The Matrix Riccati Equation for Scattering from Stratified Chiral Spheres

Dwight L. Jaggard, *Fellow, IEEE*, and John C. Liu

***Abstract*—Angular scattering from radially stratified spherical chiral objects is investigated. Based on the principles of invariant imbedding, we formulate a matrix Riccati equation that can be used to examine basic scattering properties of spherical chiral structures with radial inhomogeneities in permittivity, permeability, and chirality. High- and low-frequency limits as well as weak reflection and constant impedance cases for this equation are examined. We show that in the limit of large radii of curvature, this formulation yields the planar result.**

***Index Terms*— Chiral media, electromagnetic scattering, nonhomogeneous media, spherical scatterers.**

I. Introduction

FEW practical methods exist for determining the electromagnetic scattering from arbitrarily inhomogeneous chiral objects. We examine the particular case of scattering from chiral spheres with radial inhomogeneities using the method of invariant embedding. This gives rise to a matrix Riccati equation for chiral spheres. Our research is motivated by the work of Ambartsumian [1] who introduced the concept of invariant embedding in scattering circa 1940, by the work of Bellman and Kalaba [2] who derived the Riccati equation for planar achiral media over 40 years ago, and by the work of Latham [3] who derived the Riccati equation for cylindrical and spherical achiral media in the late 1960's.

Recent work in examining the scattering properties of homogeneous planar chiral layers [4]–[7], cylinders [8]–[10] and spheres [10]–[12] have given us a good understanding of these processes. It has been shown that the chiral Riccati equation for planar layers and its associated jump condition is a useful and practical technique for determining the exact reflection from stratified planar chiral media [7]. We develop a matrix Riccati equation for continuous variations and the jump condition for discontinuous variations. This matrix Riccati equation transforms the boundary-value scattering problem to an initial value problem and provides a simple, elegant means to calculate the angular scattering from radially stratified spherical objects.

Section II provides a brief electromagnetic description of chirality. Section III describes the chiral wave functions applicable to spherical geometry and these functions are utilized in Section IV to solve for the scattering coefficients. Section V develops the matrix Riccati equation for chiral spheres from these scattering coefficients. In Section VI, we examine basic properties and limiting cases of this equation.

Manuscript received December 27, 1995; revised May 21, 1996. This work was supported in part by NATO Collaborative Research Grant 930 923 and by the Complex Media Laboratory, University of Pennsylvania.

The authors are with the Complex Media Laboratory, Moore School of Electrical Engineering, University of Pennsylvania, Philadelphia, PA 19104 USA.

Publisher Item Identifier S 0018-926X(99)07057-X.

II. Background

An isotropic, reciprocal chiral medium can be described by the following time-harmonic ($e^{-i\omega t}$) constitutive relations

$$\mathbf{D} = \varepsilon\mathbf{E} + i\xi_c\mathbf{B} \quad \mathbf{H} = (1/\mu)\mathbf{B} + i\xi_c\mathbf{E} \tag{1}$$

where $\mathbf{E}$, $\mathbf{B}$, $\mathbf{D}$, and $\mathbf{H}$ are the electromagnetic field vectors and ε, μ, and ξ_c are, in general, complex and represent the permittivity, permeability, and chirality admittance of the chiral medium, respectively.

Based on the above constitutive relations and the source-free Maxwell equations, the chiral Helmholtz equation is found to be

$$\nabla\times\nabla\times\mathbf{C} - 2\omega\mu\xi_c\nabla\times\mathbf{C} - k_m^2\mathbf{C} = 0 \tag{2}$$

where $\mathbf{C}$ is any one of the electromagnetic field vectors $\mathbf{E}$, $\mathbf{H}$, $\mathbf{B}$, and $\mathbf{D}$, with $k_m = \omega\sqrt{\mu\varepsilon}$ where ω is the radian frequency of the time-harmonic fields. The two eigenmodes of propagation, right-handed and left-handed circularly polarized (RCP and LCP), travel with a pair of wavenumbers given by

$$k_\pm = \pm\omega\mu\xi_c + \sqrt{k_m^2 + (\omega\mu\xi_c)^2}. \tag{3}$$

The chiral impedance, defined by the ratio of electric to magnetic field eigenmodes, is given by

$$\eta = \sqrt{\frac{\mu}{\varepsilon + \mu\xi_c^2}} \tag{4}$$

and is found from relation (1) and the Maxwell equations.

III. Spherical Chiral Eigenmodes

Two classical Mie modes [13] that satisfy the conventional Helmholtz equation in spherical coordinates are given by

$$\begin{aligned}\mathbf{m}_{\substack{e\\o}mn}(k) &= \frac{m}{\sin\theta}z_n(kr)P_n^m(\cos\theta)\begin{Bmatrix}-\sin(m\phi)\\+\cos(m\phi)\end{Bmatrix}\hat{\mathbf{e}}_\theta\\ &\quad - z_n(kr)\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix}\hat{\mathbf{e}}_\phi\\ \mathbf{n}_{\substack{e\\o}mn}(k) &= n(n+1)\frac{z_n(kr)}{kr}P_n^m(\cos\theta)\begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix}\hat{\mathbf{e}}_r\\ &\quad + \frac{1}{kr}\frac{\partial}{\partial r}[rz_n(kr)]\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\cdot\begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix}\hat{\mathbf{e}}_\theta\\ &\quad + \frac{m}{kr\sin\theta}\frac{\partial}{\partial r}[rz_n(kr)]P_n^m(\cos\theta)\\ &\quad \times\begin{Bmatrix}-\sin(m\phi)\\+\cos(m\phi)\end{Bmatrix}\hat{\mathbf{e}}_\phi.\end{aligned} \tag{5}$$

In the above equations, $z_n(kr)$ represents any of the spherical Bessel functions $j_n(kr)$, $n_n(kr)$, $h_n^{(1)}(kr)$, or $h_n^{(2)}(kr)$ of order n chosen to satisfy boundary conditions and $P_n^m(\cos\theta)$ is an associated Legendre function of the first kind of order n and degree m. The subscripts e and o refer to the even and odd nature of the modes. Taking a linear combination of these modes, a new set of orthogonal vector wave functions is defined that satisfy the chiral Helmholtz equation

$$\left\{\begin{array}{c}\mathbf{v}_{\substack{e\\o}mn}(k)\\ \mathbf{w}_{\substack{e\\o}mn}(k)\end{array}\right\} = k\frac{\mathbf{m}_{\substack{e\\o}mn}(k) \pm \mathbf{n}_{\substack{e\\o}mn}(k)}{\sqrt{2}}. \quad (6)$$

These functions satisfy the relations

$$\nabla\times\left\{\begin{array}{c}\mathbf{v}_{\substack{e\\o}mn}(k)\\ \mathbf{w}_{\substack{e\\o}mn}(k)\end{array}\right\} = \pm k\left\{\begin{array}{c}\mathbf{v}_{\substack{e\\o}mn}(k)\\ \mathbf{w}_{\substack{e\\o}mn}(k)\end{array}\right\}. \quad (7)$$

Since the expansion for plane waves in spherical wave functions involve only the $m = 1$ term, we restrict ourselves to only those terms. In order to simply notation, we define the quantities

$$z_n = z_n(kr) \quad \partial z_n = \frac{1}{kr}\frac{\partial}{\partial r}[rz_n(kr)]$$
$$P_n = \frac{P_n^1(\cos\theta)}{\sin\theta} \quad \partial P_n = \frac{\partial}{\partial\theta}P_n^1(\cos\theta). \quad (8)$$

For our derivation, we define a set of equivalent positive and negative chiral eigenmodes

$$\mathbf{v}_{\pm 1n}(k) = \frac{ke^{\pm i\phi}}{\sqrt{2}}\Big\{(\pm iz_nP_n + \partial z_n\partial P_n)\hat{\mathbf{e}}_\theta + (-z_n\partial P_n \pm i\partial z_nP_n)\hat{\mathbf{e}}_\phi + n(n+1)\frac{z_n}{kr}P_n\hat{\mathbf{e}}_r\Big\}$$
$$\mathbf{w}_{\pm 1n}(k) = \frac{ke^{\pm i\phi}}{\sqrt{2}}\Big\{(\pm iz_nP_n - \partial z_n\partial P_n)\hat{\mathbf{e}}_\theta - (z_n\partial P_n \pm i\partial z_nP_n)\hat{\mathbf{e}}_\phi - n(n+1)\frac{z_n}{kr}P_n\hat{\mathbf{e}}_r\Big\} \quad (9)$$

where

$$\mathbf{v}_{\pm 1n}(k) = \frac{\mathbf{v}_{e1n}(k) \pm i\mathbf{v}_{o1n}(k)}{\sqrt{2}}$$
$$\mathbf{w}_{\pm 1n}(k) = \frac{\mathbf{w}_{e1n}(k) \pm i\mathbf{w}_{o1n}(k)}{\sqrt{2}}. \quad (10)$$

If we further define the quantities

$$\alpha_z^\pm = iz_nP_n \pm \partial z_n\partial P_n \quad \beta_z^\pm = -z_n\partial P_n \pm i\partial z_nP_n \quad (11)$$

then the equivalent positive and negative chiral eigenmodes are written compactly as

$$\mathbf{v}_{\pm 1n}(k) = \frac{ke^{\pm i\phi}}{\sqrt{2}}\Big\{\pm\alpha_z^\pm\hat{\mathbf{e}}_\theta + \beta_z^\pm\hat{\mathbf{e}}_\phi + n(n+1)\frac{z_n}{kr}P_n\hat{\mathbf{e}}_r\Big\} \quad (12)$$

$$\mathbf{w}_{\pm 1n}(k) = \frac{ke^{\pm i\phi}}{\sqrt{2}}\Big\{\pm\alpha_z^\mp\hat{\mathbf{e}}_\theta + \beta_z^\mp\hat{\mathbf{e}}_\phi - n(n+1)\frac{z_n}{kr}P_n\hat{\mathbf{e}}_r\Big\}. \quad (13)$$

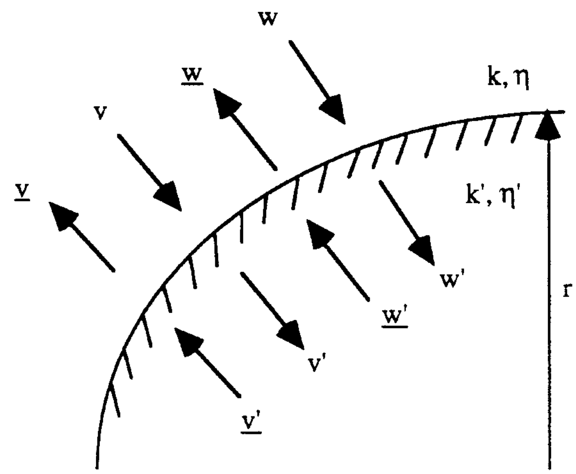


Fig. 1. Spherical interface between two chiral media. The prime denotes the inner medium and the underbar denotes outwardly traveling waves.

IV. Scattering Coefficients

Consider a spherical interface between two possibly chiral media, as in Fig. 1. On each side of the boundary, a wave can be decomposed into a set of inwardly and outwardly traveling waves of each eigenmode. Each medium is characterized by its own impedance η and wavenumbers k_+ and k_-. We determine the chiral magnitude transfer matrix at this boundary by expressing each of these waves with the wave functions defined above and applying tangential field boundary conditions.

Presently, we consider waves only of the positive mode. We can proceed analogously for waves of the negative mode to yield exactly the same results. On the outer boundary, we express inwardly traveling waves as

$$\left\{\begin{array}{c}\mathbf{E}_v\\ i\eta\mathbf{H}_v\end{array}\right\} = E_v\mathbf{v}_{+1n}(k_+) = \frac{E_v}{\sqrt{2}}[k_+\alpha_j^+\hat{\mathbf{e}}_\theta + k_+\beta_j^+\hat{\mathbf{e}}_\phi]e^{i\phi}$$
$$\left\{\begin{array}{c}\mathbf{E}_w\\ i\eta\mathbf{H}_w\end{array}\right\} = \pm E_w\mathbf{w}_{+1n}(k_-) = \pm\frac{E_w}{\sqrt{2}}[k_-\alpha_j^-\hat{\mathbf{e}}_\theta + k_-\beta_j^-\hat{\mathbf{e}}_\phi]e^{i\phi} \quad (14)$$

and outwardly traveling waves as

$$\left\{\begin{array}{c}\underline{\mathbf{E}}_v\\ i\eta\underline{\mathbf{H}}_v\end{array}\right\} = \underline{E}_v\mathbf{v}_{+1n}(k_+) = \frac{\underline{E}_v}{\sqrt{2}}[k_+\alpha_h^+\hat{\mathbf{e}}_\theta + k_+\beta_h^+\hat{\mathbf{e}}_\phi]e^{i\phi}$$
$$\left\{\begin{array}{c}\underline{\mathbf{E}}_w\\ i\eta\underline{\mathbf{H}}_w\end{array}\right\} = \pm\underline{E}_w\mathbf{w}_{+1n}(k_-) = \pm\frac{\underline{E}_w}{\sqrt{2}}[k_-\alpha_h^-\hat{\mathbf{e}}_\theta + k_-\beta_h^-\hat{\mathbf{e}}_\phi]e^{i\phi}. \quad (15)$$

Here we have ignored the radial components of the waves since we are interested only in the tangential fields. On the inner boundary, inwardly traveling waves are written

$$\left\{\begin{matrix}\mathbf{E}_{v'}\\ i\eta'\mathbf{H}_{v'}\end{matrix}\right\} = E_{v'}\mathbf{v}_{+1n}(k'_+) = \frac{E_{v'}}{\sqrt{2}}[k'_+\alpha^+_{j'}\hat{\mathbf{e}}_\theta + k'_+\beta^+_{j'}\hat{\mathbf{e}}_\phi]e^{i\phi}$$
$$\left\{\begin{matrix}\mathbf{E}_{w'}\\ i\eta'\mathbf{H}_{w'}\end{matrix}\right\} = \pm E_{w'}\mathbf{w}_{+1n}(k'_-) = \pm\frac{E_{w'}}{\sqrt{2}}[k'_-\alpha^-_{j'}\hat{\mathbf{e}}_\theta + k'_-\beta^-_{j'}\hat{\mathbf{e}}_\phi]e^{i\phi} \tag{16}$$

and outwardly traveling waves are written

$$\left\{\begin{matrix}\underline{\mathbf{E}}_{v'}\\ i\eta'\underline{\mathbf{H}}_{v'}\end{matrix}\right\} = \underline{E}_{v'}\mathbf{v}_{+1n}(k'_+) = \frac{\underline{E}_{v'}}{\sqrt{2}}[k'_+\alpha^+_{h'}\hat{\mathbf{e}}_\theta + k'_+\beta^+_{h'}\hat{\mathbf{e}}_\phi]e^{i\phi}$$
$$\left\{\begin{matrix}\underline{\mathbf{E}}_{w'}\\ i\eta'\underline{\mathbf{H}}_{w'}\end{matrix}\right\} = \pm\underline{E}_{w'}\mathbf{w}_{+1n}(k'_-) = \pm\frac{\underline{E}_{w'}}{\sqrt{2}}[k'_-\alpha^-_{h'}\hat{\mathbf{e}}_\theta + k'_-\beta^-_{h'}\hat{\mathbf{e}}_\phi]e^{i\phi}. \tag{17}$$

Matching tangential electric and magnetic fields along the $\hat{\mathbf{e}}_\theta$ and $\hat{\mathbf{e}}_\phi$ directions yields the boundary condition matrix relation

$$\Omega\mathbf{E} = \vartheta\mathbf{E}' \tag{18}$$

which is more explicitly written

$$\begin{bmatrix} k_+\alpha^+_j & k_-\alpha^-_j & k_+\alpha^+_h & k_-\alpha^-_h \\ k_+\beta^+_j & k_-\beta^-_j & k_+\beta^+_h & k_-\beta^-_h \\ k_+\alpha^+_j & -k_-\alpha^-_j & k_+\alpha^+_h & -k_-\alpha^-_h \\ k_+\beta^+_j & -k_-\beta^-_j & k_+\beta^+_h & -k_-\beta^-_h \end{bmatrix}\begin{bmatrix}E_v\\E_w\\ \underline{E}_v\\ \underline{E}_w\end{bmatrix} = \begin{bmatrix} k'_+\alpha^+_{j'} & k'_-\alpha^-_{j'} & k'_+\alpha^+_{h'} & k'_-\alpha^-_{h'} \\ k'_+\beta^+_{j'} & k'_-\beta^-_{j'} & k'_+\beta^+_{h'} & k'_-\beta^-_{h'} \\ \gamma k'_+\alpha^+_{j'} & -\gamma k'_-\alpha^-_{j'} & \gamma k'_+\alpha^+_{h'} & -\gamma k'_-\alpha^-_{h'} \\ \gamma k'_+\beta^+_{j'} & -\gamma k'_-\beta^-_{j'} & \gamma k'_+\beta^+_{h'} & -\gamma k'_-\beta^-_{h'} \end{bmatrix}\begin{bmatrix}E_{v'}\\E_{w'}\\ \underline{E}_{v'}\\ \underline{E}_{w'}\end{bmatrix} \tag{19}$$

where $\gamma = \eta/\eta'$ is the ratio of the impedances of the outer and inner media. Solving for the chiral magnitude matrix $\mathbf{M} = \Omega^{-1}\vartheta$ and noting that

$$\beta^+_h\alpha^\pm_{j'} - \alpha^+_h\beta^\pm_{j'} = \pm(P^2_n - \partial P^2_n)(h_+\partial j'_\pm \mp j'_\pm\partial h_+)$$
$$\beta^-_h\alpha^\pm_{j'} - \alpha^-_h\beta^\pm_{j'} = \pm(P^2_n - \partial P^2_n)(h_-\partial j'_\pm \pm j'_\pm\partial h_-) \tag{20}$$

where

$$z_\pm = z_n(k_\pm r) \quad \partial z_\pm = \frac{1}{k_\pm r}\frac{\partial}{\partial r}[rz_n(k_\pm r)]$$
$$z'_\pm = z_n(k'_\pm r) \quad \partial z'_\pm = \frac{1}{k'_\pm r}\frac{\partial}{\partial r}[rz_n(k'_\pm r)] \tag{21}$$

we find (22) at the bottom of the page.

A. Reflection and Transmission Matrices

Given the chiral magnitude matrix in the form

$$\begin{bmatrix}E_v\\E_w\\ \underline{E}_v\\ \underline{E}_w\end{bmatrix} = \begin{bmatrix}A_{11} & A_{12} & B_{11} & B_{12}\\ A_{21} & A_{22} & B_{21} & B_{22}\\ C_{11} & C_{12} & D_{11} & D_{12}\\ C_{21} & C_{22} & D_{21} & D_{22}\end{bmatrix}\begin{bmatrix}E_{v'}\\E_{w'}\\ \underline{E}_{v'}\\ \underline{E}_{w'}\end{bmatrix} = \begin{bmatrix}\mathbf{M}_A & \mathbf{M}_B\\ \mathbf{M}_C & \mathbf{M}_D\end{bmatrix}\begin{bmatrix}E_{v'}\\E_{w'}\\ \underline{E}_{v'}\\ \underline{E}_{w'}\end{bmatrix} \tag{23}$$

we solve for the reflection and transmission matrices for an inwardly or outwardly traveling wave incident on the boundary. For inward incidence $\mathbf{M}_B = \mathbf{M}_D = 0$, which reduces to

$$\begin{bmatrix}E_v\\E_w\end{bmatrix} = \mathbf{M}_A\begin{bmatrix}E_{v'}\\E_{w'}\end{bmatrix} \quad \begin{bmatrix}\underline{E}_v\\ \underline{E}_w\end{bmatrix} = \mathbf{M}_C\begin{bmatrix}E_{v'}\\E_{w'}\end{bmatrix}. \tag{24}$$

Solving these two equations, we find for the transmission and reflection matrices for an inwardly traveling wave to be

$$\mathbf{T} = \begin{bmatrix}t^{vv} & t^{vw}\\ t^{wv} & t^{ww}\end{bmatrix} = \mathbf{M}_A^{-1} \quad \mathbf{R} = \begin{bmatrix}r^{vv} & r^{vw}\\ r^{wv} & r^{ww}\end{bmatrix} = \mathbf{M}_C\mathbf{M}_A^{-1}. \tag{25}$$

For outwardly traveling waves incident on the boundary, we use the complementary chiral transfer matrix $\mathbf{M}' = \mathbf{M}^{-1}$. In this case, $\mathbf{M}'_A = \mathbf{M}'_C = 0$ and we have

$$\begin{bmatrix}E_{v'}\\E_{w'}\\ \underline{E}_{v'}\\ \underline{E}_{w'}\end{bmatrix} = \begin{bmatrix}\mathbf{M}'_A & \mathbf{M}'_B\\ \mathbf{M}'_C & \mathbf{M}'_D\end{bmatrix}\begin{bmatrix}E_v\\E_w\\ \underline{E}_v\\ \underline{E}_w\end{bmatrix} = \begin{bmatrix}0 & 0 & B'_{11} & B'_{12}\\ 0 & 0 & B'_{21} & B'_{22}\\ 0 & 0 & D'_{11} & D'_{12}\\ 0 & 0 & D'_{21} & D'_{22}\end{bmatrix}\begin{bmatrix}E_v\\E_w\\ \underline{E}_v\\ \underline{E}_w\end{bmatrix} \tag{26}$$

which reduces to

$$\begin{bmatrix}\underline{E}_{v'}\\ \underline{E}_{w'}\end{bmatrix} = \mathbf{M}'_D\begin{bmatrix}\underline{E}_v\\ \underline{E}_w\end{bmatrix} \quad \begin{bmatrix}E_{v'}\\E_{w'}\end{bmatrix} = \mathbf{M}'_B\begin{bmatrix}\underline{E}_v\\ \underline{E}_w\end{bmatrix}. \tag{27}$$

Solving these two equations, we find for the transmission and reflection matrices for an outwardly traveling wave to be

$$\underline{\mathbf{T}} = \mathbf{M}'^{-1}_D \quad \underline{\mathbf{R}} = \mathbf{M}'_B\mathbf{M}'^{-1}_D. \tag{28}$$

We find the transmission and reflection matrices to be

$$\mathbf{T} = \frac{1}{\Psi}\begin{bmatrix} \frac{1+\gamma}{2}\frac{k'_-}{k_-}\frac{j'_-\partial h_- - h_-\partial j'_-}{j_-\partial h_- - h_-\partial j_-} & -\frac{1-\gamma}{2}\frac{k'_-}{k_+}\frac{j'_-\partial h_+ + h_+\partial j'_-}{j_+\partial h_+ - h_+\partial j_+} \\ -\frac{1-\gamma}{2}\frac{k'_+}{k_-}\frac{j'_+\partial h_- + h_-\partial j'_+}{j_-\partial h_- - h_-\partial j_-} & \frac{1+\gamma}{2}\frac{k'_+}{k_+}\frac{j'_+\partial h_+ - h_+\partial j'_+}{j_+\partial h_+ - h_+\partial j_+} \end{bmatrix}$$

$$\underline{\mathbf{T}} = \frac{1}{\underline{\Psi}}\begin{bmatrix} \frac{1+\gamma}{2\gamma}\frac{k_-}{k'_-}\frac{j'_-\partial h_- - h_-\partial j'_-}{j'_-\partial h'_- - h'_-\partial j'_-} & -\frac{1-\gamma}{2\gamma}\frac{k_-}{k'_+}\frac{j'_+\partial h_- + h_-\partial j'_+}{j'_+\partial h'_+ - h'_+\partial j'_+} \\ -\frac{1-\gamma}{2\gamma}\frac{k_+}{k'_-}\frac{j'_-\partial h_+ + h_+\partial j'_-}{j'_-\partial h'_- - h'_-\partial j'_-} & \frac{1+\gamma}{2\gamma}\frac{k_+}{k'_+}\frac{j'_+\partial h_+ - h_+\partial j'_+}{j'_+\partial h'_+ - h'_+\partial j'_+} \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} \frac{1+\gamma}{2}\frac{k'_+}{k_+}\frac{j'_+\partial h_+ - h_+\partial j'_+}{j_+\partial h_+ - h_+\partial j_+} & \frac{1-\gamma}{2}\frac{k'_-}{k_+}\frac{j'_-\partial h_+ + h_+\partial j'_-}{j_+\partial h_+ - h_+\partial j_+} & \frac{1+\gamma}{2}\frac{k'_+}{k_+}\frac{h'_+\partial h_+ - h_+\partial h'_+}{j_+\partial h_+ - h_+\partial j_+} & \frac{1-\gamma}{2}\frac{k'_-}{k_+}\frac{h'_-\partial h_+ + h_+\partial h'_-}{j_+\partial h_+ - h_+\partial j_+} \\ \frac{1-\gamma}{2}\frac{k'_+}{k_-}\frac{j'_+\partial h_- + h_-\partial j'_+}{j_-\partial h_- - h_-\partial j_-} & \frac{1+\gamma}{2}\frac{k'_-}{k_-}\frac{j'_-\partial h_- - h_-\partial j'_-}{j_-\partial h_- - h_-\partial j_-} & \frac{1-\gamma}{2}\frac{k'_+}{k_-}\frac{h'_+\partial h_- + h_-\partial h'_+}{j_-\partial h_- - h_-\partial j_-} & \frac{1+\gamma}{2}\frac{k'_-}{k_-}\frac{h'_-\partial h_- - h_-\partial h'_-}{j_-\partial h_- - h_-\partial j_-} \\ -\frac{1+\gamma}{2}\frac{k'_+}{k_+}\frac{j'_+\partial j_+ - j_+\partial j'_+}{j_+\partial h_+ - h_+\partial j_+} & -\frac{1-\gamma}{2}\frac{k'_-}{k_+}\frac{j'_-\partial j_+ + j_+\partial j'_-}{j_+\partial h_+ - h_+\partial j_+} & -\frac{1+\gamma}{2}\frac{k'_+}{k_+}\frac{h'_+\partial h_- - j_+\partial h'_+}{j_+\partial h_+ - h_+\partial j_+} & -\frac{1-\gamma}{2}\frac{k'_-}{k_+}\frac{h'_-\partial j_+ + j_+\partial h'_-}{j_+\partial h_+ - h_+\partial j_+} \\ -\frac{1-\gamma}{2}\frac{k'_+}{k_-}\frac{j'_+\partial j_- + j_-\partial j'_+}{j_-\partial h_- - h_-\partial j_-} & -\frac{1+\gamma}{2}\frac{k'_-}{k_-}\frac{j'_-\partial j_- - j_-\partial j'_-}{j_-\partial h_- - h_-\partial j_-} & -\frac{1-\gamma}{2}\frac{k'_+}{k_-}\frac{h'_+\partial j_- + j_-\partial h'_+}{j_-\partial h_- - h_-\partial j_-} & -\frac{1+\gamma}{2}\frac{k'_-}{k_-}\frac{h'_-\partial j_- + j_-\partial h'_-}{j_-\partial h_- - h_-\partial j_-} \end{bmatrix} \tag{22}$$

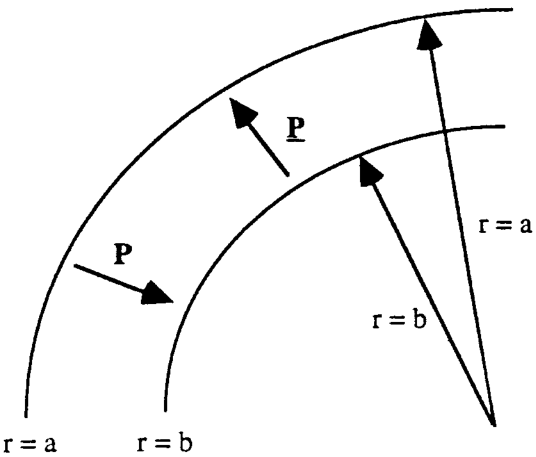


Fig. 2. Propagation matrices.

(see (29) at the bottom of the page) with

$$\begin{aligned}\Psi &= \frac{(1+\gamma)^2}{4}\frac{k'_+k'_-}{k_+k_-}\frac{j'_+\partial h_+ - h_+\partial j'_+}{j_+\partial h_+ - h_+\partial j_+}\frac{j'_-\partial h_- - h_-\partial j'_-}{j_-\partial h_- - h_-\partial j_-}\\ &\quad - \frac{(1-\gamma)^2}{4}\frac{k'_-k'_+}{k_+k_-}\frac{j'_-\partial h_+ + h_+\partial j'_-}{j_+\partial h_+ - h_+\partial j_+}\frac{j'_+\partial h_- + h_-\partial j'_+}{j_-\partial h_- - h_-\partial j_-}\\ \underline{\Psi} &= \frac{(1+\gamma)^2}{4\gamma^2}\frac{k_+k_-}{k'_+k'_-}\frac{j'_+\partial h_+ - h_+\partial j'_+}{j'_+\partial h'_+ - h'_+\partial j'_+}\frac{j'_-\partial h_- - h_-\partial j'_-}{j'_-\partial h'_- - h'_-\partial j'_-}\\ &\quad - \frac{(1-\gamma)^2}{4\gamma^2}\frac{k_-k_+}{k'_+k'_-}\frac{j'_-\partial h_+ + h_+\partial j'_-}{j'_+\partial h'_+ - h'_+\partial j'_+}\frac{j'_+\partial h_- + h_-\partial j'_+}{j'_-\partial h'_- - h'_-\partial j'_-}.\end{aligned} \tag{30}$$

Note that the reflection coefficients given above represent the scattered field for a homogeneous chiral sphere immersed in another possibly chiral medium.

B. Propagation Matrices

In order to express the propagation of inwardly and outwardly traveling spherical waves from on point to another, we must use propagation matrices (see Fig. 2). The propagation matrix $\mathbf{P}$ and complementary propagation matrix $\underline{\mathbf{P}}$ are determined from the chiral transfer and complementary transfer matrices $\mathbf{M}$ and $\mathbf{M}'$, respectively. Defined as

$$\begin{aligned}\begin{bmatrix}E_v(b)\\E_w(b)\end{bmatrix} &= \begin{bmatrix}P_{11} & P_{12}\\ P_{21} & P_{22}\end{bmatrix}\begin{bmatrix}E_v(a)\\E_w(a)\end{bmatrix} = \mathbf{P}\begin{bmatrix}E_v(a)\\E_w(a)\end{bmatrix}\\ \begin{bmatrix}\underline{E}_v(a)\\\underline{E}_w(a)\end{bmatrix} &= \begin{bmatrix}\underline{P}_{11} & \underline{P}_{12}\\ \underline{P}_{21} & \underline{P}_{22}\end{bmatrix}\begin{bmatrix}\underline{E}_v(b)\\\underline{E}_w(b)\end{bmatrix} = \underline{\mathbf{P}}\begin{bmatrix}\underline{E}_v(b)\\\underline{E}_w(b)\end{bmatrix}\end{aligned} \tag{31}$$

we find

$$\begin{aligned}\mathbf{P} &= \begin{bmatrix}\frac{j_+\partial h_+ - h_+\partial j_+}{j'_+\partial h_+ - h_+\partial j'_+} & 0\\ 0 & \frac{j_-\partial h_- - h_-\partial j_-}{j'_-\partial h_- - h_-\partial j'_-}\end{bmatrix}\\ \underline{\mathbf{P}} &= \begin{bmatrix}\frac{j'_+\partial h'_+ - h'_+\partial j'_+}{j'_+\partial h_+ - h_+\partial j'_+} & 0\\ 0 & \frac{j'_-\partial h'_- - h'_-\partial j'_-}{j'_-\partial h_- - h_-\partial j'_-}\end{bmatrix}.\end{aligned} \tag{32}$$

with the definitions

$$\begin{aligned}z_\pm &= z_n(k_\pm a) \quad \partial z_\pm = \frac{1}{k_\pm a}\frac{\partial}{\partial r}[rz_n(k_\pm r)]\bigg|_{r=a}\\ z'_\pm &= z_n(k_\pm b) \quad \partial z'_\pm = \frac{1}{k_\pm b}\frac{\partial}{\partial r}[rz_n(k_\pm r)]\bigg|_{r=b}.\end{aligned} \tag{33}$$

V. Spherical Chiral Riccati Equation

Similar to the spirit of Latham's work for achiral spheres in [3], we apply Ambartsumian's invariance principles [1], [2] to the radially stratified chiral sphere in order to derive a matrix Riccati equation. We wish to obtain an equation for the rate of change of reflected waves as we traverse through the sphere and experience changing material characteristics (radial inhomogeneity). We begin by investigating the reflection from a spherical shell of thickness Δr. The first-, second-, and third-order reflections or bounces, as indicated in Fig. 3, are given by

$$\begin{aligned}\mathbf{r}_1(r) &= \mathbf{R}\\ \mathbf{r}_2(r) &= \mathbf{TPr}(r-\Delta r)\underline{\mathbf{P}}\,\underline{\mathbf{T}}\\ \mathbf{r}_3(r) &= \mathbf{TPr}(r-\Delta r)\underline{\mathbf{P}}\,\underline{\mathbf{R}}\mathbf{Pr}(r-\Delta r)\underline{\mathbf{P}}\,\underline{\mathbf{T}}\end{aligned} \tag{34}$$

where $\mathbf{r}(r-\Delta r)$ is the reflection at the inner surface of the shell. Note that the reflection of the $(q+2)$th bounce is given by,

$$\mathbf{r}_{q+2}(r) = \mathbf{TPr}(r-\Delta r)\underline{\mathbf{P}}\,\underline{\mathbf{T}}[\underline{\mathbf{R}}\mathbf{Pr}(r-\Delta r)\underline{\mathbf{P}}]^q \quad \text{for } q>0. \tag{35}$$

The total reflection at the outer boundary is expressed as the sum

$$\mathbf{R} = \sum_{q=1}^{\infty}\mathbf{r}_q. \tag{36}$$

We keep only the terms of the series of order Δr. Expanding the coefficients to first order in Δr, and taking the limit as

$$\begin{aligned}\mathbf{R} &= \frac{1}{\Psi}\begin{bmatrix} \frac{(1-\gamma)^2}{4}\frac{k'_-k'_+}{k_+k_-}\frac{j'_-\partial j_+ + j_+\partial j'_-}{j_+\partial h_+ - h_+\partial j_+}\frac{j'_+\partial h_- + h_-\partial j'_+}{j_-\partial h_- - h_-\partial j_-} - \frac{(1+\gamma)^2}{4}\frac{k'_+k'_-}{k_+k_-}\frac{j'_+\partial j_+ - j_+\partial j'_+}{j_+\partial h_+ - h_+\partial j_+}\frac{j'_-\partial h_- - h_-\partial j'_-}{j_-\partial h_- - h_-\partial j_-} & \frac{1-\gamma^2}{4}\frac{k'_+k'_-}{k_+^2}\frac{j'_+\partial j'_- + j'_-\partial j'_+}{j_+\partial h_+ - h_+\partial j_+}\\ -\frac{1-\gamma^2}{4}\frac{k'_+k'_-}{k_-^2}\frac{j'_+\partial j'_- + j'_-\partial j'_+}{j_-\partial h_- - h_-\partial j_-} & \frac{(1-\gamma)^2}{4}\frac{k'_+k'_-}{k_+k_-}\frac{j'_+\partial j_- + j_-\partial j'_+}{j_+\partial h_+ - h_+\partial j_+}\frac{j'_-\partial h_+ + h_+\partial j'_-}{j_-\partial h_- - h_-\partial j_-} - \frac{(1+\gamma)^2}{4}\frac{k'_-k'_+}{k_+k_-}\frac{j'_-\partial j_- - j_-\partial j'_-}{j_+\partial h_+ - h_+\partial j_+}\frac{j'_+\partial h_+ - h_+\partial j'_+}{j_-\partial h_- - h_-\partial j_-}\end{bmatrix}\\ \underline{\mathbf{R}} &= \frac{1}{\underline{\Psi}}\begin{bmatrix} \frac{(1-\gamma)^2}{4\gamma^2}\frac{k_-k_+}{k'_+k'_-}\frac{h'_+\partial h_- + h_-\partial h'_+}{j'_+\partial h'_+ - h'_+\partial j'_+}\frac{j'_-\partial h_+ + h_+\partial j'_-}{j'_-\partial h'_- - h'_-\partial j'_-} - \frac{(1+\gamma)^2}{4\gamma^2}\frac{k_+k_-}{k'_+k'_-}\frac{h'_+\partial h_+ - h_+\partial h'_+}{j'_+\partial h'_+ - h'_+\partial j'_+}\frac{j'_-\partial h_- - h_-\partial j'_-}{j'_-\partial h'_- - h'_-\partial j'_-} & \frac{1-\gamma^2}{4\gamma^2}\frac{k_+k_-}{k'^2_-}\frac{h_-\partial h_+ + h_+\partial h_-}{j'_+\partial h'_+ - h'_+\partial j'_+}\\ -\frac{1-\gamma^2}{4\gamma^2}\frac{k_-k_+}{k'^2_-}\frac{h_-\partial h_+ + h_+\partial h_-}{j'_-\partial h'_- - h'_-\partial j'_-} & \frac{(1-\gamma)^2}{4\gamma^2}\frac{k_+k_-}{k_+k_-}\frac{h'_-\partial h_+ + h_+\partial h'_-}{j'_+\partial h'_+ - h'_+\partial j'_+}\frac{j'_+\partial h_- + h_-\partial j'_+}{j'_-\partial h'_- - h'_-\partial j'_-} - \frac{(1+\gamma)^2}{4\gamma^2}\frac{k_-k_+}{k'_+k'_-}\frac{h'_-\partial h_- - h_-\partial h'_-}{j'_+\partial h'_+ - h'_+\partial j'_+}\frac{j'_+\partial h_+ - h_+\partial j'_+}{j'_-\partial h'_- - h'_-\partial j'_-}\end{bmatrix}\end{aligned} \tag{29}$$

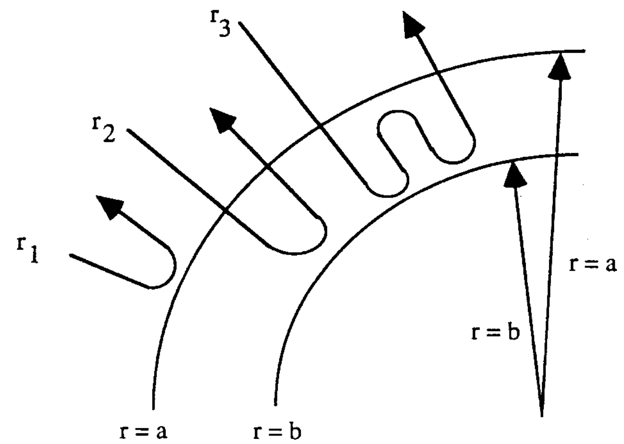


Fig. 3. Multiple bounces used to formulate matrix Riccati equation for chiral spheres.

$\Delta r \to 0$, we arrive at

$$\begin{aligned}
&\lim_{\Delta r\to 0} \mathbf{R} \\
&= \boldsymbol{\chi} = \begin{bmatrix} \frac{dj_+\partial j_+ - j_+ d\partial j_+}{j_+\partial h_+ - h_+\partial j_+} & \frac{1}{2\eta}\frac{d\eta}{dr}\frac{k_-}{k_+}\frac{j_+\partial j_- + j_-\partial j_+}{j_+\partial h_+ - h_+\partial j_+} \\ \frac{1}{2\eta}\frac{d\eta}{dr}\frac{k_+}{k_-}\frac{j_+\partial j_- + j_-\partial j_+}{j_-\partial h_- - h_-\partial j_-} & \frac{dj_-\partial j_- - j_- d\partial j_-}{j_-\partial h_- - h_-\partial j_-} \end{bmatrix} \\
&\lim_{\Delta r\to 0} \underline{\mathbf{R}} \\
&= \boldsymbol{\zeta} = \begin{bmatrix} \frac{dh_+\partial h_+ - h_+ d\partial h_+}{j_+\partial h_+ - h_+\partial j_+} & \frac{1}{2\eta}\frac{d\eta}{dr}\frac{k_-}{k_+}\frac{h_+\partial h_- + h_-\partial h_+}{j_+\partial h_+ - h_+\partial j_+} \\ \frac{1}{2\eta}\frac{d\eta}{dr}\frac{k_+}{k_-}\frac{h_+\partial h_- + h_-\partial h_+}{j_-\partial h_- - h_-\partial j_-} & \frac{dh_-\partial h_- - h_- d\partial h_-}{j_-\partial h_- - h_-\partial j_-} \end{bmatrix} \\
&\lim_{\Delta r\to 0} \mathbf{P} \\
&= \mathbf{P} = \begin{bmatrix} 1 + \frac{Dj_+\partial h_+ - h_+ D\partial j_+}{j_+\partial h_+ - h_+\partial j_+} & 0 \\ 0 & 1 - \frac{Dj_-\partial h_- - h_- D\partial j_-}{j_-\partial h_- - h_-\partial j_-} \end{bmatrix} \\
&= \begin{bmatrix} 1 + \Delta_+^j & 0 \\ 0 & 1 + \Delta_-^j \end{bmatrix} \\
&\lim_{\Delta r\to 0} \underline{\mathbf{P}} \\
&= \underline{\mathbf{P}} = \begin{bmatrix} 1 + \frac{Dh_+\partial j_+ - j_+ D\partial h_+}{j_+\partial h_+ - h_+\partial j_+} & 0 \\ 0 & 1 - \frac{Dh_-\partial j_- - j_- D\partial h_-}{j_-\partial h_- - h_-\partial j_-} \end{bmatrix} \\
&= \begin{bmatrix} 1 + \Delta_+^h & 0 \\ 0 & 1 + \Delta_-^h \end{bmatrix}
\end{aligned} \tag{37}$$

with the notation

$$\begin{aligned}
dj_\pm &= \frac{dj_\pm(x)}{dx}\frac{dx}{dr}\bigg|_{r=\text{const}} = r\frac{dk}{dr}\partial j_\pm \\
Dj_\pm &= \frac{dj_\pm(x)}{dx}\frac{dx}{dr}\bigg|_{k=\text{const}} = k\partial j_\pm \\
d\partial j_\pm &= \frac{d\partial j_\pm(x)}{dx}\frac{dx}{dr}\bigg|_{r=\text{const}} = r\frac{dk}{dr}\partial^2 j_\pm \\
D\partial j_\pm &= \frac{d\partial j_\pm(x)}{dx}\frac{dx}{dr}\bigg|_{k=\text{const}} = k\partial^2 j_\pm.
\end{aligned} \tag{38}$$

Thus, we arrive at

$$\begin{aligned}
\frac{d}{dr}\mathbf{R}(r) &= \lim_{\Delta r\to 0}\frac{\mathbf{R}(r) - \mathbf{R}(r-\Delta r)}{\Delta r} \\
&= \boldsymbol{\chi} + \mathbf{R}\boldsymbol{\zeta}\mathbf{R} + \mathbf{R}\mathbf{P} + \mathbf{P}\mathbf{R}.
\end{aligned} \tag{39}$$

This is the matrix Riccati equation for spherical waves in a chiral medium that forms the major result of this paper. The first two terms on the right-hand side are associated with the magnitude of the reflection coefficient while the second two terms relate to the phase of the reflection coefficient.

Associated with this is a jump condition, used when discontinuities exist in the wave impedance η. Taking the limiting sum of the infinite series in (36), the jump condition is found to be

$$\mathbf{R}^+ = [\mathbf{R}^- - \mathbf{R}][\underline{\underline{\mathbf{I}}} - \mathbf{R}\mathbf{R}^-]^{-1} \tag{40}$$

where $\underline{\underline{\mathbf{I}}}$ is the identity matrix $\mathbf{R}^+$ is the reflection matrix beyond the discontinuity, $\mathbf{R}^-$ is the reflection matrix right on the interior of the discontinuity, and $\mathbf{R}$ is the reflection matrix for a simple boundary between two homogeneous media given in (29). We can further reduce the form of the matrix Riccati equation by defining

$$\kappa = \begin{bmatrix} \frac{\Delta_+^h + \Delta_+^j}{2} & \frac{\Delta_+^h + \Delta_-^h}{2} \\ \frac{\Delta_+^j + \Delta_-^j}{2} & \frac{\Delta_-^h + \Delta_-^j}{2} \end{bmatrix}. \tag{41}$$

This allows us to write the matrix Riccati equation as

$$\frac{d}{dr}\mathbf{R}(r) = \boldsymbol{\chi} + \mathbf{R}\boldsymbol{\zeta}\mathbf{R} + 2\varkappa * \mathbf{R} \tag{42}$$

where the star product is defined by

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} * \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{12}b_{12} \\ a_{21}b_{21} & a_{22}b_{22} \end{bmatrix}. \tag{43}$$

To apply this equation to angular scattering, an incident wave is expanded in terms of the spherical chiral eigenmodes of (12) and (13). The matrix Riccati equation is integrated through the scatterer for each eigenmode to determine the appropriate reflection coefficients. Where discontinuous jumps occur in the material characteristics, the jump condition is used to relate the reflection coefficients across the discontinuity. The scattered wave is then expressed as the series sum of the incident eigenmodes multiplied by these reflection coefficients. Care must be taken if the origin is included in the integration as in a penetrable core, where the initial condition $\mathbf{R}(0) = 0$ should be used for all eigenmodes. Intuitively, this corresponds to the absence of an outwardly traveling wave at the core.

VI. Discussion

We examine the matrix Riccati equation for chiral spheres and compare it with its planar counterpart. We also investigate the behavior of the equation in the high-frequency limit, the low-frequency limit and the small reflection limit.

A. Comparison with Planar Counterpart

It can be shown for large radii of curvature that the matrix Riccati equation reduces to its planar counterpart. Expanding the Bessel and Hankel functions with their large argument approximations, we find that the components of the matrix

Riccati equation reduce to

$$\boldsymbol{\chi} \to \begin{bmatrix} 0 & \frac{1}{2\eta}\frac{d\eta}{dr}\mathrm{e}^{-i(k_-+k_+)r} \\ \frac{1}{2\eta}\frac{d\eta}{dr}\mathrm{e}^{-i(k_-+k_+)r} & 0 \end{bmatrix}$$
$$\boldsymbol{\zeta} \to \begin{bmatrix} 0 & -\frac{1}{2\eta}\frac{d\eta}{dr}\mathrm{e}^{-i(k_-+k_+)r} \\ -\frac{1}{2\eta}\frac{d\eta}{dr}\mathrm{e}^{-i(k_-+k_+)r} & 0 \end{bmatrix} \quad (44)$$
$$\boldsymbol{\kappa} \to \begin{bmatrix} -2ik_+ & -i(k_+ + k_-) \\ -i(k_+ + k_-) & -2ik_- \end{bmatrix}$$

and the matrix Riccati equation reduces to two decoupled equations. This decoupling is expected since curvature is the mechanism for copolar scattering. Thus, in the limit of large radii of curvature, copolar scattering reduces to zero and no coupling exists. If we normalize the reflection coefficients to the incident and scattered wave amplitudes we find

$$\Gamma_n^{vw} = r_n^{vw}(r)\frac{\mathbf{w}_{\pm 1n}(k_-)^h}{\mathbf{v}_{\pm 1n}(k_+)^j} \to r_n^{vw}(r)\mathrm{e}^{2ik_c r}$$
$$\Gamma_n^{wv} = r_n^{wv}(r)\frac{\mathbf{v}_{\pm 1n}(k_+)^h}{\mathbf{w}_{\pm 1n}(k_-)^j} \to r_n^{wv}(r)\mathrm{e}^{2ik_c r} \quad (45)$$

where the average wavenumber k_c is given by $(k_+ + k_-)/2$ and where the superscript h or j refers to the use of Bessel or Hankel functions in the spherical wave expansion, respectively. We obtain

$$\frac{d}{dr}\Gamma_n^{vw}(r) = \frac{1}{2\eta}\left(1 - \left[\Gamma_n^{vw}(r)\right]^2\right)\frac{d\eta}{dr} - 2ik_c\Gamma_n^{vw}(r)$$
$$\frac{d}{dr}\Gamma_n^{wv}(r) = \frac{1}{2\eta}\left(1 - \left[\Gamma_n^{wv}(r)\right]^2\right)\frac{d\eta}{dr} - 2ik_c\Gamma_n^{wv}(r). \quad (46)$$

This is exactly the planar chiral Riccati result [7].

B. High-Frequency Limit

In the high-frequency case, the phase term in the matrix Riccati equation will dominate the scattering behavior and the equation is approximated by

$$\frac{d}{dr}\mathbf{R} = 2\boldsymbol{\varkappa} * \mathbf{R}. \quad (47)$$

This linear differential equation decouples the modes completely and the solution is readily found by integrating each scattering coefficient independently. We find solutions of the form

$$R^{vv} \approx K_{vv}\exp\left(\int \Delta_+^h + \Delta_+^j \, dr\right)$$
$$R^{vw} \approx K_{vw}\exp\left(\int \Delta_+^h + \Delta_-^h \, dr\right)$$
$$R^{wv} \approx K_{wv}\exp\left(\int \Delta_+^j + \Delta_-^j \, dr\right)$$
$$R^{ww} \approx K_{ww}\exp\left(\int \Delta_-^h + \Delta_-^j \, dr\right) \quad (48)$$

where the coefficients K_{vv}, K_{vw}, K_{wv}, and K_{ww} are determined from initial conditions. The integrands above provide the change in phase per unit length for each spherical eigenmode and the solutions are very similar to those found from the WKB method of solution.

C. Low-Frequency Limit

In the low-frequency limit, contribution by the phase terms in the matrix Riccati equation for chiral spheres is insignificant in comparison to the others. Thus, the equation becomes

$$\frac{d}{dr}\mathbf{R} = \boldsymbol{\chi} + \mathbf{R}\boldsymbol{\zeta}\mathbf{R}. \quad (49)$$

Based on this relation, it is seen that the scattering of the cross-polar modes R^{vw} and R^{wv} are dictated solely by the variation in impedance η, while the scattering of the copolar modes R^{vv} and R^{ww} are determined solely by variations in the wavenumbers k_+ and k_-, respectively. Thus, each reflection coefficient is separately controllable by the variation in material properties.

It is interesting to note that, in the planar case, the solution to the matrix Riccati equation in the low-frequency limit yields the Fresnel reflection coefficients. One would assume that the solution in the spherical case should yield a similar result. However, this is not true. In the low-frequency case here, the radius of curvature of the sphere cannot be ignored and plays a significant role in determining the reflection.

D. Weak Scattering Limit

For weak scattering, the nonlinear term in the matrix Riccati equation is insignificant in comparison to the contributions of the other terms. Thus, we write the matrix Riccati equation as

$$\frac{d}{dr}\mathbf{R} = \boldsymbol{\chi} + 2\boldsymbol{\varkappa} * \mathbf{R}. \quad (50)$$

As in the high-frequency case, the differential equation is linearized and the scattering coefficients are decoupled. The solutions for the scattering coefficients, again reminiscent of those from the WKB method are readily found to be

$$R^{vv} \approx A_{vv}(r) + K_{vv}\exp\left(\int \Delta_+^h + \Delta_+^j \, dr\right)$$
$$R^{vw} \approx A_{vw}(r) + K_{vw}\exp\left(\int \Delta_+^h + \Delta_-^h \, dr\right)$$
$$R^{wv} \approx A_{wv}(r) + K_{wv}\exp\left(\int \Delta_+^j + \Delta_-^j \, dr\right)$$
$$R^{ww} \approx A_{ww}(r) + K_{ww}\exp\left(\int \Delta_-^h + \Delta_-^j \, dr\right) \quad (51)$$

where the coefficients A_{vv}, A_{vw}, A_{wv}, A_{ww}, K_{vv}, K_{vw}, K_{wv}, and K_{ww} are determined from initial conditions. As before, the change in phase per unit length for each spherical eigenmode is provided in the integrand.

E. Constant Impedance

For a radially stratified sphere of constant impedance, we have $\partial\eta/\partial r = 0$ and the matrix Riccati equation reduces to

$$\boldsymbol{\chi} \to \begin{bmatrix} \frac{dj_+\partial j_+ - j_+ d\partial j_+}{j_+\partial h_+ - h_+\partial j_+} & 0 \\ 0 & \frac{dj_-\partial j_- - j_- d\partial j_-}{j_-\partial h_- - h_-\partial j_-} \end{bmatrix}$$
$$\boldsymbol{\zeta} \to \begin{bmatrix} \frac{dh_+\partial h_+ - h_+ d\partial h_+}{j_+\partial h_+ - h_+\partial j_+} & 0 \\ 0 & \frac{dh_-\partial h_- - h_- d\partial h_-}{j_-\partial h_- - h_-\partial j_-} \end{bmatrix}. \quad (52)$$

As in the high-frequency case, the equations decouple and we have

$$\begin{aligned}\frac{d}{dr}r_n^{vv}(r) &= \frac{dj_+\partial j_+ - j_+ d\partial j_+}{j_+\partial h_+ - h_+\partial j_+} \\ &\quad - \left[r_n^{vv}(r)\right]^2 \frac{dh_+\partial h_+ - h_+ d\partial h_+}{j_+\partial h_+ - h_+\partial j_+} \\ &\quad + \frac{1}{2}\left(\frac{\Delta_+^h + \Delta_+^j}{2}\right) r_n^{vv}(r) \\ \frac{d}{dr}r_n^{ww}(r) &= \frac{dj_-\partial j_- - j_- d\partial j_-}{j_-\partial h_- - h_-\partial j_-} \\ &\quad - \left[r_n^{ww}(r)\right]^2 \frac{dh_-\partial h_- - h_- d\partial h_-}{j_-\partial h_- - h_-\partial j_-} \\ &\quad + \frac{1}{2}\left(\frac{\Delta_-^h + \Delta_-^j}{2}\right) r_n^{ww}(r). \qquad (53)\end{aligned}$$

From these equations, it is clearly seen that variations in wavenumber are the mechanism for scattering for copolar modes. We similarly conclude that variations in impedance are the mechanism for cross-polar scattering.

VII. Conclusion

Motivated by the pioneering work of Ambartsumian, Bellman, and Latham, we derived, for the first time, the matrix Riccati equation for chiral spheres in this paper, useful for calculating the exact scattered fields from continuously or discontinuously stratified chiral spheres. We derived an associated set of jump conditions used to link fields across discontinuities in permittivity, permeability, or chirality. This matrix Riccati equation converts the boundary value scattering problem into an initial value problem amenable to efficient numerical solution.

The matrix Riccati equation, formulated as a coupled matrix nonlinear differential equation, is not easily solvable by analytic solution. However, in the high-frequency limit, the relation decouples into two differential equations that are directly solvable. This is also true in the weak scattering and constant impedance limits. At large radii, the equation was shown to reduce to the planar result as expected, based on physical insight.

Further research with this equation is continuing and the application of this equation to investigate chiral spherical Luneberg lenses as well as lossy inhomogeneous screens for anti-reflection is forthcoming in a following paper.

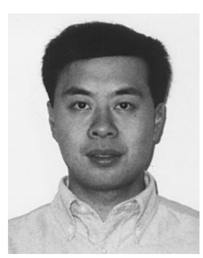
References

[1] V. A. Ambartsumian, "Diffuse reflection of light by a foggy medium," *Comp. Rend. Acad. Sci. U.R.S.S.*, vol. 38, pp. 229–232, 1943.
[2] R. Bellman and R. Kalaba, "Functional equations, wave propagation and invariant imbedding," *J. Math Mech.*, vol. 8, pp. 683–702, 1959.
[3] R. W. Latham, "Electromagnetic scattering from cylindrically and spherically stratified bodies," *Can. J. Phys.*, vol. 46, pp. 1463–1468, 1968.
[4] S. Bassiri, C. H. Papas, and N. Engheta, "Electromagnetic wave propagation through a dielectric-chiral interface and through a chiral slab," *J. Opt. Soc. Amer. A*, vol. 5, pp. 1450–1459, 1988.
[5] M. I. Oksanen, S. A. Tretyakov, and I. V. Lindell, "Vector circuit theory for isotropic and chiral slabs," *J. Electromagn. Waves Applicat.*, vol. 4, pp. 613–643, 1990.
[6] J. C. Liu and D. L. Jaggard, "Chiral layers on planar surfaces," *J. Electromagn. Wave Applicat.*, vols. 5/6, pp. 651–668, 1992.
[7] D. L. Jaggard and X. Sun, "Theory of chiral multilayers," *J. Opt. Soc. Amer. A*, vol. 5, pp. 804–813, 1992.
[8] C. F. Bohren, "Scattering of electromagnetic waves by an optically active cylinder," *J. Colloid Interface Sci.*, vol. 66, pp. 105–109, 1978.
[9] M. S. Kluskens and E. H. Newman, "Scattering by a mutlilayer chiral cylinder," *IEEE Trans. Antennas Propagat.*, vol. 39, pp. 96–99, Jan. 1991.
[10] D. L. Jaggard and J. C. Liu, "Chiral layers on curved surfaces," *J. Electromagn. Waves Applicat.*, vols. 5/6, pp. 669–694, 1992.
[11] C. F. Bohren, "Light scattering by an optically active sphere," *Chem. Phys. Lett.*, vol. 29, pp. 458–462, 1974.
[12] P. L. E. Uslenghi, "Scattering by an impedance sphere coated with a chiral layer," *Electromagn.*, vol. 10, pp. 201–211, 1990.
[13] J. A. Stratton, *Electromagnetic Theory*. New York, McGraw-Hill, 1941.

Dwight L. Jaggard (S'68–M'77–SM'86–F'91) was born in Oceanside, NY, in 1948. He received the B.S.E.E. and M.S.E.E. degrees from the University of Wisconsin, Madison, in 1971 and 1972, respectively, and the Ph.D. degree in electrical engineering and applied physics from the California Institute of Technology, Pasadena, in 1976.

From 1976 to 1978, he was a Postdoctoral Research Fellow at Caltech and a Consultant to the Jet Propulsion Laboratory, Pasadena, CA. In 1978 he joined the faculty at the University of Utah, Salt Lake City, as an Assistant Professor of electrical engineering. Since 1980 he has performed research in wave interactions with complex media, inverse scattering, and high-resolution imaging and has taught at the Moore School of Electrical Engineering, University of Pennsylvania, Philadelphia, where he is a Professor of electrical engineering and Associate Dean for Graduate Education and Research. He has served as an editor of the *Journal of Electromagnetic Wave Applications* and is the coeditor of *Recent Advances in Electromagnetic Theory* (New York: Springer-Verlag, 1990). He has also made contributions to several books, including *Symmetry in Electromagnetics* (New York: Taylor Francis, 1995). *Fractals in Engineering* (New York: Springer-Verlag, 1997), and the forthcoming *Frontiers in Electromagnetics* (Piscataway, NJ: IEEE Press, 1999). Through the Complex Media Laboratory, his research currently involves novel applications of electromagnetic chirality and fractal electrodynamics, the use of topology and symmetry in electromagnetic scattering, and inverse scattering and imaging.

Dr. Jaggard received the S. Reid Warren Award for Distinguished Teaching in 1985 and the Christian R. and Mary R. Lindback Award for Distinguished Teaching in 1987. He was elected Fellow of the Optical Society of America in 1995 for his work in wave interactions in complex media. He has served as an associate editor of the IEEE Transactions on Antennas and Propagation and was on the editorial board of the *Proceedings of the IEEE*.



John C. Liu was born in Hong Kong. He received the B.S.E.E., M.S.E.E, and Ph.D. degrees in electromagnetics from the University of Pennsylvania, Philadelphia, in 1988, 1992, and 1996, respectively.

He has worked in foreign exchange options for Bank of America, New York, and in stock index options for Lehman Brothers, New York. He is currently in charge of stock index options trading for Banque Paribas, New York.