34	0.65690724	-0.35468905×10^{1}		-3.54658672
35	-0.13524970×10^{2}	0.91870986	-1.35236736×10^{1}	
36	-0.74840978	-0.16887317×10^{2}		-1.68887646×10^{1}
37	-0.57831833×10^{2}	-0.67810897	-5.78366481×10^{1}	
38	0.10458025×10^{1}	-0.77834468×10^{2}		-7.78280644×10^{1}
39	-0.30502285×10^{3}	0.78508482	-3.05006107×10^{2}	
40	-0.12206668×10^{1}	-0.44456867×10^{3}		-4.44594709×10^{2}
41	-0.18597208×10^{4}	-0.11909475×10^{1}	-1.85977359×10^{3}	
42	0.97522540	-0.29127886×10^{4}		-2.91269330×10^{3}
43	-0.13034606×10^{5}	0.11398686×10^{1}	-1.30344632×10^{4}	
44	-0.73891601	-0.21760249×10^{5}		-2.17605509×10^{4}
45	-0.10354615×10^{6}	-0.78817533	-1.03546461×10^{5}	
46	0.92523937	-0.18335270×10^{6}		-1.83351911×10^{5}
47	-0.92341365×10^{6}	0.10417498×10^{1}	-9.23413193×10^{5}	
48	-0.96806457	-0.17271239×10^{7}		-1.72712539×10^{6}
49	-0.91708155×10^{7}	-0.12702343×10^{1}	-9.17081576×10^{6}	

(b)

TABLE II
THE FIRST 30 VALUES OF LATTICE SUMS $S_n(kd, \cos\phi_i)$ CALCULATED FOR $\lambda/d = 0.46$ AND $\phi_i = 5\pi/8$ rad. $\mathrm{Re}[S_n]$ AND $\mathrm{Im}[S_n]$ DENOTE THE REAL AND IMAGINARY PARTS OF S_n. THE RESULTS TERMED BY "HYBRID TECHNIQUE" HAVE BEEN REPRODUCED FROM [9, TABLE II]

	Present method		Hybrid technique [9]	
n	$\mathrm{Re}[S_n]$	$\mathrm{Im}[S_n]$	$\mathrm{Re}[S_n]$	$\mathrm{Im}[S_n]$
0	0.17509589×10^{1}	-0.36093071		-0.360931
1	0.89475179×10^{-1}	0.18094945×10^{1}	0.0894750	
2	-0.17623473×10^{1}	0.41827505		0.418275
3	-0.21886359	-0.19289760×10^{1}	-0.218864	
4	0.17671156×10^{1}	-0.58616202		-0.586162
5	0.56058592	0.20883932	0.560586	
6	-0.16961641×10^{1}	0.81739453		0.817395
7	-0.11547476×10^{1}	-0.20374747×10^{1}	-1.154748	
8	0.15527227×10^{1}	-0.97811658		-0.978117
9	0.16470135×10^{1}	0.14681032×10^{1}	1.647013	
10	-0.15763090×10^{1}	0.10566865×10^{1}		1.056687
11	-0.13309795×10^{1}	-0.86357884	-1.330980	
12	0.17470268×10^{1}	-0.15994934×10^{1}		-1.599493
13	0.11440242×10^{1}	0.14622680×10^{1}	1.144024	
14	-0.84241711	0.23278766×10^{1}		2.327877
15	-0.24714722×10^{1}	-0.74549368	-2.471472	
16	0.56169444	-0.11933666×10^{1}		-1.193368
17	-0.38019380	0.94680026	-0.380201	
18	-0.29538596	0.40325986×10^{1}		4.032610
19	-0.10925872×10^{2}	-0.57725945	-10.925802	
20	0.37271660	0.95525666×10^{1}		9.552479
21	-0.47474580×10^{2}	-0.26163727	-4.747525×10^{1}	
22	-0.14741115	0.77429865×10^{2}		7.743048×10^{1}
23	-0.38098195×10^{3}	0.62913403	-3.809760×10^{2}	
24	-0.35823161	0.64512325×10^{3}		6.451198×10^{2}
25	-0.36403182×10^{4}	-0.12414223	-3.640365×10^{3}	
26	0.54539488	0.68948919×10^{4}		6.894904×10^{3}
27	-0.42757079×10^{5}	0.87841418	-4.275678×10^{4}	
28	-0.14445204×10^{1}	0.88702853×10^{5}		8.870288×10^{4}
29	-0.60026345×10^{6}	-0.71529771	-6.002649×10^{5}	

TABLE III
THE FIRST 30 VALUES OF LATTICE SUMS $S_n(kd, \cos\phi_i)$ CALCULATED FOR $\lambda/d = 1.77$ AND $\phi_i = 5\pi/8$ rad. $\mathrm{Re}[S_n]$ AND $\mathrm{Im}[S_n]$ DENOTE THE REAL AND IMAGINARY PARTS OF S_n. THE RESULTS TERMED BY "HYBRID TECHNIQUE" HAVE BEEN REPRODUCED FROM [9, TABLE II]

	Present method		Hybrid technique [9]	
n	$\mathrm{Re}[S_n]$	$\mathrm{Im}[S_n]$	$\mathrm{Re}[S_n]$	$\mathrm{Im}[S_n]$
0	-0.39017104	-0.16365463		-0.163655
1	-0.43784085	-0.23337144	-0.437841	
2	0.43121420	-0.28365501×10^{1}		-0.028366
3	0.35413159	-0.56340850	0.354132	
4	$-0.55511151\times10^{-16}$	0.51775604		0.517756
5	0.20800865×10^{1}	-0.56340850	2.080086	
6	-0.43121420	0.10142624×10^{1}		1.014262
7	0.14619401×10^{2}	-0.23337144	1.461940×10^{1}	
8	-0.60982896	0.10094414×10^{2}		1.009441×10^{1}
9	0.21630980×10^{3}	0.23337144	2.163098×10^{2}	
10	-0.43121420	0.22089246×10^{3}		2.208925×10^{2}
11	0.56547325×10^{4}	0.56340850	5.654733×10^{3}	
12	-0.24374458×10^{-9}	0.73062387×10^{4}		7.306236×10^{3}
13	0.22414508×10^{6}	0.56340850	2.241451×10^{5}	
14	0.43121421	0.34636693×10^{6}		3.463672×10^{5}
15	0.12456013×10^{8}	0.23337029	1.245602×10^{7}	
16	0.60983177	0.22344388×10^{8}		2.234435×10^{7}
17	0.92197054×10^{9}	-0.23348172	9.219702×10^{8}	
18	0.43103218	0.18813547×10^{10}		1.881359×10^{9}
19	0.87580459×10^{11}	-0.53361320	8.758049×10^{10}	
20	-0.99304199×10^{-1}	0.20023677×10^{12}		2.002362×10^{11}
21	0.10378297×10^{14}	0.41621094×10^{1}	1.037829×10^{13}	
22	0.63125000×10^{1}	0.26272121×10^{14}		2.627217×10^{13}
23	0.15001095×10^{16}	-0.13023125×10^{4}	1.500109×10^{15}	
24	-0.23060000×10^{4}	0.41645451×10^{16}		4.164540×10^{15}
25	0.25970259×10^{18}	-0.20670800×10^{6}	2.597026×10^{17}	
26	-0.37081600×10^{7}	0.78444922×10^{18}		7.844496×10^{17}
27	0.53041486×10^{20}	0.67972403×10^{9}	5.304149×10^{19}	
28	-0.15229256×10^{10}	0.17316680×10^{21}		1.731668×10^{20}
29	0.12618264×10^{23}	-0.13551351×10^{12}	1.261826×10^{22}	

The hybrid technique utilizes a set of lattice sum identities which is a system of linear equations for the lattice sums. The linear system for $\{S_0, \cdots, S_{49}\}$ was numerically solved [9] by incorporating the recurrence relations and nearest-neighbors estimate for lattice sums to avoid a problem of ill condition for the system. The linear system becomes larger and the dynamic range of the matrix elements rapidly increases as the order of lattice sums concerned increases. In the present method, on the other hand, each of the lattice sums of arbitrary order is calculated separately by using (17) and (18). This difference in the calculation process may cause such a small discrepancy as mentioned above. Due to the simpler equations involved, the numerical procedure of the present method is much more efficient than that of the hybrid technique [9]. The total central processing unit (CPU) time to calculate the first 50 lattice sums by the present method was only 40 s, whereas the hybrid technique requires 1231 s for the same computation.

The first 30 values of lattice sums calculated by the present method for $\lambda/d = 0.46$ and $\lambda/d = 1.77$ are shown in Tables II and III, respectively, and compared with those of the hybrid technique [9]. The present results in Tables II and III were obtained by choosing $\Delta t = 0.01$ for $a = 20$ and $a = 35$, respectively, in the numerical integration of (17) and (18). It is seen that both results are in good agreement. Although the computation time needed for the hybrid technique has not been described in [9], the total CPU times in computation were 48 and 84 s for the present method in Tables II and III. When λ/d increases, the exponentially decaying factor involved in (16) decreases and a larger value of a is required for the integration in (17) and (18) to retain the accuracy of the results. This leads to a longer computation time as λ/d increases.

TABLE IV
COMPARISON BETWEEN DIFFERENT METHODS TO EVALUATE THE GREEN'S FUNCTION FOR $\lambda/d = 0.23$ AND $\phi_i = 5\pi/8$ rad. IN THE FIRST COLUMN, "DIRECT SUM" AND $O(1/n^3)$ INDICATE A STRAIGHTFORWARD SUM OF THE SERIES (2) AND THE CALCULATION USING A THIRD-ORDER ACCELERATION TECHNIQUE [7], RESPECTIVELY. x AND y ARE THE LOCATION OF THE OBSERVATION POINT, $\mathrm{Re}[G(x,y)]$ AND $\mathrm{Im}[G(x,y)]$ GIVE THE REAL AND IMAGINARY PARTS OF THE GREEN'S FUNCTION, N REPRESENTS THE NUMBER OF TERMS IN THE CORRESPONDING SERIES, AND T IS THE COMPUTER CPU TIME IN SECONDS. THE RESULTS OF THE DIRECT SUM, $O(1/n^3)$, AND THE HYBRID TECHNIQUE ARE REPRODUCED FROM [9, TABLE III]

	x/d	y/d	$\mathrm{Re}[G(x,y)]$	$\mathrm{Im}[G(x,y)]$	N	T
Direct Sum [9]	0.2	0.03	0.117120006144932	0.108131857633201	154	4
$O(1/n^3)$ [9]			0.117120006144931	0.108131857633201	121	8
Hybrid Technique [9]			0.117120006141860	0.108131857633197	50	1232
Present Method			0.117120006144941	0.108131857633206	50	40
Direct Sum [9]	0.2	0.003	0.115891895634567	0.103497063599642	1386	36
$O(1/n^3)$ [9]			0.115891895634567	0.103497063599643	836	48
Hybrid Technique [9]			0.115891895630095	0.103497063599643	50	1232
Present Method			0.115891895634577	0.103497063599646	50	40
Direct Sum [9]	0.2	0.0003	0.115881138140449	0.103450147416784	12681	366
$O(1/n^3)$ [9]			0.115881138140448	0.103450147416785	5416	301
Hybrid Technique [9]			0.115881138135960	0.103450147416785	50	1232
Present Method			0.115881138140457	0.103450147416788	50	40

When the lattice sums were calculated, the Green's function at any observation point (x, y) can be evaluated by using (5). The values of the Green's function evaluated by the present method for $\phi_i = 5\pi/8$ rad and $\lambda/d = 0.23$ are shown in Table IV and compared with those of other three different methods. The "direct sum" and "$O(1/n^3)$" in the first column indicate a straightforward sum of the series (2)

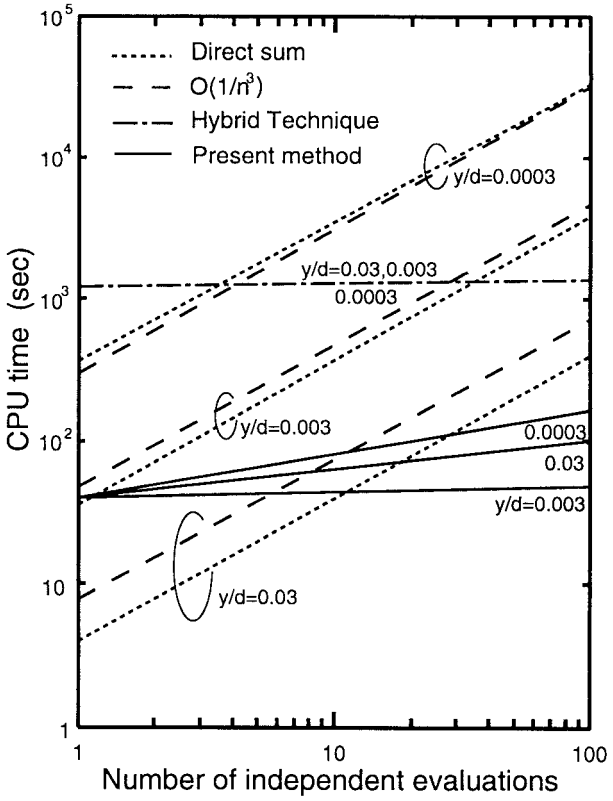

Fig. 1. Computer CPU time versus the number of independent evaluations of Green's function for four different methods. The values of parameters are the same as those given in Table IV.

and a calculation using a third-order acceleration technique [7]. $\mathrm{Re}[G(x, y)]$ and $\mathrm{Im}[G(x, y)]$ denote the real and imaginary parts of the Green's function and N represents the number of terms in the corresponding series. The results of the direct sum "$O(1/n^3)$" and the hybrid technique are reproduced from [9, Table III] by taking into account the difference in the kind of Hankel functions used. The hybrid technique and the present method used the first 50 values of lattice sums given in Table I. It is seen that the results of the present method show a closer agreement with the rigorous values of "direct sum" than those of the hybrid technique. This fact suggests that the present method based on (17) and (18) provides a more accurate evaluation of the lattice sums than the hybrid technique [9].

Fig. 1 shows the computer CPU time versus the number of independent evaluations of Green's function for four different methods. The values of parameters are the same as those given in Table IV. When the configuration parameters λ/d and ϕ_i for a periodic structure are specified, the lattice sums are determined independently of the location of observation point. The Green's function at any observation point can be evaluated from (5) using the same set of the lattice sums. Then the computation time is remarkably reduced, when the repeated evaluations of the Green's function at different points are required. This is an advantage of the hybrid technique and the present method for evaluating the Green's function. However, the hybrid technique needs much more time for the computation of the lattice sums, though the MATHEMATICA program has been used in [9]. It is seen that for 100 independent evaluations of Green's function, the total CPU time in the hybrid technique is dominated by the time (1231 s) for computing the lattice sums. In contrast, the present method requires far less computation time for the evaluations of the Green's function as well as the lattice sums, even when the observation point is located near the plane $(y = 0)$ of the periodic sources. The comparisons in Tables I, IV, and Fig. 1 demonstrate that the present method is very accurate and computationally efficient one in the evaluation of the free-space periodic Green's function.

IV. Conclusion

The free-space Green's function for a 1-D periodic array of line sources may be expressed by the Neumann series with the lattice sums as the expansion coefficients. However, the lattice sums are given by a semi-infinite sum of Hankel functions which is very slowly converging. In this paper, an efficient method to calculate the lattice sums of arbitrary order with high accuracy has been presented. The method is based on the recurrence relations for Hankel functions and the Fourier integral representation of the zeroth-order Hankel function. Then the real and imaginary parts of the lattice sums can be simultaneously evaluated using a simple trapezoidal formula of numerical integration. This greatly simplifies the computational procedure and, therefore, drastically reduces the computation time. Numerical results have shown that the proposed method provides a computationally efficient way of evaluating the Green's functions when the observation point is located near plane of the array and a highly accurate result is required.

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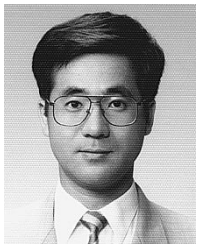

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