Radiation and Low-Frequency Scattering of EM Waves in a General Anisotropic Homogeneous Medium

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***Abstract*— This paper focuses on a boundless homogeneous medium with general anisotropy of electromagnetic properties. The explicit exact form of the four spectral Green's dyads $\hat{G}_{\nu\xi}(\mathbf{k})$ is obtained and a coordinate-free representation of the four spatial Green's functions $\hat{G}_{\nu\xi}(\mathbf{x}-\mathbf{x}')$ in terms of one scalar potential $W(\mathbf{x}-\mathbf{x}')$ is developed. On this basis, asymptotic expressions for the radiation field due to arbitrary sources are derived that show that the associated modes and the eigenmodes determine radiation along a singular optic axis and in all other directions, respectively. Using an integral equation approach and the theory of Newtonian potential, the problem of low-frequency scattering by a small anisotropic ellipsoidal body immersed into an anisotropic medium is solved analytically.**

***Index Terms*— Anisotropic media, electromagnetic radiation, electromagnetic scattering.**

I. Introduction

SINCE an ever-growing number of electromagnetic (EM) problems find their origin rooted in (bi)anisotropic media (including crystals, plasmas, composite materials), the development of analytical tools that facilitate the analysis of such media is essential. In this regard, a concept of dyadic Green's functions (GF's) has proven to be especially fruitful because it offers a flexible way to relate impressed or induced sources to the EM field [1]. Exact closed-form solutions for space-domain GF's of certain anisotropic media have been exhaustively reviewed in [2], and useful analytic representations for GF's of gyrotropic, biaxial anisotropic, affinely transformable uniaxial bianisotropic, and axially bianisotropic media have been elaborated in [3]–[7], respectively.

Concerning the general anisotropic media, the major problem here is that the space-domain GF's are unavailable in closed form. Employing the Fourier transformation, one can express GF's as three-dimensional (3-D) Fourier integrals. A closed-form solution for the four spectral Green's functions (SGF's) of an arbitrarily anisotropic medium has been exhibited in [8], which covers the case of bianisotropic materials by allowing for spatial dispersion in the medium. Original procedures for developing the SGF's of bianisotropic media have been proposed in [9] and [10]. Asymptotic behavior of the far-zone field in anisotropic and bianisotropic media has been studied, respectively, in [8], [11], [12], and [9]. Asymptotic calculation of Green's dyads for bianisotropic media in the source region has been accomplished in [6], [9], and [13].

The present paper is an extended and modified version of our reports [8], [14]. Its main contribution to prior knowledge is fourfold. Namely, here we:

- derive an explicit solution (13)–(18) for the four dyadic GF's $\hat{G}_{\alpha\beta}(\mathbf{k})$ in the spectral domain referring to an unbounded medium with general anisotropy of electromagnetic properties;
- give a simple method for the determination of plane-wave modes and associated modes by knowledge of SGFs;
- obtain asymptotic expressions (49)–(51) for the radiation field of sources of finite extent which account for the excitation of associated modes;
- illustrate the usefulness of the Green's functions technique by solving the scattering of a time-harmonic plane wave from an anisotropic ellipsoid immersed into an anisotropic region, under a low-frequency approximation.

The problem of low-frequency scattering in a (bi)anisotropic environment has been attracting growing interest in recent years due to its obvious applications for estimating electromagnetic properties of particle-laden media such as artificial dielectrics, polymer composites, and sea ice. For a small spherical or ellipsoidal scatterer, closed-form expressions for the internal field and the polarizability dyads have been calculated, e.g., in [15]–[18], of which the last paper is the most relevant to the present work. In that paper, a simultaneous change of field variables and spatial coordinates has been devised to substitute an original quasistatic problem for an anisotropic ellipsoid in an anisotropic medium by a simpler one involving an anisotropic ellipsoid placed in a vacuum. The aforementioned transformation of variables preserves the "electrostatic" makeup of both differential equations as well as the boundary conditions on the scatterer's surface, a merit not shared by the solution of [15]. An affine transfomation of [18] is always possible when the ambient medium is characterized by the real symmetric positive definite constitutive dyads. However, when this requirement is not fulfilled, as is typical of lossy materials or nonreciprocal media such as magnetized plasmas or ferrites, the said transformation cannot be accomplished. Applying an integral equation technique combined

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with the theory of Newtonian potentials [19], [20] allows the present authors to lift out some of the limitations inherent in [18]. Namely, our solution to the low-frequency scattering is obtained under an assumption that the symmetric parts of the ambient medium's constitutive dyads are real positive definite or negative definite dyads while the constitutive dyads themselves may be neither real, nor symmetric, nor definite.

In the present paper, a time dependence $\exp(-i\omega t)$ is understood and supressed throughout, geometric vectors are in boldfaced type, dyads are accompanied by a cap, a 3×3 matrix comprising elements A_{jk} is designated as $\underline{\underline{A}}$, a bar under the quantity signifies algebraic vectors that are regarded here as column matrices, e.g., $\underline{x} = [x_1, x_2, x_3]^{\mathrm{T}}$, and T designates the matrix or dyad transposition operation.

II. Dyadic Green's Functions for an Anisotropic Medium

A. Basic Definitions

We shall start with the excitation problem in an unbound dielectric-magnetic homogeneous medium bestowed with anisotropy and dissipative losses. Maxwell's equations for the EM field vectors $\mathbf{E}(\mathbf{x})$ and $\mathbf{H}(\mathbf{x})$ in such medium read as

$$\begin{aligned} \nabla \times \mathbf{H}(\mathbf{x}) + ik_0\hat{\varepsilon} \cdot \mathbf{E}(\mathbf{x}) &= \frac{4\pi}{c}\mathbf{J}(\mathbf{x}) \\ \nabla \times \mathbf{E}(\mathbf{x}) - ik_0\hat{\mu} \cdot \mathbf{H}(\mathbf{x}) &= -\frac{4\pi}{c}\mathbf{M}(\mathbf{x}). \end{aligned} \tag{1}$$

In these equations, $\mathbf{x}$ is the position vector, $k_0 = \omega/c$, c is the speed of light in free-space, and the permittivity and permeability dyads $\hat{\varepsilon}$, $\hat{\mu}$ when expressed in a Cartesian coordinate system x_1, x_2, x_3 possess all nine components each, which are allowed to take complex values.

Maxwell's equations (1) are supplemented with the absorption condition, which requires that $\mathbf{E}(\mathbf{x})$ and $\mathbf{H}(\mathbf{x})$ vanish at infinitely remote points, viz.

$$\mathbf{E}(\mathbf{x}) = o\left(\frac{1}{|\mathbf{x}|}\right), \quad \mathbf{H}(\mathbf{x}) = o\left(\frac{1}{|\mathbf{x}|}\right), \quad (|\mathbf{x}| \to +\infty). \tag{2}$$

This condition is known to ensure a unique solution to the excitation problem involving lossy media and sources of compact support [21].

The space-domain GF's $\hat{G}_{\nu\xi}(\mathbf{x}-\mathbf{x}')$, $(\nu, \xi = e, m)$ yield the electromagnetic field $\mathbf{E}(\mathbf{x})$, $\mathbf{H}(\mathbf{x})$ in terms of sources $\mathbf{J}(\mathbf{x})$, $\mathbf{M}(\mathbf{x})$ as follows:

$$\begin{aligned} \mathbf{E}(\mathbf{x}) &= \int d^3x' [\hat{G}_{ee}(\mathbf{x} - \mathbf{x}') \cdot \mathbf{J}(\mathbf{x}') \\ &\quad + \hat{G}_{em}(\mathbf{x} - \mathbf{x}') \cdot \mathbf{M}(\mathbf{x}')], \\ \mathbf{H}(\mathbf{x}) &= \int d^3x' [\hat{G}_{me}(\mathbf{x} - \mathbf{x}') \cdot \mathbf{J}(\mathbf{x}') \\ &\quad + \hat{G}_{mm}(\mathbf{x} - \mathbf{x}') \cdot \mathbf{M}(\mathbf{x}')]. \end{aligned} \tag{3}$$

Here and below, integration is performed within infinite limits if the domain of integration is not indicated explicitly.

For spatially harmonic impressed sources $\mathbf{J}(\mathbf{x}) \equiv \mathbf{J}_{\mathbf{k}}(\mathbf{x})$, $\mathbf{M}(\mathbf{x}) \equiv \mathbf{M}_{\mathbf{k}}(\mathbf{x})$,

$$\begin{aligned} \mathbf{J}_{\mathbf{k}}(\mathbf{x}) &= \mathbf{J}(\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{x}), \\ \mathbf{M}_{\mathbf{k}}(\mathbf{x}) &= \mathbf{M}(\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{x}) \end{aligned} \tag{4}$$

homogeneity of the medium entitles us to expand the excited field $\mathbf{E}(\mathbf{x}) \equiv \mathbf{E}_{\mathbf{k}}(\mathbf{x})$, $\mathbf{H}(\mathbf{x}) \equiv \mathbf{H}_{\mathbf{k}}(\mathbf{x})$ as

$$\begin{aligned} \mathbf{E}_{\mathbf{k}}(\mathbf{x}) &= \mathbf{E}(\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{x}) \\ \mathbf{H}_{\mathbf{k}}(\mathbf{x}) &= \mathbf{H}(\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{x}) \end{aligned} \tag{5}$$

where an $\exp(i\mathbf{k} \cdot \mathbf{x})$ dependence has been inherited from that in (4). In these formulas $\mathbf{k}$ is a wavevector (arbitrary vectorial quantity, maybe complex-valued), $\mathbf{J}(\mathbf{k})$, $\mathbf{M}(\mathbf{k})$ and $\mathbf{E}(\mathbf{k})$, $\mathbf{H}(\mathbf{k})$ are the source and the field spectral amplitudes, respectively. The SGF's $\hat{G}_{\nu\xi}(\mathbf{k})$, $(\nu, \xi = e, m)$ may be identified as the dyads relating the source amplitudes to the field amplitudes, viz.

$$\begin{aligned} \mathbf{E}(\mathbf{k}) &= \hat{G}_{ee}(\mathbf{k}) \cdot \mathbf{J}(\mathbf{k}) + \hat{G}_{em}(\mathbf{k}) \cdot \mathbf{M}(\mathbf{k}) \\ \mathbf{H}(\mathbf{k}) &= \hat{G}_{me}(\mathbf{k}) \cdot \mathbf{J}(\mathbf{k}) + \hat{G}_{mm}(\mathbf{k}) \cdot \mathbf{M}(\mathbf{k}). \end{aligned} \tag{6}$$

Space-domain GF's are expressible through SGF's as 3-D Fourier integrals

$$\hat{G}_{\nu\xi}(\mathbf{x} - \mathbf{x}') = (2\pi)^{-3} \int d^3k \exp[i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')]\hat{G}_{\nu\xi}(\mathbf{k}). \tag{7}$$

Here, the 3-D Fourier integrals should be interpreted in the spirit of generalized functions theory since $\hat{G}_{ee}(\mathbf{k})$, $\hat{G}_{mm}(\mathbf{k})$ do not vanish as $k \to +\infty$—cf. later (13). Also, the singularities of $\hat{G}_{\nu\xi}(\underline{k})$ are presumed to occur at complex locations due to dissipative losses, artificial if necessary, of the medium under consideration.

B. Explicit Solution for Spectral Green's Functions

In this section, we first solve for $\mathbf{E}(\mathbf{k})$, $\mathbf{H}(\mathbf{k})$, and then find the desired SGF's by comparing the result with (6).

Inserting (4) and (5) into Maxwell's equations (1) leads to equations for the field spectral amplitudes in the form

$$\begin{aligned} \mathbf{k} \times \mathbf{H}(\mathbf{k}) + k_0\hat{\varepsilon} \cdot \mathbf{E}(\mathbf{k}) &= -\frac{4\pi i}{c}\mathbf{J}(\mathbf{k}). \\ \mathbf{k} \times \mathbf{E}(\mathbf{k}) - k_0\hat{\mu} \cdot \mathbf{H}(\mathbf{k}) &= \frac{4\pi i}{c}\mathbf{M}(\mathbf{k}). \end{aligned} \tag{8}$$

In [8], these equations were solved by splitting the spectral amplitudes $\mathbf{E}(\mathbf{k})$, $\mathbf{H}(\mathbf{k})$ into transverse and longitudinal components. Here, we adopt a more straightforward approach which is based on the direct inversion of two uncoupled equations for $\mathbf{E}(\mathbf{k})$ and $\mathbf{H}(\mathbf{k})$

$$\begin{aligned} \hat{V}_e(\mathbf{k}) \cdot \mathbf{E}(\mathbf{k}) &= -\frac{4\pi i}{c}\mathbf{R}_e(\mathbf{k}) \\ \hat{V}_m(\mathbf{k}) \cdot \mathbf{H}(\mathbf{k}) &= -\frac{4\pi i}{c}\mathbf{R}_m(\mathbf{k}) \end{aligned} \tag{9}$$

which are obtainable from (8) by eliminating one unknown in favor of the other. Here

$$\hat{V}_e(\mathbf{k}) = \mathbf{k} \times \hat{\mu}^{-1} \times \mathbf{k} + k_0^2\hat{\varepsilon} \tag{10}$$

$$\hat{R}_e(\mathbf{k}) = k_0\mathbf{J}(\mathbf{k}) + \mathbf{k} \times \hat{\mu}^{-1} \cdot \mathbf{M}(\mathbf{k}). \tag{11}$$

Expressions for $\hat{V}_m(\mathbf{k})$ and $\hat{R}_m(\mathbf{k})$ are formally obtainable from (10) and (11) via the following replacements: $\hat{\mu} \leftrightarrow \hat{\varepsilon}$, $\mathbf{J} \rightarrow \mathbf{M}$, $\mathbf{M} \rightarrow -\mathbf{J}$; they are omitted here for the sake of brevity.

A formal solution to (9),

$$\begin{aligned}\mathbf{E}(\mathbf{k}) &= -\frac{4\pi i}{c}\frac{\operatorname{adj}\hat{V}_e(\mathbf{k})}{\det\hat{V}_e(\mathbf{k})}\cdot\mathbf{R}_e(\mathbf{k})\\ \mathbf{H}(\mathbf{k}) &= -\frac{4\pi i}{c}\frac{\operatorname{adj}\hat{V}_m(\mathbf{k})}{\det\hat{V}_m(\mathbf{k})}\cdot\mathbf{R}_m(\mathbf{k})\end{aligned} \tag{12}$$

can be converted into explicit expressions by standard manipulations [22], [23, p. 14]. Here adj and det denote, respectively, the adjoint and the determinant of a dyad.

A requirement that the obtained expressions for $\mathbf{E}(\mathbf{k})$ and $\mathbf{H}(\mathbf{k})$ match to (6) leads us to an explicit determination for the spectral Green's dyads that we are seeking

$$\begin{aligned}\hat{G}_{ee}(\mathbf{k}) &= -\frac{4\pi i}{ck_0\Delta(\mathbf{k})}\hat{D}_{ee}(\mathbf{k})\\ \hat{G}_{mm}(\mathbf{k}) &= -\frac{4\pi i}{ck_0\Delta(\mathbf{k})}\hat{D}_{mm}(\mathbf{k})\\ \hat{G}_{em}(\mathbf{k}) &= \frac{4\pi i}{c\Delta(\mathbf{k})}\hat{D}_{em}(\mathbf{k})\\ \hat{G}_{me}(\mathbf{k}) &= -\frac{4\pi i}{c\Delta(\mathbf{k})}\hat{D}_{me}(\mathbf{k}).\end{aligned} \tag{13}$$

Here the following designations are adopted:

$$\begin{aligned}\hat{D}_{ee}(\mathbf{k}) = {}& k_0^4(\det\hat{\mu})\operatorname{adj}\hat{\varepsilon} + (\mathbf{k}\cdot\hat{\mu}\cdot\mathbf{k})\mathbf{kk}\\ &+ k_0^2\hat{\varepsilon}^{-1}\cdot[(\mathbf{k}\cdot\hat{\alpha}\cdot\mathbf{k})\hat{I} - \mathbf{k}\cdot\hat{\mu}\mathbf{k}\cdot\hat{\gamma}\\ &+ (\mathbf{k}\cdot\hat{\mu}\cdot\mathbf{k})(\hat{\gamma} - \hat{I}\operatorname{Tr}\hat{\gamma})]\end{aligned} \tag{14}$$

$$\begin{aligned}\hat{D}_{mm}(\mathbf{k}) = {}& k_0^4(\det\hat{\varepsilon})\operatorname{adj}\hat{\mu} + (\mathbf{k}\cdot\hat{\varepsilon}\cdot\mathbf{k})\mathbf{kk}\\ &+ k_0^2\hat{\mu}^{-1}\cdot[(\mathbf{k}\cdot\hat{\beta}\cdot\mathbf{k})\hat{I} - \mathbf{k}\cdot\hat{\varepsilon}\mathbf{k}\cdot\hat{\delta}\\ &+ (\mathbf{k}\cdot\hat{\varepsilon}\cdot\mathbf{k})(\hat{\delta} - \hat{I}\operatorname{Tr}\hat{\delta})]\end{aligned} \tag{15}$$

$$\begin{aligned}\hat{D}_{em}(\mathbf{k}) &= k_0^{-2}\hat{D}_{ee}(\mathbf{k})\cdot\mathbf{k}\times\hat{\mu}^{-1}\\ &= k_0^{-2}\hat{\varepsilon}^{-1}\cdot\mathbf{k}\times\hat{D}_{mm}(\mathbf{k})\end{aligned} \tag{16}$$

$$\begin{aligned}\hat{D}_{me}(\mathbf{k}) &= k_0^{-2}\hat{D}_{mm}(\mathbf{k})\cdot\mathbf{k}\times\hat{\varepsilon}^{-1}\\ &= k_0^{-2}\hat{\mu}^{-1}\cdot\mathbf{k}\times\hat{D}_{ee}(\mathbf{k})\end{aligned} \tag{17}$$

$$\begin{aligned}\Delta(\mathbf{k}) = {}& k_0^4(\det\hat{\varepsilon})(\det\hat{\mu}) + (\mathbf{k}\cdot\hat{\varepsilon}\cdot\mathbf{k})(\mathbf{k}\cdot\hat{\mu}\cdot\mathbf{k})\\ &+ k_0^2(\mathbf{k}\cdot\hat{\alpha}\cdot\mathbf{k} - \mathbf{k}\cdot\hat{\mu}\cdot\mathbf{k}\operatorname{Tr}\hat{\gamma})\end{aligned} \tag{18}$$

$$\begin{aligned}\hat{\alpha} &= \hat{\mu}\cdot(\operatorname{adj}\hat{\varepsilon}^{\mathrm{T}})\cdot\hat{\mu}\\ \hat{\beta} &= \hat{\varepsilon}\cdot(\operatorname{adj}\hat{\mu}^{\mathrm{T}})\cdot\hat{\varepsilon},\\ \hat{\gamma} &= \hat{\mu}^{\mathrm{T}}\cdot\operatorname{adj}\hat{\varepsilon},\\ \hat{\delta} &= \hat{\varepsilon}^{\mathrm{T}}\cdot\operatorname{adj}\hat{\mu}.\end{aligned} \tag{19}$$

Therein, $\hat{I}$ is the identity dyad and Tr signifies the trace of a dyad.

One can derive, through simple manipulations, a host of other representations for $\hat{D}_{\nu\xi}(\mathbf{k})$, of which we render a few which find application in the following analysis:

$$\begin{aligned}\hat{D}_{ee}(\mathbf{k}) = {}& \frac{\mathbf{nn}}{\varepsilon_l(\mathbf{n})}\Delta(\mathbf{k}) - \frac{\mathrm{k}_0^2}{\varepsilon_l^2(\mathbf{n})\mu_l(\mathbf{n})}[\varepsilon_l(\mathbf{n})\hat{I} - \mathbf{nn}\cdot\hat{\varepsilon}]\\ &\cdot\hat{t}(\mathbf{k})\cdot(\operatorname{adj}\hat{\mu})^{\mathrm{T}}\cdot[\varepsilon_l(\mathbf{n})\hat{I} - \hat{\varepsilon}\cdot\mathbf{nn}]\end{aligned} \tag{20}$$

$$\hat{D}_{mm}(\mathbf{k}) = \hat{\mu}^{-1}\cdot[i\Delta(\mathbf{k})\hat{I} - \mathbf{k}\times\hat{D}_{em}(\mathbf{k})] \tag{21}$$

$$\begin{aligned}\hat{t}(\mathbf{k}) = {}& k^2\varepsilon_l(\mathbf{n})\mu_l(\mathbf{n})\hat{I}_\perp(\mathbf{n}) + k_0^2\hat{I}_\perp(\mathbf{n})\cdot(\operatorname{adj}\hat{\varepsilon})\\ &\cdot\mathbf{n}\times(\operatorname{adj}\hat{\mu})\times\mathbf{n}.\end{aligned} \tag{22}$$

Here, $\mathbf{n}$ is a unit vector in the direction of vector $\mathbf{k}$, $\varepsilon_l(\mathbf{n})$ and $\mu_l(\mathbf{n})$ are the "longitudinal" permittivity and permeability of a medium, and $\hat{I}_\perp(\mathbf{n})$ is the identity dyad in a plane perpendicular to $\mathbf{n}$

$$\mathbf{n} = \mathbf{k}/k, \quad (k^2 = \mathbf{k}\cdot\mathbf{k}) \tag{23}$$

$$\varepsilon_l(\mathbf{n}) = \mathbf{n}\cdot\hat{\varepsilon}\cdot\mathbf{n} \tag{24}$$

$$\begin{aligned}\mu_l(\mathbf{n}) &= \mathbf{n}\cdot\hat{\mu}\cdot\mathbf{n}\\ \hat{I}_\perp(\mathbf{n}) &= \hat{I} - \mathbf{nn}.\end{aligned} \tag{25}$$

The expressions of (13) are valid under the restrictions that the constitutive dyads are nonsingular, i.e., $\det\hat{\mu} \neq 0$, $\det\hat{\varepsilon} \neq 0$, and that $\Delta(\mathbf{k}) \neq 0$. The last requirement means that $\mathbf{k}$ must not coincide with the wavevector of a plane wave mode of the electromagnetic field (see Section II-C).

It can be easily checked that all the preceding treatment holds for arbitrary homogeneous medium with spatial dispersion as well if one views $\hat{\varepsilon}$, $\hat{\mu}$ as pseudodifferential operators $\hat{\varepsilon}(-i\nabla)$, $\hat{\mu}(-i\nabla)$ in Maxwell's equations (1), and as functions $\hat{\varepsilon}(\mathbf{k})$, $\hat{\mu}(\mathbf{k})$ of spectral parameter $\mathbf{k}$ in the remaining formulas, which involve the constitutive parameters. The role played by spatial dispersion is particularly well known for magnetoactive plasma and in crystal optics [24]. Also, a bianisotropic medium can be regarded as a spatially dispersive medium by virtue of Drude–Born–Fedorov constitutive relations.

C. Plane-Wave Modes and Associated Modes

Here, we determine the eigensolutions to source-free Maxwell's equations using a SGF's technique. With this aim we set

$$\begin{aligned}\mathbf{J}(\mathbf{k}) &= \frac{i\omega}{4\pi}\Delta(\mathbf{k})\mathbf{P}_e,\\ \mathbf{M}(\mathbf{k}) &= \frac{i\omega}{4\pi}\Delta(\mathbf{k})\mathbf{P}_m\end{aligned} \tag{26}$$

where $\mathbf{P}_e$ and $\mathbf{P}_m$ are arbitrary vectors which may depend on $\mathbf{k}$ and are independent of $\mathbf{x}$. Recalling (6) we find that (5) take the form

$$\begin{aligned}\mathbf{E}_\mathbf{k}(\mathbf{x}) &= \mathbf{e}(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{x}),\\ \mathbf{H}_\mathbf{k}(\mathbf{x}) &= \mathbf{h}(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{x})\end{aligned} \tag{27}$$

where

$$\begin{aligned}\mathbf{e}(\mathbf{k}) &= \hat{D}_{ee}(\mathbf{k})\cdot\mathbf{P}_e - k_0\hat{D}_{em}(\mathbf{k})\cdot\mathbf{P}_m\\ \mathbf{h}(\mathbf{k}) &= \hat{D}_{mm}(\mathbf{k})\cdot\mathbf{P}_m + k_0\hat{D}_{me}(\mathbf{k})\cdot\mathbf{P}_e.\end{aligned} \tag{28}$$

Let $\mathbf{k}$ be a root of the dispersion equation

$$\Delta(\mathbf{k}) = 0. \tag{29}$$

For such value of $\mathbf{k}$, the impressed sources (4) identically vanish by virtue of (26), (29), and (27) obey the source-free Maxwell's equations. Thus, relations (27) yield general representation for a plane wave mode of the EM field in an anisotropic medium. In the following, the direction $\mathbf{n}$ of vector

$\mathbf{k}$ is taken to be prescribed: $\mathbf{k} = k\mathbf{n}$ and (29) is viewed as an equation in k. Its solutions are discussed in Section II-D.

Let us assume that k is a multiple root of the dispersion equation satisfying the requirements

$$\Delta(\mathbf{k}) = 0, \quad \frac{\partial \Delta(\mathbf{k})}{\partial k} = 0. \tag{30}$$

According to the existing nomenclature, every direction of $\mathbf{n}$ that permits a multiple root of the dispersion equation is called an optic axis of the anisotropic medium [23, sec. 5.3]. Since for conventional anisotropic media $\Delta(\mathbf{k})$ is an even function of $\mathbf{k}$, for each given direction of optic axis $\mathbf{n}$ the opposite direction $-\mathbf{n}$ will be an optic axis as well [23]. Based on (30) and (26) we can show that $\partial \mathbf{J}_{\mathbf{k}}(\mathbf{x})/\partial k = 0$ and $\partial \mathbf{M}_{\mathbf{k}}(\mathbf{x})/\partial k = 0$. Performing the operation $\partial/\partial k$ in (27), we obtain that

$$\begin{aligned} \mathbf{E}_1(\mathbf{x}) &\equiv \frac{\partial \mathbf{E}_{\mathbf{k}}(\mathbf{x})}{\partial k} \\ &= \left[i(\mathbf{n}\cdot\mathbf{x})\mathbf{e}(\mathbf{k}) + \frac{\partial \mathbf{e}(\mathbf{k})}{\partial k} \right] \exp(i\mathbf{k}\cdot\mathbf{x}) \\ \mathbf{H}_1(\mathbf{x}) &\equiv \frac{\partial \mathbf{H}_{\mathbf{k}}(\mathbf{x})}{\partial k} \\ &= \left[i(\mathbf{n}\cdot\mathbf{x})\mathbf{h}(\mathbf{k}) + \frac{\partial \mathbf{h}(\mathbf{k})}{\partial k} \right] \exp(i\mathbf{k}\cdot\mathbf{x}) \end{aligned} \tag{31}$$

is a source-free solution to Maxwell's equations.

An optic axis for which an additional requirement

$$\hat{t}(\mathbf{k}) = 0 \tag{32}$$

holds, is called nonsingular [22], [24]. Equations (16), (17), (20), and (21) imply that $\mathbf{e}(\mathbf{k}) = \mathbf{h}(\mathbf{k}) = 0$ and (31) becomes a plane-wave mode

$$\begin{aligned} \mathbf{E}_1(\mathbf{x}) &= \frac{\partial \mathbf{e}(\mathbf{k})}{\partial k} \exp(i\mathbf{k}\cdot\mathbf{x}) \\ \mathbf{H}_1(\mathbf{x}) &= \frac{\partial \mathbf{h}(\mathbf{k})}{\partial k} \exp(i\mathbf{k}\cdot\mathbf{x}). \end{aligned} \tag{33}$$

It can be readily verified that its amplitude factors are interrelated as follows:

$$\begin{aligned} \frac{\partial \mathbf{e}(\mathbf{k})}{\partial k} &= -\frac{\hat{\varepsilon}^{-1}}{k_0} \cdot \mathbf{k} \times \frac{\partial \mathbf{h}(\mathbf{k})}{\partial k} \\ \frac{\partial \mathbf{h}(\mathbf{k})}{\partial k} &= \frac{\hat{\mu}^{-1}}{k_0} \cdot \mathbf{k} \times \frac{\partial \mathbf{e}(\mathbf{k})}{\partial k}. \end{aligned} \tag{34}$$

The argument in [24] shows that for a nonsingular optic axis there correspond two linearly independent plane waves propagating along this axis. Their general representation derivable from equations for the transverse components of spectral field amplitudes [8] is rendered by

$$\begin{aligned} \mathbf{E}_{\mathbf{k}}(\mathbf{x}) &= [\varepsilon_l(\mathbf{n})\hat{I} - \mathbf{n}\mathbf{n}\cdot\hat{\varepsilon}] \cdot \mathbf{a}_\perp \exp(i\mathbf{k}\cdot\mathbf{x}) \\ k_0\mathbf{H}_{\mathbf{k}}(\mathbf{x}) &= \varepsilon_l(\mathbf{n})\hat{\mu}^{-1} \cdot \mathbf{k} \times \mathbf{a}_\perp \exp(i\mathbf{k}\cdot\mathbf{x}) \end{aligned} \tag{35}$$

where $\mathbf{a}_\perp$ is an arbitrarily fixed vector in a plane perpendicular to the optic axis.

If the solution $\mathbf{k}$ to (30) does not satisfy the condition (32), an optic axis is called singular [22], [24]. In this case $\mathbf{e}(\mathbf{k}) \neq 0$, $\mathbf{h}(\mathbf{k}) \neq 0$, and the structure of the medium permits the existence of a plane wave mode (27) as well as that of an associated mode (31). Unlike plane wave modes, (31) shows that in an associated mode the amplitude factors are (linear) functions of $\mathbf{x}$. For a detailed examination of conditions which permit the existence of singular optic axis and associated modes, an interested reader is referred to [24]. Principally, they presuppose the presence of losses in an anisotropic medium.

D. Analytic Representation for Space-Domain Green's Functions

Following the lines of [11], one readily finds from (7) and (13) that the space-domain GF's are represented by one scalar potential $W(\mathbf{x}-\mathbf{x}')$

$$W(\mathbf{x}-\mathbf{x}') = \frac{1}{(2\pi)^3} \int d^3k \frac{\exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')]}{\Delta(\mathbf{k})} \tag{36}$$

as follows:

$$\begin{aligned} \hat{G}_{ee}(\mathbf{x}-\mathbf{x}') &= -\frac{4\pi i}{ck_0}\hat{D}_{ee}(-i\nabla)W(\mathbf{x}-\mathbf{x}') \\ \hat{G}_{mm}(\mathbf{x}-\mathbf{x}') &= -\frac{4\pi i}{ck_0}\hat{D}_{mm}(-i\nabla)W(\mathbf{x}-\mathbf{x}') \\ \hat{G}_{em}(\mathbf{x}-\mathbf{x}') &= \frac{4\pi i}{c}\hat{D}_{em}(-i\nabla)W(\mathbf{x}-\mathbf{x}') \\ \hat{G}_{me}(\mathbf{x}-\mathbf{x}') &= -\frac{4\pi i}{c}\hat{D}_{me}(-i\nabla)W(\mathbf{x}-\mathbf{x}'). \end{aligned} \tag{37}$$

The computation of the 3-D Fourier integral in (36) proceeds as follows: [9], [13], [23, p. 345], [25]. First, introduce in the $\mathbf{k}$ space a Cartesian coordinate system k_1, k_2, k_3 the k_3 axis of which is set along a unit vector

$$\mathbf{n}_{\rm ob} = \frac{\mathbf{x}-\mathbf{x}'}{L}. \tag{38}$$

Therein L is the distance from the source point $\mathbf{x}'$ to the observation point $\mathbf{x}$

$$L = |\mathbf{x}-\mathbf{x}'|. \tag{39}$$

The other axes k_1 and k_2 must be perpedicular to each other and the k_3 axis and are otherwise arbitrarily oriented. Second, change in (36) the Cartesian variables k_1, k_2, k_3 into spherical variables k, θ, φ by letting

$$\begin{aligned} &k_1 = k\sin\theta\cos\varphi, \quad k_2 = k\sin\theta\sin\varphi \\ &k_3 = k\cos\theta, \quad d^3k = \sin\theta\, dk\, d\varphi\, d\theta. \end{aligned} \tag{40}$$

In the resulting expression integration over k, θ, and φ is carried out within usual limits $0 < k + \infty$, $0 < \theta < \pi$, and $0 < \varphi < 2\pi$. Third, by recognizing that the concurrent reduction of integration over θ to an interval $0 < \theta < \pi/2$ and the extension of ranges for k to $-\infty < k < +\infty$ do not affect the integral involved, bring $W(\mathbf{x}-\mathbf{x}')$ to a form

$$W(\mathbf{x}-\mathbf{x}') = \frac{1}{(2\pi)^3} \int_{\Sigma_+} d\Sigma \int_{-\infty}^{+\infty} \frac{k^2\, dk}{\Delta(k\mathbf{n})} e^{ik\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}')}. \tag{41}$$

Here, $\mathbf{n} \equiv \mathbf{n}(\theta,\varphi)$ plays the part of a position vector to a point on the sphere $\Sigma = \{\mathbf{k} : k_1^2 + k_2^2 + k_3^2 = 1\}$ of unit radius in the $\mathbf{k}$ space, and the exterior surface integral is performed over the hemisphere Σ_+ of sphere Σ which is seen by an observer when the sphere is viewed looking in the direction opposite to the direction of vector $\mathbf{n}_{\rm ob}$. Mathematically, Σ_+ is defined by the requirements $k=1$, $0<\theta<\pi/2$, $0<\varphi<2\pi$.

Finally, we calculate the interior integral over k. Having regard for the respective integral in (41), one can easily verify that the integrand has poles in the complex k plane, which are the roots of (29) taken at $\mathbf{k}=k\mathbf{n}$. There will be four simple poles at $k=k_j(\mathbf{n})$, $(j=\overline{1,4})$ if both $D(\mathbf{n})\neq 0$ and $A(\mathbf{n})\neq 0$ and two second-order poles at $k=\pm k_a(\mathbf{n})$ if $D(\mathbf{n})=0$ but $A(\mathbf{n})\neq 0$. In the explicit form, one has

$$\begin{aligned} k_{1,2}(\mathbf{n}) &= k_0 N_\pm(\mathbf{n}) \\ k_{3,4}(\mathbf{n}) &= -k_{1,2}(\mathbf{n}) \\ N_\pm(\mathbf{n}) &= \sqrt{\frac{B(\mathbf{n})\pm\sqrt{D(\mathbf{n})}}{A(\mathbf{n})}} \end{aligned} \tag{42}$$

$$\begin{aligned} k_a(\mathbf{n}) &= k_0 N_a(\mathbf{n}), \\ N_a(\mathbf{n}) &= \sqrt{\frac{B(\mathbf{n})}{A(\mathbf{n})}}. \end{aligned} \tag{43}$$

Here and in the remainder of the present paper the branch of the square root is defined to satisfy the requirement $0\leq \arg\sqrt{\ }<\pi$ so that $0\leq \arg N_\pm(\mathbf{n}), N_a(\mathbf{n})<\pi$

$$\begin{aligned} A(\mathbf{n}) &= (\mathbf{n}\cdot\hat{\varepsilon}\cdot\mathbf{n})(\mathbf{n}\cdot\hat{\mu}\cdot\mathbf{n}), \\ C &= (\det\hat{\varepsilon})(\det\hat{\mu}) \\ 2B(\mathbf{n}) &= (\mathbf{n}\cdot\hat{\varepsilon}\cdot\mathbf{n})\,\mathrm{Tr}\,\hat{\delta}-\mathbf{n}\cdot\hat{\beta}\cdot\mathbf{n} \end{aligned} \tag{44}$$

$$D(\mathbf{n}) = B^2(\mathbf{n}) - A(\mathbf{n})C \tag{45}$$

$N_+(\mathbf{n})$ and $N_-(\mathbf{n})$ are the indexes of refraction in the direction $\mathbf{n}$, which permits the existence of two isonormal plane waves (i.e., the waves propagating in the same direction but having different propagation constants) and $N_a(\mathbf{n})$ is the index of refraction along an optic axis where the two indexes of refraction $N_+(\mathbf{n})$ and $N_-(\mathbf{n})$ coalesce [22], [23, sec. 5.5]. We remind the reader that the aforementioned poles have been presumed to lie off the real axis in the complex k plane (e.g., due to dissipative losses in the medium). Then the poles $k=k_{1,2}$ and $k=k_a$ will be located in the upper half and $k=k_{3,4}$ and $k=-k_a$ in the lower half of the complex k plane. It is important to notice that a factor $k^2/\Delta(k\mathbf{n})$ decays as k^{-2} with $k\to+\infty$, and a term $\exp[ik\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}')]$ vanishes as k approaches infinity in the upper half of the complex k plane because $\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}')>0$ for $\mathbf{n}\in\Sigma_+$. This allows us to close the real-axis path of integration by an infinite semicircle in the upper half-plane and compute the integral via conventional residue calculus. Implementation of this procedure furnishes analytic representation for $W(\mathbf{x}-\mathbf{x}')$ that we are questing (46), shown at the bottom of the page. It should be mentioned for the sake of rigour that whenever $\mathbf{n}$ becomes parallel to an optic axis of the medium both $D(\mathbf{n})$ and a term in the brackets on the right of (46) tend to zero; hence, a limit $D\to 0$ has to be taken to calculate the integrand properly.

E. Far-Field Zone

We shall now engage in evaluating the radiation field of a point source with electric dipole moment $\mathbf{p}_e$ and magnetic dipole moment $\mathbf{p}_m$ located at the source point $\mathbf{x}'$. In this case the current distribution is given by

$$\begin{aligned} \mathbf{J}(\mathbf{x}) &= -i\omega\mathbf{p}_e\delta(\mathbf{x}-\mathbf{x}') \\ \mathbf{M}(\mathbf{x}) &= -i\omega\mathbf{p}_m\delta(\mathbf{x}-\mathbf{x}') \end{aligned} \tag{47}$$

where $\delta(\mathbf{x}-\mathbf{x}')$ is the Dirac delta function. Our purpose is to calculate the EM field (3) for large L where L is defined in (39). To this end, we insert (41) into (37) and further into (3) and interchange the order of the (θ,φ) integration and the k integration. The integration with respect to θ and φ in the resulting equations is carried out according to an approximate formula

$$\begin{aligned} &\int_{\Sigma_+} F(\mathbf{n})e^{ik\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}')}d\Sigma \\ &\quad\approx 2\pi F(k\mathbf{n}_{\rm ob})\frac{e^{ikL}}{ikL} \\ &\qquad -\frac{1}{ikL}\int_0^{2\pi} d\varphi F(k\mathbf{n})|_{\theta=\pi/2}+\mathcal{O}\left(\frac{1}{L^2}\right) \end{aligned} \tag{48}$$

which follows from the stationary phase method. The remaining integral over k may be evaluated via conventional residue calculus. Then performing differentiation with respect to $\mathbf{x}$ and retaining the leading terms we get: if $D(\mathbf{n}_{\rm ob})\neq 0$ and $A(\mathbf{n}_{\rm ob})\neq 0$ then

$$\begin{aligned} \mathbf{E}(\mathbf{x}) &\approx \frac{1}{2k_0^2 L D^{1/2}(\mathbf{n}_{\rm ob})}\big[\tilde{\mathbf{e}}(\mathbf{k}_2^{\rm ob})e^{ik_2(\mathbf{n}_{\rm ob})L} \\ &\quad - \tilde{\mathbf{e}}(\mathbf{k}_1^{\rm ob})e^{ik_1(\mathbf{n}_{\rm ob})L}\big] \\ \mathbf{H}(\mathbf{x}) &\approx \frac{1}{2k_0^2 L D^{1/2}(\mathbf{n}_{\rm ob})}\big[\tilde{\mathbf{h}}(\mathbf{k}_2^{\rm ob})e^{ik_2(\mathbf{n}_{\rm ob})L} \\ &\quad - \tilde{\mathbf{h}}(\mathbf{k}_1^{\rm ob})e^{ik_1(\mathbf{n}_{\rm ob})L}\big] \end{aligned} \tag{49}$$

if $D(\mathbf{n}_{\rm ob})=0$, $A(\mathbf{n}_{\rm ob})\neq 0$ and $\hat{t}(\mathbf{k}_a^{\rm ob})=0$ then

$$\begin{aligned} \mathbf{E}(\mathbf{x}) &\approx \frac{ie^{ik_a(\mathbf{n}_{\rm ob})L}}{2k_0 L A(\mathbf{n}_{\rm ob})N_a(\mathbf{n}_{\rm ob})}\left.\frac{\partial\tilde{\mathbf{e}}(\mathbf{k})}{\partial k}\right|_{\mathbf{k}=\mathbf{k}_a^{\rm ob}} \\ \mathbf{H}(\mathbf{x}) &\approx \frac{ie^{ik_a(\mathbf{n}_{\rm ob})L}}{2k_0 L A(\mathbf{n}_{\rm ob})N_a(\mathbf{n}_{\rm ob})}\left.\frac{\partial\tilde{\mathbf{h}}(\mathbf{k})}{\partial k}\right|_{\mathbf{k}=\mathbf{k}_a^{\rm ob}} \end{aligned} \tag{50}$$

$$W(\mathbf{x}-\mathbf{x}') = \frac{i}{(4\pi)^2 k_0}\int_{\Sigma_+}\frac{d\Sigma}{\sqrt{D(\mathbf{n})}}\big[N_+(\mathbf{n})e^{ik_1(\mathbf{n})\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}')}-N_-(\mathbf{n})e^{ik_2(\mathbf{n})\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}')}\big] \tag{46}$$

and for $D(\mathbf{n}_{\rm ob})=0$, $A(\mathbf{n}_{\rm ob})\neq 0$, $\hat{t}(\mathbf{k}_a^{\rm ob})\neq 0$

$$\begin{aligned}\mathbf{E}(\mathbf{x})&\approx -\frac{e^{ik_a(\mathbf{n}_{\rm ob})L}}{2k_0A(\mathbf{n}_{\rm ob})N_a(\mathbf{n}_{\rm ob})}\tilde{\mathbf{e}}(\mathbf{k}_a^{\rm ob})\\ \mathbf{H}(\mathbf{x})&\approx -\frac{e^{ik_a(\mathbf{n}_{\rm ob})L}}{2k_0A(\mathbf{n}_{\rm ob})N_a(\mathbf{n}_{\rm ob})}\tilde{\mathbf{h}}(\mathbf{k}_a^{\rm ob}).\end{aligned}\tag{51}$$

In the following:

$$\begin{aligned}\mathbf{k}_{1,2}^{\rm ob}&=k_{1,2}(\mathbf{n}_{\rm ob})\mathbf{n}_{\rm ob}\\ \mathbf{k}_a^{\rm ob}&=k_a(\mathbf{n}_{\rm ob})\mathbf{n}_{\rm ob}\end{aligned}\tag{52}$$

and $\tilde{\mathbf{e}}(\mathbf{k})$, $\tilde{\mathbf{h}}(\mathbf{k})$ are formally obtainable from $\mathbf{e}(\mathbf{k})$, $\mathbf{h}(\mathbf{k})$ in (28) after the replacement $\mathbf{P}_e\to\mathbf{p}_e$, $\mathbf{P}_m\to\mathbf{p}_m$.

Equation (49) describes radiation in a direction that does not coincide with an optic axis, whereas (50) and (51) refer to radiation along a nonsingular optic axis and a singular optic axis, respectively. Physical interpretation of these expressions follows from the analysis of phase factors encountered therein and the comparison of the vectorial structure of these equations with that of eigensolutions (27), (33), and (31). The terms on the right-hand side of (49)–(51) may be interpreted as a sum of two "isonormal" modes, a plane wave mode, or an associated mode, respectively, which propagate along the ray launched from the source point $\mathbf{x}'$ to the observation point $\mathbf{x}$. Note that (49) and (50) contain a customary factor $1/L$ pertinent to a spherical wave propagation. The absence of such factor in (51) should be ascribed to the excitation of an associated mode whose amplitude grows linearly with distance ($\sim L$), thus compensating for an algebraic decay ($1/L$) of a spherical wave.

F. Static GFs

Anticipating further needs, we conclude this section with the explicit solutions for the quasi-static parts $\hat{S}_e(\mathbf{x}-\mathbf{x}')$ and $\hat{S}_m(\mathbf{x}-\mathbf{x}')$ of the electric $\hat{G}_{ee}(\mathbf{x}-\mathbf{x}')$ and magnetic $\hat{G}_{mm}(\mathbf{x}-\mathbf{x}')$ Green's dyads. The quantities $\hat{S}_{e,m}(\mathbf{x}-\mathbf{x}')$ describe the behavior of respective GF's close to the source point. They can be calculated from the knowledge of the limiting values of $\hat{G}_{ee}(\mathbf{k})$ and $\hat{G}_{mm}(\mathbf{k})$ as $k\to+\infty$ [26]. Taking note of (13) one gets

$$\begin{aligned}\hat{G}_{ee}(\mathbf{x}-\mathbf{x}')&\approx\hat{S}_e(\mathbf{x}-\mathbf{x}')=\frac{4\pi i}{ck_0}\nabla\nabla g_e(\mathbf{x}-\mathbf{x}'),\\ \hat{G}_{mm}(\mathbf{x}-\mathbf{x}')&\approx\hat{S}_m(\mathbf{x}-\mathbf{x}')=\frac{4\pi i}{ck_0}\nabla\nabla g_m(\mathbf{x}-\mathbf{x}'),\\ &(\mathbf{x}\approx\mathbf{x}')\end{aligned}\tag{53}$$

where $g_{e,m}(\mathbf{x}-\mathbf{x}')$ are the static Green's functions defined as

$$\begin{aligned}g_e(\mathbf{x}-\mathbf{x}')&=\frac{1}{(2\pi)^3}\int d^3k\frac{\exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')]}{\mathbf{k}\cdot\hat{\varepsilon}\cdot\mathbf{k}}\\ g_m(\mathbf{x}-\mathbf{x}')&=\frac{1}{(2\pi)^3}\int d^3k\frac{\exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')]}{\mathbf{k}\cdot\hat{\mu}\cdot\mathbf{k}}.\end{aligned}\tag{54}$$

Following [9] and [13], it is expedient to define in the 3-D space a (right-handed) basis set $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3\equiv\mathbf{n}_{\rm ob}$, where $\mathbf{u}_1$ and $\mathbf{u}_2$ are arbitrarily fixed orthonormal vectors in a plane perpendicular to $\mathbf{n}_{\rm ob}$. (A reader must not confuse $\mathbf{u}_1$ and $\mathbf{u}_2$ for unit vectors along the coordinate axes k_1 and k_2 in the $\mathbf{k}$ space.) Let us introduce the elements of the constitutive dyads in this basis set as $\varepsilon_{jk}=\mathbf{u}_j\cdot\hat{\varepsilon}\cdot\mathbf{u}_k$ and $\mu_{jk}=\mathbf{u}_j\cdot\hat{\mu}\cdot\mathbf{u}_k$, $(j,k=1,2,3)$. Then the application of an integration technique of [9] and [13] to the calculation of 3-D Fourier integrals in (54) yields

$$g_{e,m}(\mathbf{x}-\mathbf{x}')=\frac{1}{4\pi R_{e,m}(\mathbf{x}-\mathbf{x}')}\begin{cases}+1 & \text{if } w_1<1 \text{ and } w_2>1 \quad \text{(a)}\\ -1 & \text{if } w_1>1 \text{ and } w_2<1 \quad \text{(b)}.\end{cases}\tag{55}$$

In the above

$$R_{e,m}(\mathbf{x}-\mathbf{x}')=[(\mathbf{x}-\mathbf{x}')\cdot\hat{\omega}_{e,m}\cdot(\mathbf{x}-\mathbf{x}')]^{1/2}\tag{56}$$

$$\hat{\omega}_e=\operatorname{adj}\hat{\varepsilon}^{(s)},\quad \hat{\omega}_m=\operatorname{adj}\hat{\mu}^{(s)}\tag{57}$$

$$\hat{\varepsilon}^{(s)}=\frac{1}{2}(\hat{\varepsilon}+\hat{\varepsilon}^{\rm T}),\quad \hat{\mu}^{(s)}=\frac{1}{2}(\hat{\mu}+\hat{\mu}^{\rm T}).\tag{58}$$

In the case of the static GF $g_e(\mathbf{x}-\mathbf{x}')$, the quantities w_1 and w_2 are given by

$$w_{1,2}(\mathbf{n}_{\rm ob})=\left|\frac{\varepsilon_{11}+\varepsilon_{22}\pm\sqrt{\varepsilon_{11}\varepsilon_{22}-(\varepsilon_{12}+\varepsilon_{21})^2/4}}{\varepsilon_{11}-\varepsilon_{22}-i(\varepsilon_{12}+\varepsilon_{21})}\right|.\tag{59}$$

Similar expressions for w_1 and w_2 referring to the magnetic GF $g_m(\mathbf{x}-\mathbf{x}')$ are obtainable from (59) after the replacement $\varepsilon\to\mu$.

It can be shown that the quantities $w_{1,2}(\mathbf{n}_{\rm ob})$ remain unchanged when the triad $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ is rotated about the $\mathbf{u}_3$ axis. This makes clear that the obtained solution for $g_{e,m}(\mathbf{x}-\mathbf{x}')$ is, in fact, independent upon the choice of vectors $\mathbf{u}_1$ and $\mathbf{u}_2$, as should be anticipated on physical grounds. It is appropriate to mention that an equivalent form of expression (55a) was derived earlier in [13, eq. (39)].

The analysis of what physical situation stands behind alternatives (a) and (b) in (55) goes beyond the scope of the present paper. We may, however, remark that the top (bottom) expression on the right of (55) holds true in the case where the symmetric part of the permittivity dyad $\hat{\varepsilon}^{(s)}$ belongs to a class of real positive-definite (negative-definite) dyads. We note that in the aforementioned case the permittivity dyad itself need not be real, symmetric, or definite.

III. Low-Frequency Scattering by an Anisotropic Ellipsoid

A. Quasi-Static Approximation

We here consider the problem of scattering of an electromagnetic wave propagating in a homogeneous anisotropic medium of permittivity $\hat{\varepsilon}$ and permeability $\hat{\mu}$, which impinges on a small homogeneous anisotropic scatterer V_b with the constitutive parameters $\hat{\varepsilon}_b$, $\hat{\mu}_b$, and maximum diameter a. The incident field $\mathbf{E}_{\rm in}(\mathbf{x})$, $\mathbf{H}_{\rm in}(\mathbf{x})$ is assumed to have a plane wave form described in Section II-D and not discussed here.

In the presence of the scatterer, the total EM field $\mathbf{E}(\mathbf{x})$, $\mathbf{H}(\mathbf{x})$ consists of an incident field $\mathbf{E}_{\rm in}(\mathbf{x})$, $\mathbf{H}_{\rm in}(\mathbf{x})$, and a scattered field $\mathbf{E}_{\rm sc}(\mathbf{x})$, $\mathbf{H}_{\rm sc}(\mathbf{x})$ which may be generated by the induced electric and magnetic currents of densities [17], [19], [20]

$$\mathbf{J}_{\rm ind}(\mathbf{x}) = -(ik_0c/4\pi)(\hat{\varepsilon}_b - \hat{\varepsilon}) \cdot \mathbf{E}(\mathbf{x}),$$
$$\mathbf{M}_{\rm ind}(\mathbf{x}) = -(ik_0c/4\pi)(\hat{\mu}_b - \hat{\mu}) \cdot \mathbf{H}(\mathbf{x}) \tag{60}$$

distributed over the region V_b. Relating the scattered field to the induced currents (60) with the help of the dyadic GF's yields a system of two Lippman–Schwinger integrodifferential equations. On expanding the unknowns as well as the kernels in the aforementioned equations in powers of k_0a taking (53) into account and retaining the lowest order terms only we arrive at the well-known relations of quasi-statics

$$\mathbf{E}^{(0)}(\mathbf{x}) = \mathbf{E}_{\rm in}(\underline{x}_c) + \nabla\nabla \cdot \int_{V_b} g_e(\mathbf{x}-\mathbf{x}')(\hat{\varepsilon}_b - \hat{\varepsilon}) \cdot \mathbf{E}^{(0)}(\mathbf{x}')\, d^3x' \tag{61}$$

$$\mathbf{H}^{(0)}(\mathbf{x}) = \mathbf{H}_{\rm in}(\underline{x}_c) + \nabla\nabla \cdot \int_{V_b} g_m(\mathbf{x}-\mathbf{x}')(\hat{\mu}_b - \hat{\mu}) \cdot \mathbf{H}^{(0)}(\mathbf{x}')\, d^3x' \tag{62}$$

which govern the leading-order terms $\mathbf{E}^{(0)}(\mathbf{x})$, $\mathbf{H}^{(0)}(\mathbf{x})$ of the aforementioned Rayleigh series for the total EM field in the vicinity of a small scatterer. Here, $\underline{x}_c$ is an arbitrarily fixed point within V_b. By allowing $\mathbf{x}$ to lie in V_b, (61) and (62) lead to a system of two volume integrodifferential equations for the quasistatic interior field $\mathbf{E}^{(0)}(\mathbf{x})$, $\mathbf{H}^{(0)}(\mathbf{x})$.

B. Solution Procedure

The above formulation (61) and (62) holds for a small scatterer of arbitrary shape. Here we restrict ourselves to the case where the scatterer V_b has the form of an ellipsoid with semiaxes a_1, a_2 and a_3, $(a_1 \geq a_2 \geq a_3)$ is centered at the origin of the principal frame x_1, x_2, x_3, and is aligned along coordinate axes of a local orthogonal frame x_1^l, x_2^l, x_3^l. An equation of the domain V_b in the local frame has a canonical form

$$\left(\frac{x_1^l}{a_1}\right)^2 + \left(\frac{x_2^l}{a_2}\right)^2 + \left(\frac{x_3^l}{a_3}\right)^2 \equiv (\underline{x}^l)^{\rm T} \cdot \underline{\underline{A}} \cdot \underline{x}^l < 1 \tag{63}$$

where $\underline{\underline{A}}$ stands for a diagonal matrix $\mathrm{diag}[1/a_1^2, 1/a_2^2, 1/a_3^2]$. For the sake of convenience, the point $\underline{x}_c$ is chosen to coincide with a symmetry center $\underline{x} = 0$ of the ellipsoid. Orientation of the ellipsoid is determined by a real orthogonal matrix $\underline{\underline{\mathcal{O}}} = [\mathcal{O}_{jk}]$, which establishes a linear relationship between the unit coordinate vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ of the principal frame and those of a local frame $\mathbf{e}_1^l$, $\mathbf{e}_2^l$, $\mathbf{e}_3^l$, viz. $\mathbf{e}_j = \sum_{k=1}^3 \mathcal{O}_{jk}\mathbf{e}_k^l$, $(j = 1, 2, 3)$. With the aid of a relation $\underline{x}^l = \underline{\underline{\mathcal{O}}}^{\rm T} \cdot \underline{x}$ [27] we can write (63) in terms of the global coordinates x_1, x_2, x_3 as

$$B(\underline{x}, \underline{x}) \equiv \underline{x}^{\rm T} \cdot \underline{\underline{B}} \cdot \underline{x} < 1 \tag{64}$$

$$\underline{\underline{B}} = \underline{\underline{\mathcal{O}}} \cdot \underline{\underline{A}} \cdot \underline{\underline{\mathcal{O}}}^{\rm T} \tag{65}$$

where $\underline{\underline{B}}$ is a real symmetric positive definite matrix and $B(\underline{x}, \underline{x})$ is a correlative quadratic form.

At this point we shall further assume that both the symmetric part of the permittivity $\hat{\varepsilon}^{(s)}$ and that of the permeability $\hat{\mu}^{(s)}$ referring to the surrounding medium are real positive definite or negative definite dyads. Note that the constitutive dyads themselves, $\hat{\varepsilon}$ and $\hat{\mu}$, may be neither real nor definite.

Under these conditions, separate consideration in the Appendix shows that the following relations hold [14]:

$$\nabla\nabla \int_{V_b} g_e(\mathbf{x}-\mathbf{x}')\, d^3x' = -\hat{N}_e$$
$$\nabla\nabla \int_{V_b} g_m(\mathbf{x}-\mathbf{x}')\, d^3x' = -\hat{N}_m \tag{66}$$

provided that $\mathbf{x}$ lies inside V_b. Constant dyads $\hat{N}_{e,m}$ are determined by the permittivity $(\hat{N}_e)$ or permeability $(\hat{N}_m)$ of the ambient medium, orientation, and dimensions of the ellipsoid and are independent of the constitutive parameters of the latter. Their specification is relegated to the Appendix.

Accounting of the constancy of the right-hand members $\mathbf{E}_{\rm in}(0)$ and $\mathbf{H}_{\rm in}(0)$ in (61) and (62), and making use of (66) enables us to develop an explicit solution for $\mathbf{E}^{(0)}(\mathbf{x})$ and $\mathbf{H}^{(0)}(\mathbf{x})$ in the form:

$$\mathbf{E}^{(0)}(\mathbf{x}) = [\hat{I} + \hat{N}_e \cdot (\hat{\varepsilon}_b - \hat{\varepsilon})]^{-1} \cdot \mathbf{E}_{\rm in}(0)$$
$$\mathbf{H}^{(0)}(\mathbf{x}) = [\hat{I} + \hat{N}_m \cdot (\hat{\mu}_b - \hat{\mu})]^{-1} \cdot \mathbf{H}_{\rm in}(0), \quad (\mathbf{x} \in V_b). \tag{67}$$

It is clearly seen from these expressions that to the order adopted in (61) and (62), the interior field within the ellipsoid is uniform in full agreement with a previously known result [18].

With the assistance of (60), an electric dipole moment $\mathbf{p}_e$ and a magnetic dipole moment $\mathbf{p}_m$ of the scatterer defined by

$$\mathbf{p}_e = -\frac{1}{i\omega} \int_{V_b} \mathbf{J}_{\rm ind}(\mathbf{x})\, d^3x'$$
$$\mathbf{p}_m = -\frac{1}{i\omega} \int_{V_b} \mathbf{M}_{\rm ind}(\mathbf{x})\, d^3x' \tag{68}$$

can be written as

$$\mathbf{p}_e = \hat{\alpha}_e \cdot \mathbf{E}_{\rm in}(0), \quad \mathbf{p}_m = \hat{\alpha}_m \cdot \mathbf{H}_{\rm in}(0) \tag{69}$$

wherein $\hat{\alpha}_e$ and $\hat{\alpha}_m$ are the polarizability dyads

$$\hat{\alpha}_e = \frac{V}{4\pi}(\hat{\varepsilon}_b - \hat{\varepsilon}) \cdot [\hat{I} + \hat{N}_e \cdot (\hat{\varepsilon}_b - \hat{\varepsilon})]^{-1}$$
$$\hat{\alpha}_m = \frac{V}{4\pi}(\hat{\mu}_b - \hat{\mu}) \cdot [\hat{I} + \hat{N}_m \cdot (\hat{\mu}_b - \hat{\mu})]^{-1} \tag{70}$$

and $V = (4\pi/3)a_1a_2a_3$ is the volume of the ellipsoid.

Once the EM field throughout the body is solved, the scattered field $\mathbf{E}_{\rm sc}(\mathbf{x}) \equiv \mathbf{E}(\mathbf{x}) - \mathbf{E}_{\rm in}(\mathbf{x})$, $\mathbf{H}_{\rm sc}(\mathbf{x}) \equiv \mathbf{H}(\mathbf{x}) - \mathbf{H}_{\rm in}(\mathbf{x})$ can be calculated by evaluating the integrals in (61) and (62) using expressions for Newtonian potentials at the external points of an ellipsoid [28] if the observation point is located in the neighborhood of V_b and using the asymptotic formulas (49)–(51) when the observation point is located far away from the scatterer.

IV. Concluding Remarks

In summary, after obtaining an explicit solution (13) for the four spectral GF's $\hat{G}_{\nu\xi}(\mathbf{k})$ referring to an unbounded medium with general anisotropy of electromagnetic properties (and spatial dispersion), we have shown that the four spatial GF's $\hat{G}_{\nu\xi}(\mathbf{x}-\mathbf{x}')$ can be expressed via (37) in terms of one scalar function $W(\mathbf{x}-\mathbf{x}')$ for which an analytic representation (46) as a two-dimensional integral over angular variables has been developed. On this basis, invariant asymptotic expressions (49)–(51) for the far-zone field of point sources have been derived, with special attention being drawn to the radiation along optic axes. As well, a coordinate-free form (55) of the static Green's function $g_e(\mathbf{x}-\mathbf{x}')$ has been given. As an example of application of these results, rigorous analytic solutions for the interior EM field (67) and the polarizabilities (70) of a small homogeneous anisotropic ellipsoid in an anisotropic environment have been found referring to the case where the symmetric parts of the permittivity and permeability dyads in the ambient medium are real positive definite or negative definite dyads.

The emphasis of this article is put on anisotropic materials which are describable in terms of the electric permittivity and the magnetic permeability dyads $\hat{\varepsilon}$ and $\hat{\mu}$. It would be interesting to extend the present methodology to the case of general bianisotropic materials whose electromagnetic properties are describable in terms of four constitutive dyads. The present authors' attempt to achieve this goal will be reported elsewhere.

Appendix

In this Appendix we have included, for the readers' benefit, an analytical scheme by which the dyads $\hat{N}_{e,m}$ in (66), (67), and (70) can be calculated for an ellipsoidal scatterer. In the following, only the quantity $\hat{N}_e$ is considered since the solution for $\hat{N}_m$ is obtained automatically by replacing in subsequent formulas the symbol of permittivity to that of permeability.

Consulting (55) supplies us with an equality

$$\int_{V_b} g_e(\mathbf{x}-\mathbf{x}')\, d^3x' = \pm(1/4\pi)\Phi(\mathbf{x}). \tag{71}$$

Here and in later (93), the top (bottom) sign applies if $\hat{\varepsilon}^{(s)}$ is a real positive definite (negative definite) dyad

$$\Phi(\mathbf{x}) \equiv \int_{V_b} \frac{d^3x'}{\omega^{1/2}(\underline{x}-\underline{x}', \underline{x}-\underline{x}')} \tag{72}$$

$$\omega(\underline{x}-\underline{x}', \underline{x}-\underline{x}') \equiv (\underline{x}-\underline{x}')^{\mathrm{T}} \cdot \underline{\underline{\omega}} \cdot (\underline{x}-\underline{x}') \tag{73}$$

and $\underline{\underline{\omega}}$ is a shorthand notation for the matrix of a dyad $\hat{\omega}_e = \operatorname{adj} \varepsilon^{(s)}$ in the principal frame $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$. Note that in view of the constraints imposed on $\hat{\varepsilon}^{(s)}$ the matrix $\underline{\underline{\omega}}$ turns out to be real, symmetric, and positive definite.

A real symmetric dyad $\hat{\varepsilon}^{(s)}$ possesses three real orthonormal eigenvectors $\mathbf{e}_1^s$, $\mathbf{e}_2^s$, and $\mathbf{e}_3^s$ belonging to real eigenvalues ε_1^s, ε_2^s, and ε_3^s [27]. Since $\hat{\varepsilon}^{(s)}$ is a definite dyad, the products $\varepsilon_1^s\varepsilon_2^s$, $\varepsilon_1^s\varepsilon_3^s$, and $\varepsilon_2^s\varepsilon_3^s$ will be positive values. Let $\underline{\underline{\mathcal{D}}} = [\mathcal{D}_{jk}]$ stand for a 3 ×3 matrix which determines transition from the basis set $\mathbf{e}_1^s$, $\mathbf{e}_2^s$, $\mathbf{e}_3^s$ to the global basis set $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ according to $\mathbf{e}_j = \sum_{k=1}^3 \mathcal{D}_{jk}\mathbf{e}_k^s$, $(j=1,2,3)$. For further use, we shall introduce real symmetric positive definite matrices

$$\underline{\underline{R}} = \operatorname{diag}\left[\sqrt{\varepsilon_2^s\varepsilon_3^s}, \sqrt{\varepsilon_1^s\varepsilon_3^s}, \sqrt{\varepsilon_1^s\varepsilon_2^s}\right] \tag{74}$$

$$\underline{\underline{F}} = \underline{\underline{\mathcal{D}}} \cdot \underline{\underline{R}} \cdot \underline{\underline{\mathcal{D}}}^{\mathrm{T}} \tag{75}$$

$$\underline{\underline{S}} = \underline{\underline{F}}^{-1} \cdot \underline{\underline{B}} \cdot \underline{\underline{F}}^{-1} \tag{76}$$

with $\underline{\underline{B}}$ having the same meaning as in (64) and consider the eigenvalue problem defined by

$$\underline{\underline{S}} \cdot \underline{u} = \lambda \underline{u}. \tag{77}$$

It is worthwhile to remark that the matrix $\underline{\underline{R}}$ represents a square root of a positive definite dyad $\hat{\omega}$ in the basis set $\mathbf{e}_1^s$, $\mathbf{e}_2^s$, $\mathbf{e}_3^s$.

The problem (77) has three positive eigenvalues λ_1, λ_2, and λ_3, and the corresponding real eigenvectors $\underline{v}_1$, $\underline{v}_2$, and $\underline{v}_3$ can always be chosen to satisfy the orthogonality condition [27]

$$\underline{v}_j^{\mathrm{T}} \cdot \underline{v}_k = \delta_{jk}, \quad (j,k=1,2,3) \tag{78}$$

δ_{jk} being the Kronecker delta. It can be verified easily that

$$\underline{z}_j = \underline{\underline{F}}^{-1} \cdot \underline{v}_j, \quad (j=1,2,3) \tag{79}$$

are the eigenvectors and λ_j the eigenvalues of a general eigenvalue problem

$$\underline{\underline{B}} \cdot \underline{z} = \lambda \underline{\underline{\omega}} \cdot \underline{z}. \tag{80}$$

Furthermore, due to (78) the following general orthogonality condition holds:

$$\underline{z}_j^{\mathrm{T}} \cdot \underline{\underline{\omega}} \cdot \underline{z}_k = \delta_{jk}, \quad (j,k=1,2,3). \tag{81}$$

We shall need a real 3 × 3 matrix $\underline{\underline{Z}} = [\underline{z}_1, \underline{z}_2, \underline{z}_3]$ whose column j comprises vector $\underline{z}_j$, $(j=1,2,3)$. Because [27, sec. 6.3]

$$\begin{aligned} \underline{\underline{Z}}^{\mathrm{T}} \cdot \underline{\underline{B}} \cdot \underline{\underline{Z}} &= \operatorname{diag}[\lambda_1, \lambda_2, \lambda_3] \\ \underline{\underline{Z}}^{\mathrm{T}} \cdot \underline{\underline{\omega}} \cdot \underline{\underline{Z}} &= \operatorname{diag}[1,1,1] \end{aligned} \tag{82}$$

the substitution

$$\underline{x} = \underline{\underline{Z}} \cdot \underline{y}, \quad \underline{x}' = \underline{\underline{Z}} \cdot \underline{y}' \tag{83}$$

carries the quadratic forms $B(\underline{x}', \underline{x}')$, and $\omega(\underline{x}-\underline{x}', \underline{x}-\underline{x}')$ into a sum of squares

$$\begin{aligned} B(\underline{x}', \underline{x}') &= \lambda_1 y_1'^2 + \lambda_2 y_2'^2 + \lambda_3 y_3'^2 \\ \omega(\underline{x}-\underline{x}', \underline{x}-\underline{x}') &= (y_1-y_1')^2 + (y_2-y_2')^2 \\ &\quad + (y_3-y_3')^2 \equiv |\underline{y}-\underline{y}'|. \end{aligned} \tag{84}$$

The values λ_1, λ_2, and λ_3 will henceforth be arranged in the following order: $0 < \lambda_1 \le \lambda_2 \le \lambda_3$ so that the reciprocals of the square roots of these eigenvalues $b_j = 1/\sqrt{\lambda_j}$ fulfill the requirement $b_1 \ge b_2 \ge b_3 > 0$. By using (83) together with the relations

$$\begin{aligned} \frac{D(x_1', x_2', x_3')}{D(y_1', y_2', y_3')} &= \det \underline{\underline{Z}} \\ (\det \underline{\underline{Z}})^2 \left(\det \hat{\varepsilon}^{(s)}\right)^2 &= 1 \end{aligned} \tag{85}$$

we find that

$$\Phi(\mathbf{x}) = \frac{1}{\left|\det \hat{\varepsilon}^{(s)}\right|} \int_{W_b} \frac{d^3y'}{|\underline{y}-\underline{y}'|} \tag{86}$$

where

$$W_b = \left\{ \underline{y}' : \left(\frac{y_1'}{b_1}\right)^2 + \left(\frac{y_2'}{b_2}\right)^2 + \left(\frac{y_3'}{b_3}\right)^2 < 1 \right\}. \quad (87)$$

The above expression (86) determines the Newtonian potential of an ellipsoid W_b. The potential at an inner point $\underline{y}$ of an ellipsoid W_b is known to be given by [28, sec. 10]

$$\Phi(\mathbf{x}) = \frac{2\pi}{\left|\det \hat{\varepsilon}^{(s)}\right|}\left(L_0 - L_1 y_1^2 - L_2 y_2^2 - L_3 y_3^2\right) \quad (88)$$

where L_1, L_2, and L_3 are the geometric factors of an ellipsoid W_b with semiaxes b_1, b_2, b_3

$$L_j = \frac{b_1 b_2 b_3}{2} \int_0^{+\infty} \frac{dq}{(q^2 + b_j^2) f(q)}, \quad (j = 1, 2, 3) \quad (89)$$

$$f(q) = \left[\left(q^2 + b_1^2\right)\left(q^2 + b_2^2\right)\left(q^2 + b_3^2\right)\right]^{1/2} \quad (90)$$

and $L_0 = b_1^2 L_1 + b_2^2 L_2 + b_3^2 L_3$. When the numbers b_1, b_2, and b_3 are all different, the quantities L_1, L_2, and L_3 may be expressed in terms of incomplete elliptic integrals of the first and second kinds [28, sec. 5]. In the particular case where at least two of the said numbers coincide, the geometric factors are expressible in terms of elementary functions [28].

On reverting via (83) to the original variables x_1, x_2, and x_3, the Newtonian potential becomes

$$\Phi(\mathbf{x}) = \frac{2\pi}{\left|\det \hat{\varepsilon}^{(s)}\right|}\left(L_0 - \underline{x}^{\mathrm{T}} \cdot \underline{\underline{M}}_e \cdot \underline{x}\right), \quad (\mathbf{x} \in V_b) \quad (91)$$

where

$$\underline{\underline{M}}_e = (\underline{\underline{Z}}^{-1})^{\mathrm{T}} \cdot \underline{\underline{L}} \cdot \underline{\underline{Z}}^{-1} \quad (92)$$

and $\underline{\underline{L}} = \mathrm{diag}[L_1, L_2, L_3]$.

Finally, if we act on (91) with the $\nabla\nabla$ operator, the ensuing expression will read

$$\nabla\nabla\Phi(\mathbf{x}) = \mp 4\pi \hat{N}_e. \quad (93)$$

Here $\hat{N}_e$ denotes a dyad which is characterized in the global basis set by matrix

$$\underline{\underline{N}}_e = \left(\det \hat{\varepsilon}^{(s)}\right)^{-1} \underline{\underline{M}}_e. \quad (94)$$

This concludes the derivation of $\hat{N}_e$. Accounting for (93) and (71) leads us to a representation (66) that we are seeking.

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