Accurate and Efficient Simulation of Antennas Using Hierarchical Mixed-Order Tangential Vector Finite Elements for Tetrahedra

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***Abstract*— Hierarchical mixed-order tangential vector finite elements (TVFE's) for tetrahedral elements are attractive for accurate and efficient finite-element method simulation of complicated electromagnetic problems. They provide versatility in the geometric modeling of physical structures, guarantee solutions free of spurious modes, and allow local increase of resolution by selective expansion of the unknown electromagnetic field, i.e., by combination of mixed-order TVFE's of different orders within a computational domain. For a realistic antenna radiation problem, this paper demonstrates that field expansion using lowest and higher order hierarchical mixed-order TVFE's selectively is vastly superior [in terms of accuracy, memory, as well as central processing unit (CPU)-time] to field expansion using a lowest order mixed-order TVFE only.**

***Index Terms*—Finite-element method, hierarchical basis functions, tangential vector finite elements.**

I. Introduction

The finite-element method (FEM) has proven attractive for simulation of complicated electromagnetic problems. Use of tetrahedral meshes provides versatility in the geometric modeling of physical structures and field expansions based on mixed-order tangential vector finite elements (TVFE's) guarantee solutions free of spurious modes and provide for easy enforcement of boundary conditions. Hierarchical mixed-order TVFE's (where the vector basis functions forming a mixed-order TVFE of a given order are a subset of the vector basis functions forming mixed-order TVFE's of higher orders) have the additional advantage of permitting selective field expansion. That is, they allow for combination of mixed-order TVFE's of different orders within a computational domain for efficient expansion of the unknown electromagnetic field. This is relevant for simulating a large class of realistic electromagnetic problems characterized by disjoint regions with high and low field variation. Thus, FEM analysis with fields expanded using hierarchical mixed-order TVFE's for tetrahedral elements is attractive for accurate and efficient solution of certain classes of electromagnetic problems.

Hierarchical mixed-order TVFE's for tetrahedral elements have been proposed up to and including order 1.5 by Webb and Forghani [1] and up to and including order 2.5 by Andersen and Volakis [2], [3]. However, no test has yet been carried out that numerically demonstrates the potential of selective field expansion for realistic electromagnetic problems. The purpose of this paper is to do so using the hierarchical mixed-order TVFE's of order 0.5 and 1.5 for tetrahedral elements proposed in [2] and [3]. For eigenvalue computation, the convergence rate of the hierarchical mixed-order TVFE of order 1.5 is shown to be comparable to that of a nonhierarchical mixed-order TVFE of order 1.5 [4]. For a realistic antenna radiation problem, field expansion using the hierarchical mixed-order TVFE's of order 0.5 and 1.5 selectively is shown to be vastly superior [in terms of accuracy, memory as well as central processing unit (CPU) time] to field expansion using the mixed-order TVFE of order 0.5 only.

This paper is organized as follows. Section II presents the hierarchical mixed-order TVFE's of order 0.5 and 1.5 for tetrahedral elements that will be used for field expansion. Section III presents the numerical results. Section IV concludes the paper.

II. Presentation of TVFE's

We consider a tetrahedral element with nodes 1, 2, 3, and 4. The volume of the tetrahedron is denoted by V. Simplex (or volume) coordinates ζ_1, ζ_2, ζ_3, and ζ_4 at a point P are defined in the usual manner, viz. $\zeta_n = V_n/V$ where V_n denotes the volume of the tetrahedron formed by P and the nodes of the triangular face opposite to node n. Below, we review the hierarchical mixed-order TVFE's of order 0.5 and 1.5 for tetrahedral elements that will be used for field expansion. For more details, see [2] and [3]. We note that the vector basis functions presented in this paper are not normalized. Furthermore, the indexes i, j, and k in (1)–(3) are implicitly assumed to belong to the set $\{1, 2, 3, 4\}$.

A mixed-order TVFE of order 0.5 providing constant tangential/linear normal variation along element edges and linear variation at element faces and inside the element is characterized by six linearly independent vector basis functions. Whitney [5] initially presented such a TVFE. It is characterized by the six edge-based vector basis functions

$$\zeta_i \nabla \zeta_j - \zeta_j \nabla \zeta_i, \qquad i < j. \tag{1}$$

A mixed-order TVFE of order 1.5 providing linear tangential/quadratic normal variation along element edges and quadratic variation at element faces and inside the element is characterized by 20 linearly independent vector basis functions. Andersen and Volakis [2], [3] presented a mixed-

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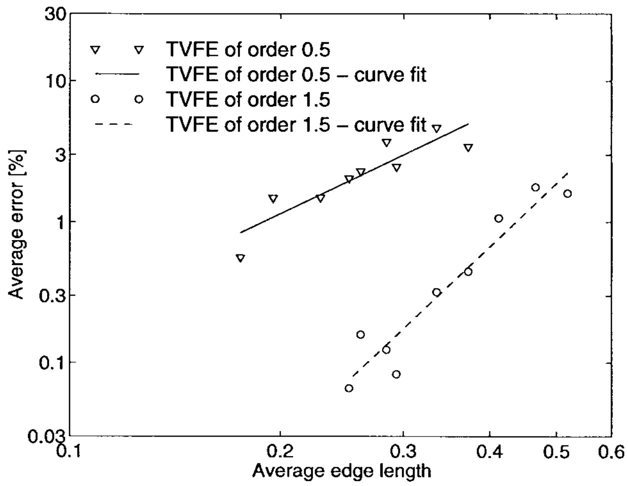

Fig. 1. Convergence rate for expansion of field within homogeneous andisotropic rectangular cavity using hierarchical mixed-order TVFE's of order 0.5 and 1.5.

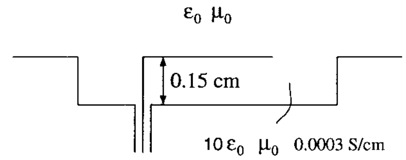

Fig. 2. Side view of square metallic patch antenna backed by a dielectric-filled rectangular cavity recessed in an infinite metallic ground plane.

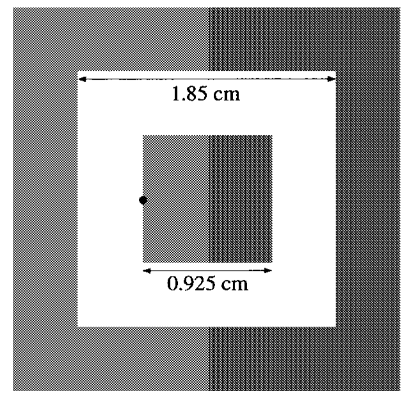

Fig. 3. Top view of square metallic patch antenna backed by a dielectric-filled rectangular cavity recessed in an infinite metallic ground plane.

order TVFE of order 1.5 that compares hierarchically to the mixed-order TVFE of order 0.5 presented by Whitney [5]. In addition to the six edge-based vector basis functions (1), it is characterized by the six edge-based vector basis functions

$$(\zeta_i - \zeta_j)(\zeta_i \nabla \zeta_j - \zeta_j \nabla \zeta_i), \qquad i < j \tag{2}$$

and the eight face-based vector basis functions

$$\left.\begin{array}{l}\zeta_k(\zeta_i \nabla \zeta_j - \zeta_j \nabla \zeta_i)\\ \zeta_j(\zeta_k \nabla \zeta_i - \zeta_i \nabla \zeta_k)\end{array}\right\}, \qquad i < j < k. \tag{3}$$

III. Numerical Results

The objective of this section is to numerically demonstrate the potential of the hierarchical mixed-order TVFE's of order 0.5 and 1.5 proposed in [2] and [3] and summarized above. The eigenvalues of a homogeneous and isotropic rectangular cavity are determined numerically for different uniform tetrahedral meshes to show that the convergence rate of the hierarchical mixed-order TVFE of order 1.5 is comparable to that of a nonhierarchical mixed-order TVFE of order 1.5 [4]. The input impedance (and hereby the resonant frequency) of a probe-fed square metallic patch antenna backed by a dielectric-filled cavity recessed in an infinite metallic ground plane is determined numerically for different uniform tetrahedral meshes to show that field expansion using the hierarchical mixed-order TVFE's of order 0.5 and 1.5 selectively is vastly superior (in terms of accuracy, memory as well as CPU time) to field expansion using the mixed-order TVFE of order 0.5 only.

Consider a homogeneous, isotropic rectangular cavity of normalized dimensions $1 \times 0.75 \times 0.5$. The exact eigenvalues for this geometry are well known [6]. A FEM solution for the eigenvalues of the cavity is carried out for various uniform tetrahedral meshes of different average edge length with the hierarchical mixed-order TVFE's of order 0.5 and 1.5 used for field expansion (for the formulation, see for instance [7]).

The convergence rate for the two cases is illustrated in Fig. 1 where the average error of the first eight eigenvalues is plotted in percent as a function of the average edge length in the mesh (log–log plot). The approximate distribution around a straight line suggests that the average error decreases as x^p for a decreasing average edge length. For the mixed-order TVFE of order 0.5, the exponent is $p = 2.37$, which is slightly larger than the expected value of two [4]. This is due to the very low average error 0.56% for the average edge length 0.175. Similarly, for the hierarchical mixed-order TVFE of order 1.5, the exponent is $p = 4.66$, which is again larger than the expected value of four [4] and the exponent $p = 3.86$ found in [4] for a different and nonhierarchical mixed-order TVFE of order 1.5. This demonstrates that the hierarchical mixed-order TVFE of order 1.5 in [2] and [3] has slightly better convergence properties than the nonhierarchical one in [4] for this particular geometry and for the employed meshes. However, a relatively large uncertainty range can be expected for such numerically obtained exponents and, thus, no general statement can be made regarding the rigor of results based on the two different mixed-order TVFE's of order 1.5.

Consider a square metallic patch antenna backed by a rectangular cavity recessed in an infinite metallic ground plane, as illustrated in Fig. 2 (side view) and Fig. 3 (top view). The cavity-backed patch antenna is situated in free-space characterized by the permittivity ε_0 and the permeability μ_0. The cavity is of dimensions 1.85 cm $\times$ 1.85 cm $\times$ 0.15 cm and filled with a dielectric material of permittivity $10\epsilon_0$ and conductivity 0.0003 S/cm. The patch is of sidelength 0.925 cm and centered in the cavity aperture. It is fed by a vertical coaxial line whose outer conductor is attached to the ground plane and whose inner conductor is attached to the patch at the mid point of an edge, as illustrated in Figs. 2 and 3. The coaxial feed will be modeled as a vertical probe of constant current.

An almost identical antenna was considered by Schuster and Luebbers [8]. In [8], the cavity walls and the ground plane was removed and a similar patch on a similar but finite grounded dielectric substrate was analyzed using the finite-difference

TABLE I
COMPUTATIONAL EFFORT FOR CASE 1–6 FOR ANTENNA IN FIGS. 2 AND 3

Case	TVFE order(s)	Average edge length [cm]	Resonant frequency [GHz]	# of unknowns	# of BI unknowns	# of matrix entries	Time per frequency point [sec]
1	0.5	0.260	3.974	345	120	17119	7.52
2	0.5	0.188	4.147	817	288	89695	44.78
3	0.5	0.153	4.258	1489	528	291359	222.92
4	0.5	0.133	4.302	2361	840	725791	771.59
5	0.5/1.5	0.260	4.323	827	120	30675	17.33
6	0.5/1.5	0.188	4.437	1467	288	107963	77.28

time-domain (FDTD) method. In spite of these geometrical differences, the two antennas are expected to have the same input impedance and, consequently, the same resonant frequency since the dominant fields are confined to a volume under and in the near vicinity of the patch. The resonant frequency was found in [8] to be 4.43 GHz. The resistance at resonance was found to be 400 Ω while the reactance was in the range of 230 Ω to −170 Ω close to resonance. We note that the results in [8] were found with an extremely fine discretization and hence can be considered accurate.

The patch antenna is analyzed using the finite-element boundary-integral (FE-BI) method (for the formulation, see, for instance, [7]) in conjunction with an iterative quasi-minimal residual solver [9]. We discretize the cavity into tetrahedral elements and consequently discretize the surface forming the boundary between the cavity and free-space into triangular faces. Two different TVFE options are applied. The first TVFE option is to use the mixed-order TVFE of order 0.5 throughout the mesh. For a mesh of average edge length 0.260 cm (Case 1), the input impedance is determined as a function of frequency and the resonant frequency of the patch is predicted. The coarse discretization of Case 1 means that this resonant frequency is most likely not accurate. For meshes of average edge lengths of 0.188 cm (Case 2), 0.153 cm (Case 3), and 0.133 cm (Case 4), more accurate resonant frequencies but also higher computational costs can be expected. The second TVFE option is to use the mixed-order TVFE of order 1.5 close to the radiating edges (where we expect high field variation) and the mixed-order TVFE of order 0.5 elsewhere (where we expect little field variation). For the meshes of average edge length 0.260 cm (Case 5) and 0.188 cm (Case 6), the input impedance is again determined and the resonant frequency is again predicted. The effectiveness of this approach (Case 5–6) in terms of accuracy/CPU time/memory requirements is compared to the previous one (Case 1–4). The six cases are summarized in Table I.

Real and imaginary parts of the input impedance as a function of frequency are given in Figs. 4 and 5 for Case 1–6 and corresponding resonant frequencies are provided in Table I. For Case 1–4, a larger and larger resonant frequency

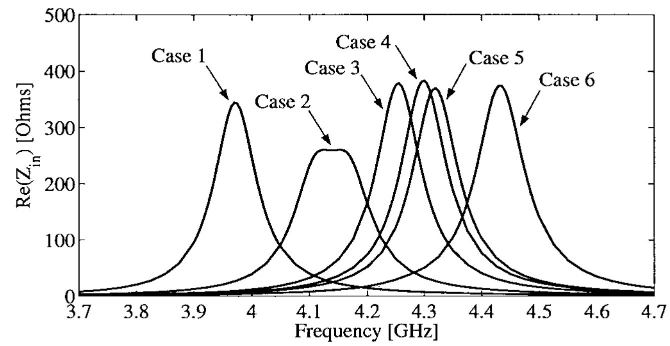

Fig. 4. Real part of the input impedance of the antenna in Figs. 2 and 3 for Case 1–6.

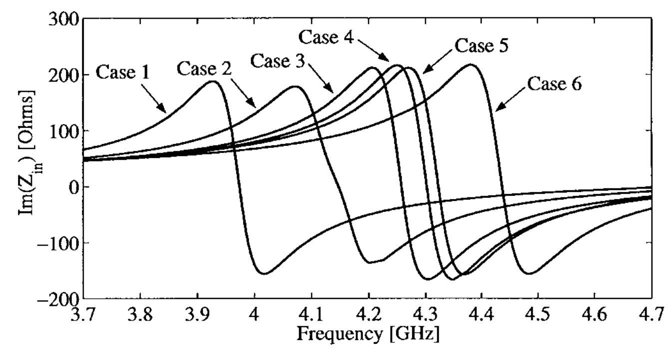

Fig. 5. Imaginary part of the input impedance of the antenna in Figs. 2 and 3 for Case 1–6.

is observed as the mesh becomes denser and denser. However, even for Case 4, the error as compared to the result obtained by Schuster and Luebbers is quite large (2.98%) for resonant frequency computation. Use of selective field expansion (Case 5–6) leads to a significant accuracy improvement. Case 5 (error 2.42%) gives a more accurate result than Case 1–4 and Case 6 (error 0.16%) matches the result by Schuster and Luebbers almost exactly. The computational cost (number of unknowns, number of boundary integral (BI) unknowns, number of nonzero matrix entries (memory usage) and CPU time per frequency point) to obtain these results are also given in Table I. It is evident that the second TVFE option corresponding to Case 5–6 is significantly more attractive than the first TVFE option corresponding to Case 1–4. Case 5 gives a more accurate result than Case 4 but uses only 4.22% of the memory and 2.15% of the CPU time that Case 4 does. The accuracy of Case 6 is vastly superior to that of Case 4 and yet Case 6 uses only 14.88% of the memory and 10.02% of the CPU time that Case 4 does. We note that the savings in Case 5 and Case 6 are reached in part because coarse meshes with higher order TVFE's lead to significantly smaller BI portions of the resulting matrix equation systems than fine meshes with lowest order TVFE's.

IV. CONCLUSION

The potential of the hierarchical mixed-order TVFE's of order 0.5 and 1.5 for tetrahedral elements proposed by Andersen and Volakis [2], [3] was demonstrated. For eigenvalue computation, the convergence rate of the hierarchical mixed-order TVFE of order 1.5 was shown to be comparable to that of a nonhierarchical mixed-order TVFE of order 1.5 [4]. For a realistic antenna radiation problem, field expansion using the hierarchical mixed-order TVFE's of order 0.5 and 1.5 selectively was shown to be vastly superior (in terms of accuracy, memory, as well as CPU time) to field expansion using the mixed-order TVFE of order 0.5 only.

REFERENCES

[1] J. P. Webb and B. Forghani, "Hierarchal scalar and vector tetrahedra," *IEEE Trans. Magn.*, vol. 29, pp. 1495–1498, Mar. 1993.

[2] L. S. Andersen and J. L. Volakis, "Hierarchical tangential vector finite elements for tetrahedra," *IEEE Microwave Guided Wave Lett.*, vol. 8, pp. 127–129, Mar. 1998.

[3] ——, "Hierarchical tangential vector finite elements for tetrahedra," in *Proc. IEEE Antennas Propagat. Soc. Int. Symp.*, Atlanta, GA, June 1998, vol. 1, pp. 240–243.

[4] J. S. Savage and A. F. Peterson, "Higher-order vector finite elements for tetrahedral cells," *IEEE Trans. Microwave Theory Tech.*, vol. 44, pp. 874–879, June 1996.

[5] H. Whitney, *Geometric Integration Theory*. Princeton, NJ: Princeton Univ. Press, 1957.

[6] R. F. Harrington, *Time Harmonic Electromagnetic Fields*. New York: McGraw-Hill, 1961.

[7] J. L. Volakis, A. Chatterjee, and L. C. Kempel, *Finite Element Method for Electromagnetics*. Piscataway, NJ: IEEE Press, 1998.

[8] J. W. Schuster and R. Luebbers, private communication.

[9] Y. Saad, *Iterative Methods for Sparse Linear Systems*. Boston, MA: PWS, 1996.

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