On Iterative Approaches for Electromagnetic Rough-Surface Scattering Problems

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***Abstract*— Iterative techniques developed for solving general systems of linear equations have been applied to systems resulting from electromagnetic rough-surface scattering problems. Recently used iterative procedures that model the multiple scattering of the electromagnetic energy are shown to be mathematically equivalent to the application of stationary iterative procedures to the system of equations resulting from the standard moment method. Convergence difficulties that are sometimes observed with these approaches are due to the inherent limitations of the stationary techniques. The performances of the stationary approaches are compared with that of several conjugate-direction-based nonstationary iterative procedures through the application to a series of scattering surfaces that yield rapidly changing conditioning of the moment-method interaction matrix. The stationary algorithms give the quickest convergence when applied to the systems with the best conditioning, but the nonstationary techniques prove much more robust in other more ill-conditioned situations.**

***Index Terms*—Convergence of numerical methods, electromagnetic scattering by rough surface.**

I. Introduction

THE primary factor limiting the use of the moment method in the calculation of electromagnetic scattering from rough surfaces is that a linear system of equations must be solved to yield the currents induced on the scatterer. Direct solution methods such as LU decomposition are $O(N^3)$ operations, where N is the number of unknowns in the discretized representation of the the surface current. As the size of the scatterer increases the computational expense of these operations becomes prohibitive. This has led to the development of iterative schemes that solve for the surface current in $O(N^2)$ steps.

Two different general approaches have recently been followed. In the first the surface current is initially approximated by the Kirchhoff (or physical optics) approximation applied to the incident field [1]–[3]. The current is then updated by applying the surface boundary conditions to the scattered field associated with the previous iteration's current. Using the "extended Kirchhoff approximation" approach of Liszka and McCoy [4], Ericson and Lyzenga [3] update all the current values from the previous iteration simultaneously. Kapp and Brown [1] and Holliday *et al.* [2] use the updated current values as they become available, choosing the ordering of the updates to follow the multiple scattering paths on the surface. This led to Kapp and Brown and Holliday *et al.* terming the functionally identical approaches the method of ordered multiple interactions (MOMI) and the forward–backward technique, respectively. These approaches have proven very effective in solving the systems that result from magnetic field integral equation (MFIE) treatments of scattering from perfectly electrically conducting (PEC) surfaces that are single valued and rough in one dimension. However, Tran [5] applied MOMI to scattering from a two-dimensionally rough surface and found that the convergence of the iterative process depended strongly on the order in which the current elements were updated (with some orderings leading to divergence), and observed a failure to converge in one case even when optimal ordering was used. Torrungrueng and Newman [6] introduced a technique that is similar to MOMI, termed the multiple sweep method of moments (MSMM), and found that it yields rapid convergence when applied to scattering from a resistively loaded flat plate but diverges when applied to a PEC closed cylinder. Adams and Brown [7] then applied MOMI to a circular cylindrical scatterer and showed that it diverges independent of the interaction ordering used when a straight MFIE formulation is used and the cylinder is more than a few tenths of a wavelength in radius.

The second class of iterative approaches used to solve moment method systems of linear equations are nonstationary techniques. These are extensions of the standard conjugate gradient method that were developed to solve general asymmetric/non-Hermitian systems of equations [8] and therefore do not attempt to model the physical multiple scatterings of the electromagnetic energy directly. Examples of this are given by Smith *et al.* [9], who used the biconjugate gradient (BICG) method, Cao and Macaskill [10], who used a generalized conjugate gradient (GCG) method, Donohue *et al.* [11], who used a preconditioned multigrid generalized conjugate residual (GCR) approach, and Chen [12], who used the quasi-minimum residual method (QMR).

In this paper, the performance of iterative techniques in solving moment-method systems of linear equations under various conditions is examined. We first show that the extended Kirchoff approximation and MOMI algorithms are mathematically equivalent to well-known stationary iteration methods applied to the moment-method linear system of equations. The performance of these approaches and that of several state-of-the-art nonstationary iteration algorithms that have been developed in the last several years for general

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complex systems is examined through the application to a series of surface profiles that approximate breaking water waves. Considerations for the choice of an iterative routine for general scattering problems are then discussed.

II. Iterative Techniques

Iterative solution techniques are applied to systems of the form

$$\boldsymbol{Z}\boldsymbol{j} = \boldsymbol{j}^{\boldsymbol{i}}. \tag{1}$$

For electromagnetic scattering problems $\boldsymbol{Z}$ is the $N \times N$ interaction matrix and $\boldsymbol{j}^{\boldsymbol{i}}$ is the N-length source vector, both of which result from the discretization of an appropriate field integral equation. When an MFIE is used for rough surface scattering, $\boldsymbol{j}^{\boldsymbol{i}}$ is equal to the physical optics (or Kirchhoff approximation) current without correction for shadowing. $\boldsymbol{j}$ is a length N vector that represents the unknown (discretized) current to be found. For simplicity we will assume that the diagonal elements of $\boldsymbol{Z}$ are unity. This is naturally occurring in many formulations where the MFIE is applied to a perfectly conducting scattering surfaces [1], [3] and any system of the type in (1) can be cast into this form using point-Jacobi preconditioning, where the rows of $\boldsymbol{Z}$ and the corresponding entries of $\boldsymbol{j}^{\boldsymbol{i}}$ are normalized to the value of the diagonal entry in that row [8].

A. Stationary Algorithms

Stationary iterative schemes to solve (1) can be written in the form [13]

$$\boldsymbol{j}^{(n+1)} = \boldsymbol{G}\boldsymbol{j}^{(n)} + \boldsymbol{k} \tag{2}$$

where

$$\boldsymbol{G} = \boldsymbol{I} - \boldsymbol{Q}^{-1}\boldsymbol{Z} \tag{3}$$

$$\boldsymbol{k} = \boldsymbol{Q}^{-1}\boldsymbol{j}^{\boldsymbol{i}}. \tag{4}$$

$\boldsymbol{I}$ is the identity matrix, $\boldsymbol{Q}$ is the "splitting" matrix, and $\boldsymbol{G}$ is the "iteration" matrix. Different definitions of $\boldsymbol{Q}$ define the different stationary iteration techniques. Recursively substituting for the $\boldsymbol{j}^{(n)}$ terms in (2) gives the iteration scheme

$$\boldsymbol{j}^{(n+1)} = \boldsymbol{G}^n\boldsymbol{j}^{(0)} + \sum_{m=0}^{n-1} \boldsymbol{G}^m\boldsymbol{k}. \tag{5}$$

Stationary iterative algorithms have received considerable attention the literature [14]–[17]. They were formulated primarily under the assumption that the iteration matrix $\boldsymbol{Z}$ is symmetric (in the real case) or Hermitian (in the complex case) [18]. Equation (2) results from the Neumann series expansion

$$(\boldsymbol{I} - \boldsymbol{G})^{-1} = \boldsymbol{I} + \boldsymbol{G} + \boldsymbol{G}^2 + \boldsymbol{G}^3 + \cdots \tag{6}$$

so a necessary and sufficient condition for the convergence of stationary iterative procedures is that the eigenvalues of the iteration matrix $\boldsymbol{G}$ lie within the unit circle on the complex plane [14]. Following are two definitions of $\boldsymbol{Q}$ that lead to iterative schemes used in the literature recently.

1) Jacobi Iteration: The most straightforward iteration scheme is Jacobi iteration given by [13]

$$\boldsymbol{Q} = \boldsymbol{D} \tag{7}$$

where $\boldsymbol{D}$ is a diagonal matrix of the same diagonal elements as $\boldsymbol{Z}$. ($\boldsymbol{D} = \boldsymbol{I}$ for the preconditioned system considered here.) Substituting into (2) therefore gives

$$\boldsymbol{j}^{(n+1)} = (\boldsymbol{I} - \boldsymbol{Z})\boldsymbol{j}^{(n)} + \boldsymbol{j}^{\boldsymbol{i}}. \tag{8}$$

Letting the initial guess $\boldsymbol{j}^{(0)} = 0$ and using $\boldsymbol{A} = \boldsymbol{I} - \boldsymbol{Z}$ gives

$$\begin{aligned} \boldsymbol{j}^{(1)} &= \boldsymbol{j}^{\boldsymbol{i}} \\ \boldsymbol{j}^{(n_1)} &= \boldsymbol{A}\boldsymbol{j}^{(n)} + \boldsymbol{j}^{\boldsymbol{i}} \end{aligned} \tag{9}$$

Equation (9) is the iterative scheme given in (4) of Liszka and McCoy [3]. The extended Kirchhoff approximation approach is therefore mathematically equivalent to Jacobi iteration of the moment-method system using a zero vector as the initial guess. Tran [5] has shown conditions under which this approach does not converge.

2) SSOR Iteration: Symmetric successive over-relaxation (SSOR) iteration is obtained by setting [13]

$$\boldsymbol{Q} = \frac{\omega}{2-\omega}\left(\frac{1}{\omega}\boldsymbol{D} - \boldsymbol{L}\right)\boldsymbol{D}^{-1}\left(\frac{1}{\omega}\boldsymbol{D} - \boldsymbol{U}\right) \tag{10}$$

where ω is the "relaxation factor" $(0 < \omega < 2)$ and $\boldsymbol{L}$ and $\boldsymbol{U}$ are lower and upper triangular matrices, respectively, found from $\boldsymbol{Z} = \boldsymbol{D} - \boldsymbol{L} - \boldsymbol{U}$. Setting $\omega = 1$, again using $\boldsymbol{D} = \boldsymbol{I}$, and substituting into (5) yields

$$\begin{aligned} \boldsymbol{j}^{(n+1)} = {} & [\boldsymbol{I} - (\boldsymbol{I} - \boldsymbol{U})^{-1}(\boldsymbol{I} - \boldsymbol{L})^{-1}]^n\boldsymbol{j}^{(0)} \\ & + \sum_{m=0}^{n-1} [\boldsymbol{I} - (\boldsymbol{I} - \boldsymbol{U})^{-1}(\boldsymbol{I} - \boldsymbol{L})^{-1}\boldsymbol{Z}]^m \\ & \cdot (\boldsymbol{I} - \boldsymbol{U})^{-1}(\boldsymbol{I} - \boldsymbol{L})^{-1}\boldsymbol{j}^{\boldsymbol{i}}. \end{aligned} \tag{11}$$

Letting the initial guess $\boldsymbol{j}^{(0)} = 0$ simplifies this to

$$\begin{aligned} \boldsymbol{j}^{(n+1)} = {} & \sum_{m=0}^{n-1} [\boldsymbol{I} - (\boldsymbol{I} - \boldsymbol{U})^{-1}(\boldsymbol{I} - \boldsymbol{L})^{-1}\boldsymbol{Z}]^m \\ & \times (\boldsymbol{I} - \boldsymbol{U})^{-1}(\boldsymbol{I} - \boldsymbol{L})^{-1}\boldsymbol{j}^{\boldsymbol{i}}. \end{aligned} \tag{12}$$

Following Kapp and Brown [1] we now use

$$\boldsymbol{Z} = \boldsymbol{I} - \boldsymbol{L} - \boldsymbol{U} = (\boldsymbol{I} - \boldsymbol{L})(\boldsymbol{I} - \boldsymbol{U}) - \boldsymbol{L}\boldsymbol{U}. \tag{13}$$

Substituting (13) into (12) gives

$$\begin{aligned} \boldsymbol{j}^{(n+1)} = {} & \sum_{m=0}^{n-1} [(\boldsymbol{I} - \boldsymbol{U})^{-1}(\boldsymbol{I} - \boldsymbol{L})^{-1}\boldsymbol{L}\boldsymbol{U}]^m \\ & \times (\boldsymbol{I} - \boldsymbol{U})^{-1}(\boldsymbol{I} - \boldsymbol{L})^{-1}\boldsymbol{j}^{\boldsymbol{i}}. \end{aligned} \tag{14}$$

This is the iteration scheme given in (17) of Kapp and Brown [1]. MOMI (and, hence, forward–backward) is therefore mathematically equivalent to SSOR with a relaxation factor of $\omega = 1$ and a zero initial guess vector.

Note that changing the order that the surface interactions incorporated in the interaction matrix $\boldsymbol{Z}$ ultimately changes the eigenvalues of the iteration matrix $\boldsymbol{G}$. The convergence of the SSOR is therefore strongly affected by ordering used, as demonstrated by Tran [5]. Modification of the relaxation factor ω has similar effects.

B. Nonstationary Algorithms

Nonstationary iterative procedures are those that generate a new iteration matrix $\boldsymbol{G}^{(n)}$ for each iteration. Here we consider four nonstationary algorithms that are members of the general conjugate direction class of algorithms developed for asymmetric/non-Hermitian matrices: biconjugate gradient–stable (BICGSTAB), quasi-minimum residual (QMR), general minimal residual (GMRES), and conjugate gradient–normal equation (CGNR). Discussion of the theory behind these algorithms is beyond the scope of this paper, but overviews of each are included in Barrett *et al.* [18]. BICGSTAB and QMR address numerical instability and numerical breakdown limitations of the original biconjugate gradient algorithm, respectively, but neither guarantee convergence in exact arithmetic for general matrices. GMRES is an orthogonalizing Krylov-subspace algorithm that guarantees convergence in N iterations for general systems. However, the increasing memory requirements and computational overhead with each iteration require that GMRES be periodically restarted, so convergence may never be achieved. CGNR is the application of the standard conjugate gradient approach to the system

$$\boldsymbol{Z}^H\boldsymbol{Z}\boldsymbol{j} = \boldsymbol{Z}^H\boldsymbol{j}^{\boldsymbol{i}} \tag{15}$$

where the superscript H implies the Hermitian transpose, and was one of the first nonstationary algorithms applied to general electromagnetic scattering problems [19]. Since $\boldsymbol{Z}^H\boldsymbol{Z}$ is always Hermitian positive definite CGNR is guaranteed to converge after N iterations in exact arithmetic. However, the convergence can be quite slow and roundoff errors may lead to instabilities and even divergence.

The BICGSTAB routine used in this work was written by the authors from the complex form of the algorithm given by Gutknecht [20]. QMR iteration was performed using the routine ZUCPL from the QMRPACK numerical library [21]. The GMRES implementation of [22] was used. The CGNR routine was also written by the authors [23].

III. Tests

The iterative algorithms were tested and compared through the application to linear systems resulting from the discretization of the MFIE when applied to scatterers that are uniform in one dimension. These tests were limited to perfectly conducting scatterers, corresponding to the original implementation of the stationary algorithms to surface scattering problems. Issues concerning the discretization of the integral equations appropriate for finite-conductivity scatterers are briefly discussed in [24] and will be addressed in greater depth in a subsequent paper.

A. Convergence History

The convergence history of the algorithms was determined by examining the normalized residual of the solution given by

$$R_N = \frac{||\boldsymbol{r}||}{||\boldsymbol{j}^{\boldsymbol{i}}||} \tag{16}$$

where $\boldsymbol{r} = \boldsymbol{j}^{\boldsymbol{i}} - \boldsymbol{Z}\boldsymbol{j}$ is the residual vector. This bounds the "backward error" of the system and corresponds to "Stopping Criterion 2" of Barrett *et al.* [8] and "Stopping Test (10)" of Oppe *et al.* [18]. It is a desirable stopping criterion since the solution errors are tied directly to the accuracy of the elements in the interaction matrix $\boldsymbol{Z}$ and source vector $\boldsymbol{j}^{\boldsymbol{i}}$. That is, if the elements are accurate to n significant figures, the iteration should be stopped when $R_N < 10^{-n}$ as any further iterations will not improve the overall solution of the original integral equation [8], [22]. Also, the residual can always be directly calculated to insure that cumulative roundoff errors are not affecting any quantities propagated by the iteration algorithms [17], [22], [25].

A zero vector was used as the starting guess for each algorithm, both to match the implementations of the stationary techniques as described in Section II-A and also to eliminate the overhead of calculating the initial residuals in the nonstationary techniques. As described above, point-Jacobi preconditioning was used to normalize the systems to unity diagonals in all cases. Of the more sophisticated preconditioning algorithms available [8], [18], incomplete factorization is designed for use with sparse matrices. SSOR preconditioning is intended for symmetric matrices and can lead to worse performance if the optimal relaxation factor is not used (which is generally difficult to find). It also makes interpretation of the residuals errors less straightforward and is not well suited to parallel implementation. The banded preconditioner used by Donohue *et al.* [11] was introduced for use with single-valued surfaces where strong interactions between surface points occur only over limited ranges and is also poorly suited to parallel processing. We therefore rely on point-Jacobi preconditioning. This actually has very little effect on the convergence of the techniques since the diagonals are already very nearly equal in the discretized MFIE formulations used here.

A fair comparison of the algorithms requires that each iteration plotted require approximately the same work. With the dense matrices that result from the moment method the most time consuming step is a matrix-vector multiply. BICGSTAB, QMR, and CGNR each require two matrix-vector multiplies per iteration to find both the updated solution vector and the residuals needed to perform the stopping test. We use this as the baseline for an iteration in the performance comparison. Jacobi iteration requires a single matrix-vector multiply to yield the updated solution and the residual for the previous iteration can then be found with only an additional vector–vector multiply (a computationally inexpensive step compared to a matrix–vector multiply with dense matrices). Also, the initial Jacobi iteration requires no work since the the starting vector is set to zero [as shown in (9)], offsetting the fact that the residual lags the solution by one iteration. Two Jacobi iterations are therefore presented as a single baseline iteration in the figures that follow. GMRES also requires one matrix–vector multiply per iteration, but the number of vector–vector operations and the overall storage needed to form the Krylov subspace increases with the iteration number and can actually become overwhelming if too large a restart is used. Freund and Nachtigal [26] state that the work and storage required for two iterations of GMRES is similar to that of one iteration of QMR if GMRES is restarted every 20 iterations. We therefore use that restart and count two iterations of GMRES as one iteration

of the other nonstationary techniques. (Note that the restart of 20 refers to single iterations, so in the plotted data that follows the restart occurs after every tenth baseline iteration.)

Comparison of SSOR with the other techniques is more problematic since it is not formulated in terms of matrix–vector multiplies. The MOMI implementation of SSOR allows the solution vector to be updated each iteration by two back-substitution operations that give the same number of scalar floating-point multiplies as a single matrix–vector multiply [1] However, since back-substitution requires that the results of the previous steps be used immediately it can take much longer to evaluate than a matrix–vector multiply in vector or parallel processing environments [8], [27], [28]. Moreover, SSOR/MOMI does not directly yield the residuals needed to perform the stopping test. Calculating the residuals requires an additional back-substitution operation [35]. Work can be avoided in some problems by not performing a stopping test on every iteration [27], but this results in extra iterations being unnecessarily performed in others. In the plots that follow, we count one iteration of SSOR/MOMI as being equal to one baseline iteration and point out that the actual time spent per iteration might be more or less than the other techniques, depending upon the specific algorithmic implementation, computational environment, and problem being solved.

Stationary techniques have sometimes been stopped based on the the norm of the iterative update vector [2], [5], [7]

$$\|\delta^{(n)}\| = \|\boldsymbol{j}^{(n+1)} - \boldsymbol{j}^{(n)}\|. \tag{17}$$

This does not make an acceptable stopping test for general scattering problems. Hageman and Young [16] term $\delta^{(n)}$ the pseudoresidual vector and show that the error in the solution is related to the pseudoresidual by

$$\|\epsilon^{(n)}\| = \|\boldsymbol{j} - \boldsymbol{j}^{(n)}\| \leq \|(\boldsymbol{G} - \boldsymbol{I})^{-1}\| \, \|\delta^{(n)}\|. \tag{18}$$

For general problems, one cannot predict the value of $\|\boldsymbol{G} - \boldsymbol{I})^{-1}\|$ *a priori*. Calculating it directly is an $O(N^3)$ operation and methods to estimate the value from the iteration history that are valid for symmetric/Hermitian positive definite interaction matrices fail for general asymmetric matrices [16]. (Even with symmetric positive definite matrices these techniques require that the relaxation factor ω be adaptively modified, which is not permitted by MOMI, and no test may be made for several iterations until initial transients have died out.) Quite small pseudoresiduals can be associated with large solution errors. Higham [17] bases stopping of stationary algorithms on the true residuals of (16) while acknowledging that it may require additional work each iteration to compute them directly and Oppe *et al.* [18] state that this is the safest stopping test for general asymmetric problems, so that is what we use here.

B. Wave Surfaces

The iterative algorithms were tested using the linear systems derived from the application of the MFIE to the surface profiles shown in Fig. 1. These profiles were generated by the LONGTANK numerical code [29] and represent the time evolution of a breaking water wave. The first 16 of these profiles were

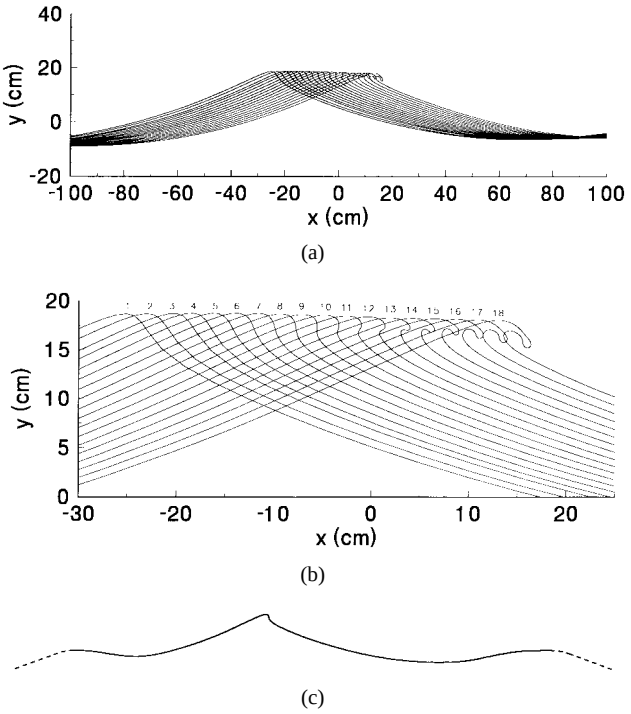

Fig. 1. Breaking wave profiles generated by the LONGTANK algorithm. (a) Full wave profiles. (b) Expanded view of crests. (c) Weighted wave including infinite extension (dashed).

previously treated in [30]. The MFIE was discretized using the hybrid MM/GTD approach described in [31] and [32]; details of that procedure are not repeated here. A raised-cosine weighting was applied to each surface from 50 to 100 cm behind the maximum displacement and from 100 to 150 cm in front of the maximum displacement to smoothly taper the displacement to zero. Infinite planar extensions sloping downward at 20° to horizontal were then connected to the front and back ends to allow the application of the MM/GTD technique. The connection was made using curved surface sections with radii of 30 cm that yielded a continuous slope, thereby limiting diffraction. The final surface corresponding to wave 7 is shown in Fig. 1(c).

The tests were performed using an electromagnetic frequency of 10 GHz with illumination from the right at a grazing angle (from horizontal) of 5°. Moment-method pulse basis functions of 0.05 λ in length, where λ is the electromagnetic wavelength, were needed to find the low cross sections resulting from the earliest waves and at the deepest multipath interference nulls of the later waves [32] to within 0.5 dB, giving approximately 2000 unknowns for each surface. Also, the same accuracy required that the matrix elements representing the interactions of the GTD basis functions on the infinite extensions with the other surface points be numerically integrated to a normalized tolerance of 10^{-4}. We therefore also use that for the convergence value for the normalized residual as explained in Section III-A. All results are presented for vertically polarized illumination. Similar results were obtained at horizontal polarization with an MFIE formulation.

The convergence histories of the iterative algorithms for five of the wave profiles are plotted in Fig. 2. Examining

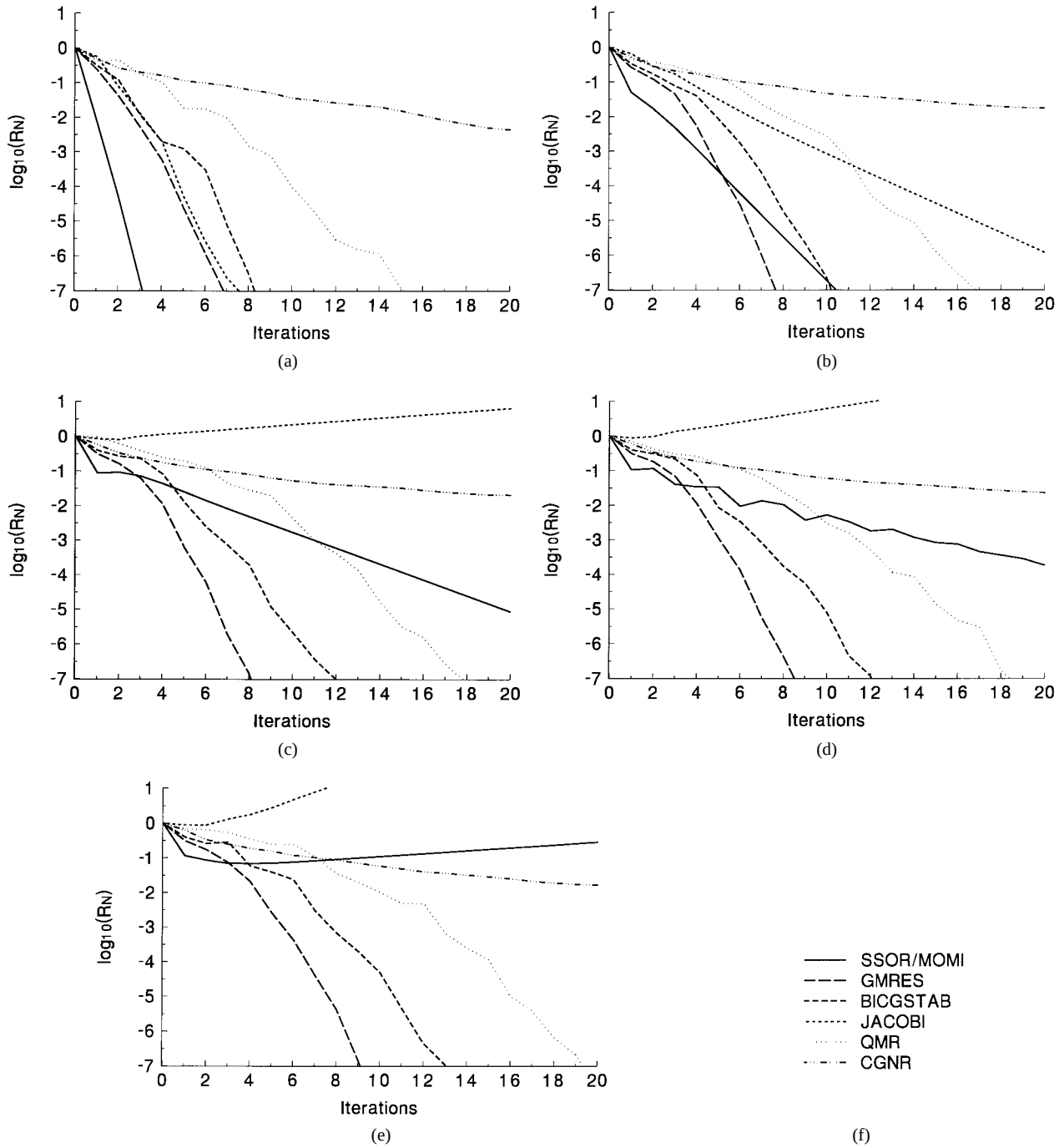

Fig. 2. Convergence history of iterative techniques applied to surface profiles of Fig. 1. Every other iteration of Jacobi and GMRES are plotted and counted to give per-iteration workloads approximately equal to those of the other stationary techniques. (a) Wave 7. (b) Wave 13. (c) Wave 15. (d) Wave 16. (e) Wave 17. (f) Legend. See text for SSOR/MOMI workload.

wave 7 in Fig. 2(a), SSOR/MOMI gives extremely rapid convergence, reaching $R_N < 10^{-4}$ after only two iterations. GMRES and Jacobi both require five baseline iterations to reach the same convergence level, and BICGSTAB needs seven. QMR is significantly slower, not reaching a suitable residual until after 11 iterations, and CGNR is slower still, needing 32 iterations. This performance is typical for waves 1–8. However, for waves beyond 8 the number of iterations required by the stationary techniques increases continuously. By wave 13 [shown in Fig. 2(b)], SSOR/MOMI has lost much of its advantage, reaching 10^{-4} after six iterations and Jacobi iteration now needs 14 iterations. The nonstationary approaches converge only slightly more slowly than with the earlier waves. With wave 15 SSOR/MOMI needs 16 iterations and Jacobi diverges. At wave 16 the SSOR/MOMI residuals oscillate but eventually converge to the desired level after 24 iterations. When applied to wave 17 SSOR/MOMI gives reducing residuals for the first three iterations, but it then diverges. Both SSOR/MOMI and Jacobi also diverge with wave 18 (not shown). The nonstationary algorithms are much less affected by the changing wave profile, with GMRES, BICGSTAB, QMR, and CGNR requiring 7, 10, 15, and 41

TABLE I
BASELINE ITERATIONS REQUIRED TO REACH A NORMALIZED RESIDUAL OF 10^{-4}. A DASH INDICATES THAT THE ALGORITHMS DIVERGED (SS = SSOR/MOMI, GM = GMRES, JA = Jacobi, BI = BICGSTAB, QM = QMR, CG = CGNR)

Wave	SS	GM	JA	BI	QM	CG
1	2	5	5	6	10	23
2	2	5	5	6	10	28
3	2	5	5	6	10	28
4	2	5	5	6	11	29
5	2	5	5	6	10	30
6	2	5	5	6	10	31
7	2	5	5	7	10	32
8	2	5	5	7	11	32
9	3	5	6	7	11	33
10	3	5	6	7	12	33
11	3	6	6	7	12	33
12	4	6	7	8	13	34
13	6	6	13	8	13	36
14	11	6	44	8	12	36
15	16	6	-	9	13	37
16	24	7	-	9	13	38
17	-	7	-	10	15	41
18	-	8	-	11	15	44

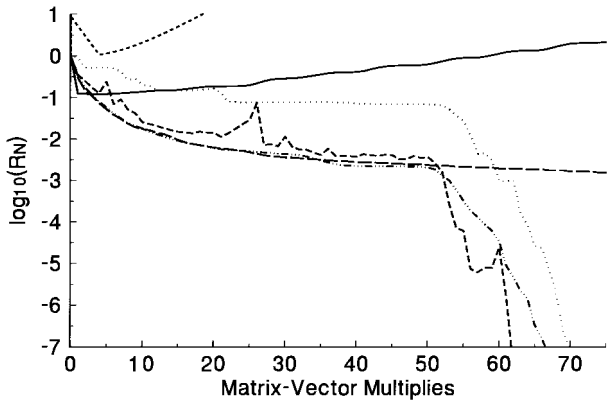

Fig. 3. Normalized residuals for conducting cylinder scattering at vertical polarization. Every other iteration of Jacobi and GMRES are plotted and counted to give per-iteration workloads approximately equal to those of the other stationary techniques. The legend is given in Fig. 2.

iterations respectively required to reach the desired normalized residuals with wave 17. Table I gives the number of iterations required to reach $R_N = 10^{-4}$ for each algorithm applied to all of the wave profiles.

As an additional test, the scattering from a 10λ radius, infinitely long, perfectly conducting circular cylinder was considered. This, of course, is not a rough-surface scattering problem, nor is the MFIE formulation optimal [7], but it does allow the performance of the algorithms to be benchmarked under extremely difficult conditions. The radius was chosen to be away from internal resonances so the discretized MFIE leads to the correct currents. (A combined electric/magnetic field integral equation (CFIE) formulation would eliminate the effects of these resonances [33].) Basis functions of 0.05λ in length were found to be sufficient in this case, yielding a system of 1257 unknowns. The iteration histories are shown in Fig. 3. Here, both SSOR/MOMI and Jacobi iteration diverge immediately. BICGSTAB, QMR, CGNR, and GMRES iteration all give slow convergence over the first 50 or so iterations, but the residuals for the first three rapidly drop after that. BICGSTAB gives the most unstable convergence at two points giving increasing residuals for 5–10 iterations before suddenly dropping back to its earlier level. GMRES continues its slow convergence beyond beyond 50 baseline iterations, reaching a normalized residual of 10^{-3} after 100 iterations (the maximum we allowed).

IV. DISCUSSION AND RECOMMENDATIONS

The results demonstrate both the strengths and weaknesses of stationary iterative procedures. SSOR/MOMI is clearly well suited to MFIE formulations of scattering from one-dimensionally rough conducting surfaces that are single valued and and perfectly conducting, requiring less than half the number of iterations needed by any other technique. This results because the MFIE is a second-type integral equation, giving large diagonal elements in the interaction matrix and the one-dimensional single-valued roughness of the surface minimizes the coupling between spatially separated points yielding small off-diagonal elements. Stationary procedures are particularly well suited to this type of system (and, in fact, a sufficient (although not necessary) condition for the convergence of Jacobi and SSOR is a diagonally dominant matrix [14]. However, stationary iterative procedures are inherently not robust, which can lead to convergence problems when the scattering geometry leads to less than ideally conditioned interaction matrices. This is demonstrated by the later breaking-wave profiles of Fig. 1. The stationary techniques quickly lose their rapid convergence characteristics when the wave becomes multivalued, resulting because strong interactions between nearby elements lead to large off-diagonal terms and eventually diverge with waves 17 and 18. The results of Tran [5] show that an MFIE treatment of two-dimensionally rough single-valued surfaces can also lead to less than optimally conditioned interaction matrices, as evidenced by the dependence of convergence on the order in which the interactions are mapped and the failure of SSOR/MOMI to converge for any ordering in one case. Due to the relatively slow convergence in the best conditioned cases and divergence under conditions where all other techniques converge the use of Jacobi iteration is not recommended for scattering problems.

Based on these results we recommend that nonstationary iterative procedures be considered for use with general scattering problems. The inherent robustness of these techniques allow their convergence speeds to slow only a small amount through geometry changes that transitioned the stationary techniques from very rapid convergence to divergence. They also have the added advantage that, unlike SSOR, their convergence properties are independent of the ordering used when forming the interaction matrix. Also, since they depend upon matrix–vector multiplies rather than back-substitution to perform the iterations they are well suited to parallel processing. The nonstationary techniques presented here have been implemented using the message-passing interface (MPI) [34] to distribute the matrix–vector multiply operation across five workstations with no interprocessor communication except to distribute the initial and final vector elements. This yields nearly optimal improvement in speed despite the fact that the workstations are connected by a standard 10-Mb/s EtherNet link. The primary disadvantage of the nonstationary techniques

is, of course, that they require more iterations to converge in the best conditioned cases. This can be of significant concern in some cases, such as if the interaction matrix does not fit within available memory and the individual elements must be recalculated each time they are used. (The MM/GTD approach used here is relatively memory efficient since single basis functions are used on surface sections that extend to infinity, so the cost of matrix–vector multiplies are typically small compared to the initial time needed to fill the interaction matrix.) The relative merits of more rapid convergence under good conditioning versus the need for robustness in more difficult situations must be carefully weighed when an iterative technique is chosen.

In examining Fig. 2, it would appear that GMRES is the most suitable nonstationary algorithm for surface scattering calculations. It provided the most rapid convergence of the nonstationary techniques for every breaking-wave case and the convergence properties are quite stable. However, the cylinder case in Fig. 3 shows that GMRES can perform very poorly in particularly difficult cases if too short a restart period is used. In rerunning this case with the restart increased to 50 baseline iterations the GMRES residuals dropped rapidly after about 30 iterations (similar to that of the other nonstationary techniques at 60 iterations), giving by far the quickest convergence in this case. Unfortunately, increasing the restart requires both more storage and more work per iteration, so it is difficult to predict the optimal restart *a priori*. A better choice for a general nonstationary iterative scheme then might be BICGSTAB. Its convergence was typically only a little slower than GMRES for the same number of matrix–vector multiplies, but it is not limited by a restart in the particularly difficult situations. Despite its name, BICGSTAB appears to give somewhat unstable convergence for some of these problems (particularly the cylinder), and it must be stressed that BICGSTAB is not guaranteed to converge (although we have not encountered a situation where it does not converge). Although QMR converges at approximately 50% the speed of BICGSTAB for these problems, it is formulated to avoid the breakdowns to which BICGSTAB is susceptible so might be considered in cases where robustness over greatly varying conditions is of primary importance and additional operations can be tolerated. As CGNR converges very slowly, we suggest that it should only be considered in cases where the other routines fail and the guaranteed convergence (in exact arithmetic) is needed.

Finally, we note that Pino *et al.* [35] have recently introduced a generalized self-interaction matrix to the original forward–backward implementation of SSOR that allows it to converge with reentrant surfaces of the type of waves 17 and 18. However, this approach requires *a priori* knowledge of the surface so that the self-interaction matrix can be tailored to the specific reentrant section and introduces additional work per iteration that can become overwhelming is the reentrant section is too large. It cannot be used effectively with closed-body problems of the type considered in Fig. 3. Also, Adams and Brown [7] have shown that use of a combined integral equation formulation can lead to very rapid convergence with SSOR/MOMI for certain two-dimensional closed-body problems if the relative weightings of the magnetic and electric field kernels are properly chosen. This approach is not as effective for the wave surfaces of Fig. 1. The rapid convergence depends upon the two kernel contributions canceling over many off-diagonal terms to improve the conditioning of the interaction matrix. This is achievable for fairly uniform scatterers where many off-diagonals are similar, but with the wave profiles a weighting that cancels the strong interactions of elements near the crest leads to large off-diagonals in other regions. A nonlinear multivariate optimization routine was used to find a complex weighting that allowed SSOR/MOMI to converge to 10^{-4} in 15 iterations for the CFIE applied to horizontally polarized scattering from wave 18. However, many hundreds of iterations were needed to identify this best weighting. (The best purely imaginary weighting found only allowed convergence to the same level after 61 iterations.) The resulting interaction matrix actually had an average of all off-diagonal values (normalized with respect to the magnitude of the diagonals) that was more than an of order magnitude larger than that of the original MFIE formulation, so the weighting could not be found by simply minimizing the off-diagonals. GMRES and BICGSTAB converged in 8 and 11 iterations, respectively, when applied to the original MFIE in this case.

V. Conclusions

The performance of several stationary and nonstationary iterative techniques in the solving of the system of linear equations that result in electromagnetic scattering problems has been examined. Recently used iterative techniques that model the multiple scattering of the incident electromagnetic energy using the Kirchhoff approximation reduce mathematically to the application of stationary iterative techniques to the discretized integral equation describing the surface current. Convergence problems sometimes associated with these approaches are due to inherent limitations of the stationary techniques. Nonstationary conjugate-direction-based iteration techniques offer a more robust alternative that can converge reasonably rapidly to the correct solution in ill-conditioned situations where the stationary approaches either converge slowly or diverge. However, stationary techniques give quickest convergence with the best conditioned systems. The choice of iteration schemes to be used depends upon the need for more rapid convergence in the well-conditioned cases versus robustness in more difficult situations.

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