A Boundary-Element Solution of the Leontovitch Problem

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***Abstract*—A boundary-element method is introduced for solving electromagnetic scattering problems in the frequency domain relative to an impedance boundary condition (IBC) on an obstacle of arbitrary shape. The formulation is based on the field approach; namely, it is obtained by enforcing the total electromagnetic field, expressed by means of the incident field and the equivalent electric and magnetic currents and charges on the scatterer surface, to satisfy the boundary condition. As a result, this formulation is well-posed at any frequency for an absorbing scatterer. Both of the equivalent currents are discretized by a boundary-element method over a triangular mesh of the surface scatterer. The magnetic currents are then eliminated at the element level during the assembly process. The final linear system to be solved keeps all of the desirable properties provided by the application of this method to the usual perfectly conducting scatterer; that is, its unknowns are the fluxes of the electric currents across the edges of the mesh and its coefficient matrix is symmetric.**

***Index Terms*— Boundary elements, boundary integral equations, impedance boundary condition, Maxwell equations, scattering.**

I. Introduction

SINCE the pioneering work of Leontovitch (see, for instance, [29]), the impedance boundary condition (IBC) has been extensively used to simplify the formulation involved in the solution of complicated electromagnetic scattering problems related to imperfectly conducting scatterers or those with a rough surface (see [23] for more details concerning this approach; in [34], two criteria are worked out to define the range of validity of the IBC for imperfect conductors). It was recognized later that this type of boundary condition can be advantageously used to get a more tractable problem in numerous complex situations of electromagnetic scattering computations (see for example [1], [2], [14], [17], [19], [31], [35]). Such a boundary condition also arises as a simple terminating boundary condition in problems related to the propagation in a waveguide (cf. [18]). It can also be involved in a domain decomposition solution of scattering problems leading in this way to efficient parallel algorithms (cf. [12], [13]). Therefore, it is important for practitioners in computational electromagnetics to have at hand numerical methods dealing with such a boundary condition as efficiently as for the usual perfect conductor boundary condition. It is the aim of this paper to devise such a method.

We sketch the main features of the approach. The total electric and magnetic fields are represented by the equivalent magnetic and electric currents on the surface of the scatterer (cf. [15]). The direct integral boundary equation is obtained by requiring the resulting field expressions to satisfy the IBC. Then, using Rumsey's reaction concept [28], we show how this integral equation can be expressed in a simple variational form with testing electric and magnetic currents. The unknown electric and magnetic currents as well as the electric and magnetic testing currents are not free, but are linked by the same relation induced by the IBC. The main trick is to consider the relation linking electric and magnetic currents as a constraint and to express it variationally by a vector field playing the role of a system of Lagrangian multipliers. Both of the electric and magnetic currents are approximated over a mesh of the surface in flat triangles by a flux (across the edges of the mesh) finite-element method (FEM) (cf. [25]). It has been well known for some time that the flux FEM is crucial to ensuring the conservation of charges, this property being decisive for the consistency of any numerical scheme dealing with these charges (among many others, one can quote [3], [22], and [25]). The introduction of a suitable finite-element approximation of the Lagrangian multipliers makes their elimination possible as well as that of the magnetic currents at the element level during the assembly process. The final linear system to be solved exactly mimics the one corresponding to the electric field integral equation (EFIE) for the perfectly conducting obstacle. It has as unknowns only the fluxes of the electric currents across the edges of the mesh and its coefficient matrix is symmetric.

The plan is as follows. In Section II, we first review some commonly used boundary integral equations for solving the IBC. As a result, the motivation of the formulation which is carried out in this paper is clearly brought out. We further establish that the formulation is well posed at any frequency as long as the IBC is relative to an absorbing obstacle. Section III is devoted to the description of the boundary-element method (BEM). In the final section, we give some numerical experiments which validate the approach by comparing the calculated radar cross section (RCS) and the exact one available from the series solution for a spherical scatterer.

Manuscript received October 7, 1997.

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Publisher Item Identifier S 0018-926X(99)09391-6.

II. Boundary Integral Equation

A. Scattering Problem

Using partly the notation in [15], we refer by D_- to the region of space embodying the scatterer; its surface is denoted by Γ and $\mathbf{n}$ is the unit normal to Γ pointing into the exterior D_+ of D_-.

The determination of the total electromagnetic field $(\mathbf{E}, \mathbf{H})$ induced within D_+ by an impinging incident field $(\mathbf{E}^{\mathrm{inc}}, \mathbf{H}^{\mathrm{inc}})$ on the impedance scatterer is governed by the following boundary-value problem:

$$\begin{cases}\nabla \times (\mathbf{E} - \mathbf{E}^{\mathrm{inc}}) - ikZ(\mathbf{H} - \mathbf{H}^{\mathrm{inc}}) = 0 \\ \nabla \times (\mathbf{H} - \mathbf{H}^{\mathrm{inc}}) + ikZ^{-1}(\mathbf{E} - \mathbf{E}^{\mathrm{inc}}) = 0,\end{cases} \quad \text{in } D_+ \tag{1}$$

$$\mathbf{n} \times (\mathbf{E} \times \mathbf{n}) = ikZ\eta\mathbf{n} \times \mathbf{H} \quad \text{on } \Gamma \tag{2}$$

and $(\mathbf{E} - \mathbf{E}^{\mathrm{inc}}, \mathbf{H} - \mathbf{H}^{\mathrm{inc}})$ satisfies a Silver-Müller radiation condition (e.g., [15], [32]); time dependence is assumed to be in $e^{-i\omega t}$. In the previous equations, k is the wave number, $Z := \sqrt{\mu_0/\varepsilon_0}$ is the intrinsic impedance of the vacuum, and η is the relative impedance of the surface scatterer Γ. The method developed below can handle the case where η is variable and even a function valued in the class of symmetric tensors of the plane tangent to Γ. For simplicity, however, we restrict ourselves to a constant η.

We make the scatterer absorbing by requiring that

$$\Im m(\eta) < 0. \tag{3}$$

Under some further assumptions relative to sufficient smoothness of the surface of the scatterer, the above boundary-value problem admits an unique solution [9].

B. Review of Some Commonly Used Integral Equations

Before introducing the formulation of interest to us, we begin with a review of some boundary integral equations commonly used to solve this problem. We focus on the formulations based on the determination of the equivalent currents

$$\mathbf{J} := \mathbf{n} \times \mathbf{H}, \quad \mathbf{M} := -\mathbf{n} \times \mathbf{E}. \tag{4}$$

For convenience, we designate the electric and magnetic charges, respectively, by $\nabla_\Gamma \cdot \mathbf{J}$ and $\nabla_\Gamma \cdot \mathbf{M}$ where $\nabla_\Gamma \cdot$ is the surface divergence of a vector field tangent to Γ (cf. [8]). Thus the electromagnetic field can be expressed in D_+ in terms of the equivalent currents and charges by the familiar Stratton-Chu formula (cf. [15])

$$\mathbf{E}(x) = \mathbf{E}^{\mathrm{inc}}(x) + ikZT\mathbf{J}(x) + K\mathbf{M}(x); \quad x \in D_+ \tag{5}$$

$$\mathbf{H}(x) = \mathbf{H}^{\mathrm{inc}}(x) - K\mathbf{J}(x) + ikZ^{-1}T\mathbf{M}(x); \quad x \in D_+ \tag{6}$$

where the respective potentials T and K are defined by

$$T\mathbf{J}(x) := \int_\Gamma \left(G(x,y)\mathbf{J}(y) + \frac{1}{k^2}\nabla_x G(x,y)\nabla_\Gamma \cdot \mathbf{J}(y)\right) d\Gamma(y) \tag{7}$$

$$K\mathbf{M}(x) := \int_\Gamma \nabla_y G(x,y) \times \mathbf{M}(y)\, d\Gamma(y) \tag{8}$$

and $G(x,y)$ is the Green kernel giving the outgoing solutions to the scalar Helmholtz equation in three dimensions

$$G(x,y) := \frac{e^{ik|x-y|}}{4\pi|x-y|}; \qquad \text{for } x \neq y \text{ in } R^3. \tag{9}$$

Using the classical jump relations (e.g., [6], [8]), we can express the respective limiting boundary tangential values of $\mathbf{E}$ and $\mathbf{H}$ by

$$\mathbf{E_t} = \mathbf{E_t}^{\mathrm{inc}} + ikZ(T\mathbf{J})_\mathbf{t} + (K\mathbf{M})_\mathbf{t} + \frac{1}{2}\mathbf{n} \times \mathbf{M} \tag{10}$$

$$\mathbf{H_t} = \mathbf{H_t}^{\mathrm{inc}} - (K\mathbf{J})_\mathbf{t} + ikZ^{-1}(T\mathbf{M})_\mathbf{t} - \frac{1}{2}\mathbf{n} \times \mathbf{J} \tag{11}$$

where the subscript t designates the tangential component $\mathbf{E_t} := \mathbf{n} \times (\mathbf{E} \times \mathbf{n})$ of the respective vector field on Γ. In the above expressions, all integrals exist in the ordinary sense as improper integrals with a weak singularity except the integrals of the form

$$\int_\Gamma \nabla_x G(x,y)\nabla_\Gamma \cdot \mathbf{J}(y)\, d\Gamma(y)$$

which have to be interpreted for the moment as Cauchy principal value integrals.

Note that the IBC results in the following relation linking $\mathbf{J}$ and $\mathbf{M}$:

$$\mathbf{n} \times \mathbf{M} = ikZ\eta\,\mathbf{J}. \tag{12}$$

The most straightforward formulation consists in directly expressing the boundary condition (2) using the boundary values (10) and (11). Then, using relation (12) to eliminate the magnetic currents $\mathbf{M}$, we get

$$\begin{aligned} &ikZ((T\mathbf{J})_\mathbf{t} - \eta(K\mathbf{n} \times \mathbf{J})_\mathbf{t} \\ &\quad + \eta\mathbf{n} \times K\mathbf{J} - k^2\eta^2\mathbf{n} \times T\mathbf{n} \times \mathbf{J}) = \mathbf{F} \end{aligned} \tag{13}$$

where $\mathbf{F}$ is expressed through the incident field by

$$\mathbf{F} = -\big(\mathbf{E_t}^{\mathrm{inc}} - ikZ\eta\mathbf{n} \times \mathbf{H}^{\mathrm{inc}}\big). \tag{14}$$

Note that the terms involving the unknowns which are not under the integral sign cancel so that the resulting equation is a Fredholm equation of the first kind. By now, it is well established that this type of integral equation has generally better stability properties than the Fredholm integral equations of the second kind, which can be obtained from alternative formulations (cf., e.g., [10], [16], [24]). However, in this form, this formulation suffers from two main flaws.

- It includes some singular integrals that have meaning only as Cauchy principal value integrals. Although some numerical procedures effectively handling such integrals have recently become available [20], it would be more efficient to avoid their occurrence in practical computations.
- Even if the magnetic charges have been expressed in terms of the electric currents, they remain present in the formulation. Since the conservation of these charges as well as the electric ones is crucial to the consistency of the numerical scheme, the electric currents must fulfill the following requirement: their tangential as well as their

normal component must be continuous at each interface between two elements of the mesh. No finite-element method is yet available which, while remaining simple enough, at the same time satisfies such a degree of continuity constraint and applies to a surface of arbitrary shape. This explains why, in our opinion, this formulation has been used only for bodies of revolution (cf. [23], [27], [30]).

The formulation of interest to us precisely consists in writing (13) of an equivalent form that overcomes the above difficulties. Before going further in this direction, we first consider the second class of formulations usually used to deal with the IBC. An almost immediate method consists in using either (10) or (11) to derive an integral equation. The resulting equations are respectively called the EFIE and the magnetic field integral equation (MFIE) (e.g., [27]). Let us consider the EFIE—the MFIE case being similar. Using relation (12), we obtain

$$\frac{1}{2}\eta\mathbf{J} - (T\mathbf{J})_{\mathbf{t}} + \eta(K\mathbf{n}\times\mathbf{J})_{\mathbf{t}} = \frac{1}{ikZ}\mathbf{E}_{\mathbf{t}}^{\mathrm{inc}}. \tag{15}$$

To the best of the authors' knowledge, it is in the work of Harrington and Mautz (e.g., [22]) that a simple variational procedure removing the Cauchy singular integral from the expression of $(T\mathbf{J})_t$ was first used. The recipe consists in doing the following integration by parts on Γ for any testing surface current $\mathbf{J}'$:

$$\begin{aligned}\int_\Gamma \nabla_x\left(\int_\Gamma G(x,y)\nabla_\Gamma\cdot\mathbf{J}(y)\,d\Gamma(y)\right)\cdot\mathbf{J}'(x)\,d\Gamma(x)\\ = -\int_\Gamma\int_\Gamma G(x,y)\nabla_\Gamma\cdot\mathbf{J}(y)\nabla_\Gamma\cdot\mathbf{J}'(x)\,d\Gamma(y)\,d\Gamma(x).\end{aligned} \tag{16}$$

Note that as a counterpart, to avoiding the strongly singular integral, the charges relative to the testing currents $\mathbf{J}'$ then join the formulation. As a result, we get the following variational equation:

$$\begin{aligned}\frac{1}{2}\int_\Gamma \eta\mathbf{J}\cdot\mathbf{J}'\,d\Gamma - \int_\Gamma T\mathbf{J}\cdot\mathbf{J}'\,d\Gamma + \int_\Gamma K\mathbf{n}\times\mathbf{J}\cdot\eta\mathbf{J}'\,d\Gamma\\ = \frac{1}{ikZ}\int_\Gamma \mathbf{E}_{\mathbf{t}}^{\mathrm{inc}}\cdot\mathbf{J}'\,d\Gamma\end{aligned} \tag{17}$$

where the double integrals are given by

$$\begin{aligned}\int_\Gamma T\mathbf{J}\cdot\mathbf{J}'\,d\Gamma = \int_\Gamma\int_\Gamma G(x,y)\Big(-\frac{1}{k^2}\nabla_\Gamma\cdot\mathbf{J}(y)\nabla_\Gamma\cdot\mathbf{J}'(x)\\ + \mathbf{J}(y)\cdot\mathbf{J}'(x)\Big)\\ d\Gamma(y)\,d\Gamma(x)\end{aligned} \tag{18}$$

$$\begin{aligned}\int_\Gamma K\mathbf{M}\cdot\mathbf{J}'\,d\Gamma = \int_\Gamma\int_\Gamma \nabla_y G(x,y)\times\mathbf{M}(y)\\ \cdot\mathbf{J}'(x)\,d\Gamma(y)\,d\Gamma(x).\end{aligned} \tag{19}$$

A main advantage of the previous formulation is that it does not include the magnetic charges. Hence, it can be solved by standard boundary flux elements [7]. Note that unlike (13), this is a Fredholm integral equation of the second kind. Despite its attractive properties, this formulation also suffers from two main drawbacks.

- It leads to a final linear system with a nonsymmetric matrix. Hence, to compute the solution requires about two times as much work as for the solution of the perfect conductor boundary condition.
- Spurious solutions can corrupt the results at the interior frequencies of the perfect conductor cavity D_- even if the case under consideration is that of an absorbing obstacle (e.g., [27]).

Clearly, the elimination of the electric currents can similarly be used to obtain the MFIE from relation (11), which, again, is subject to the same deficiencies.

To eliminate spurious resonances, the usual procedure is to make a convex combination of the two previous equations called the combined field integral equation (CFIE), which can be symbolically written as

$$\mathrm{CFIE} = \alpha\mathrm{EFIE} + \frac{1}{ik}(1-\alpha)\mathrm{MFIE}$$

(e.g., [27]). However, the accuracy of the results depends greatly on an adequate choice for the parameter α. For lack of a theoretical determination of this parameter, this method is generally impractical for an arbitrary scatterer. At the same time, eliminating the magnetic currents through relationship (12), the CFIE involves both electric and magnetic charges exactly as the direct formulation (13) does hence loosing the main advantage of either the EFIE or the MFIE.

In fact, there exists another way to devise a robust CFIE that involves no parameter [21]. Since it uses the same background as the integral equation, which is considered in this paper, we will describe it later.

C. Rumsey's Reaction Concept Formulation

Now we come to the formulation devised in this paper. Let $\mathbf{J}'$ and $\mathbf{M}'$ be a system of testing electric and magnetic currents tangential to Γ. Let $\mathbf{E}_\pm$ and $\mathbf{H}_\pm$ designate the respective electromagnetic fields induced by the equivalent currents $\mathbf{J}$ and $\mathbf{M}$ in the domains $D_\pm$. We also denote by $\mathbf{E}_\pm$ and $\mathbf{H}_\pm$ the limiting boundary values of these fields on Γ from their respective values in $D_\pm$. Rumsey's reaction [28] of the system of currents $\{\mathbf{J},\mathbf{M}\}$ on the testing currents $\{\mathbf{J}',\mathbf{M}'\}$ in $D_\pm$, respectively, is given by

$$\begin{aligned}\int_\Gamma(\mathbf{E}_\pm\cdot\mathbf{J}' - \mathbf{H}_\pm\cdot\mathbf{M}')\,d\Gamma\\ = (A\{\mathbf{J},\mathbf{M}\},\{\mathbf{J}',\mathbf{M}'\})\\ \pm\frac{1}{2}\int_\Gamma(\mathbf{n}\times\mathbf{M}\cdot\mathbf{J}' + \mathbf{n}\times\mathbf{J}\cdot\mathbf{M}')\,d\Gamma\end{aligned} \tag{20}$$

with

$$\begin{aligned}(A\{\mathbf{J},\mathbf{M}\},\{\mathbf{J}',\mathbf{M}'\})\\ = ikZ\int_\Gamma T\mathbf{J}\cdot\mathbf{J}'\,d\Gamma - ikZ^{-1}\int_\Gamma T\mathbf{M}\cdot\mathbf{M}'\,d\Gamma\\ + \int_\Gamma K\mathbf{M}\cdot\mathbf{J}'\,d\Gamma + \int_\Gamma K\mathbf{J}\cdot\mathbf{M}'\,d\Gamma.\end{aligned} \tag{21}$$

Likewise, Rumsey's reaction of the impressed currents, possibly located at infinity when the incident wave is a plane wave, on the test currents $\{\mathbf{J}', \mathbf{M}'\}$ is given by

$$\int_\Gamma (\mathbf{E}^{\text{inc}} \cdot \mathbf{J}' - \mathbf{H}^{\text{inc}} \cdot \mathbf{M}')\, d\Gamma = -V\{\mathbf{J}', \mathbf{M}'\}. \tag{22}$$

On one hand, in view of (10) and (11), the total field can be written as $\mathbf{E} = \mathbf{E}_+ + \mathbf{E}^{\text{inc}}$, $\mathbf{H} = \mathbf{H}_+ + \mathbf{H}^{\text{inc}}$. Therefore, the reaction of the currents, impressed and induced, creating the total field in D_+ on the test currents $\{\mathbf{J}', \mathbf{M}'\}$, takes the following form:

$$\begin{aligned}\int_\Gamma &(\mathbf{E} \cdot \mathbf{J}' - \mathbf{H} \cdot \mathbf{M}')\, d\Gamma \\ &= (A\{\mathbf{J}, \mathbf{M}\}, \{\mathbf{J}', \mathbf{M}'\}) - \frac{1}{2} \int_\Gamma \mathbf{n} \times \mathbf{H} \\ &\quad \cdot (\mathbf{n} \times \mathbf{M}' - ikZ\eta \mathbf{J}')\, d\Gamma - V\{\mathbf{J}', \mathbf{M}'\}.\end{aligned}$$

On the other hand, using the IBC (2), we get another expression for this reaction

$$\begin{aligned}\int_\Gamma &(\mathbf{E} \cdot \mathbf{J}' - \mathbf{H} \cdot \mathbf{M}')\, d\Gamma \\ &= \int_\Gamma (\mathbf{E_t} \cdot \mathbf{J}' - \mathbf{n} \times \mathbf{H} \cdot \mathbf{n} \times \mathbf{M}')\, d\Gamma \\ &= -\int_\Gamma \mathbf{n} \times \mathbf{H} \cdot (\mathbf{n} \times \mathbf{M}' - ikZ\eta \mathbf{J}')\, d\Gamma.\end{aligned}$$

Hence, setting $\mathbf{L} = \frac{1}{2}\mathbf{n} \times \mathbf{H}$, we readily obtain from the two expressions of the reaction that $\{\mathbf{J}, \mathbf{M}, \mathbf{L}\}$ is solution to the following variational system:

$$\begin{cases} (A\{\mathbf{J}, \mathbf{M}\}, \{\mathbf{J}', \mathbf{M}'\}) + \int_\Gamma \mathbf{L} \cdot (\mathbf{n} \times \mathbf{M}' - ikZ\eta \mathbf{J}')\, d\Gamma \\ = V\{\mathbf{J}', \mathbf{M}'\};\, |\, \int_\Gamma \mathbf{L}' \cdot (\mathbf{n} \times \mathbf{M} - ikZ\eta \mathbf{J})\, d\Gamma = 0 \end{cases} \tag{23}$$

for every system of testing currents $\{\mathbf{J}', \mathbf{M}', \mathbf{L}'\}$ tangential on Γ.

Some comments are in order.

- The variational system appears as a constrained variational equation where the unknown $\mathbf{L}$ plays the role of a Lagrangian multiplier.
- It is quite obvious from the derivation procedure of the previous formulation that any solution to (13) gives a solution $\{\mathbf{J}, \mathbf{M}, \mathbf{L}\}$ to the variational system (23). Conversely, associating with all testing currents $\mathbf{J}'$ magnetic test currents $\mathbf{M}'$ by the relation

$$\mathbf{n} \times \mathbf{M}' = ikZ\eta \mathbf{J}' \tag{24}$$

  we readily obtain that $\mathbf{J}$ is a solution to (13). Thus, system (23) is an equivalent formulation of (13).

The advantage of system (23) comes from two important properties of the variational formulation. All the involved integrals are only weakly singular and, hence, are convergent in the usual sense. The electric and the magnetic currents are not explicitly linked by relation (12). Thus, the strong interelement continuity requirement may be omitted at the level of the discretization process by a boundary-element method.

Apparently, a serious drawback of this approach is the tripling of the number of unknowns. However, both the magnetic currents $\mathbf{M}$ and the Lagrangian multiplier $\mathbf{L}$ will be eliminated at the element level during the assembly process leaving as unknown only the electric current, exactly as with the usual perfect conductor boundary condition.

In the same manner, the previously mentioned alternative form for the CFIE can be written using Rumsey's reaction concept as

$$(A\{\mathbf{J}, \mathbf{M}\}, \{\mathbf{J}', \mathbf{M}'\}) + \frac{1}{2} \int_\Gamma \left(ikZ\eta \mathbf{J} \cdot \mathbf{J}' - \frac{1}{ikZ\eta} \mathbf{M} \cdot \mathbf{M}' \right) d\Gamma = V\{\mathbf{J}', \mathbf{M}'\}. \tag{25}$$

We note that there is no explicit relationship linking either the unknown currents $\{\mathbf{J}, \mathbf{M}\}$ or the testing currents $\{\mathbf{J}', \mathbf{M}'\}$. Although this is not evident at first sight, the above system is nothing else but a coupling of the EFIE and MFIE obtained by leaving implicit the relationship (12) linking the unknown currents $\{\mathbf{J}, \mathbf{M}\}$. The drawback of this formulation is that the order of the final linear system is twice as large as in the perfect conductor case.

D. Well Posedness of Rumsey's Reaction Formulation

Since we are insured that system (23) has at least one solution, it only remains to prove that this solution cannot be corrupted by any spurious one. Thus, consider a solution $\{\mathbf{J}, \mathbf{M}, \mathbf{L}\}$ to the variational system (23) corresponding to a zero incident field. Clearly, the equivalent currents $\{\mathbf{J}, \mathbf{M}\}$ are linked by relation (12). As above, denoted by $\mathbf{E}_\pm$, $\mathbf{H}_\pm$, the electromagnetic field induced, respectively, in the domains $D_\pm$ by the currents $\{\mathbf{J}, \mathbf{M}\}$. Making use once more of the classical jump relations (20), we express the limiting boundary values of $\mathbf{E}_\pm$, $\mathbf{H}_\pm$ on Γ in terms of the currents $\{\mathbf{J}, \mathbf{M}\}$. Thus, from the very definition of a solution to (23) corresponding to $\{\mathbf{E}^{\text{inc}}, \mathbf{H}^{\text{inc}}\} = 0$, we get

$$\begin{aligned}\int_\Gamma &(\mathbf{E}_\pm \cdot \mathbf{J}' - \mathbf{H}_\pm \cdot \mathbf{M}')\, d\Gamma \\ &= -\int_\Gamma \left(\mathbf{L} \pm \frac{1}{2}\mathbf{J} \right) \cdot (\mathbf{n} \times \mathbf{M}' - ikZ\eta \mathbf{J}')\, d\Gamma.\end{aligned} \tag{26}$$

Associating with any $\mathbf{J}'$, a tangential field $\mathbf{M}'$ by relation (24), we readily see that $\{\mathbf{E}_+, \mathbf{H}_+\}$ is a solution to the impedance boundary value problem (1), (2) with $\{\mathbf{E}_+, \mathbf{H}_+\}$ further satisfying the Silver–Müller radiation condition. Since this problem as stated above is well-posed for an absorbing scatterer [owing to assumption (3)], we obtain $\mathbf{E}_+ = \mathbf{H}_+ = \mathbf{0}$ in D_+. Likewise, $\{\mathbf{E}_-, \mathbf{H}_-\}$ is a solution to the Maxwell equations in D_- and their limiting tangential components are related by the impedance boundary on Γ. A straightforward application of Green's formula gives

$$\Im m \int_\Gamma \eta |\mathbf{n} \times \mathbf{H}_-|^2\, d\Gamma = 0.$$

From condition (3), we conclude $\mathbf{n} \times \mathbf{H}_- = \mathbf{0}$ and then $(\mathbf{E}_-)_\mathbf{t} = 0$. As a result, the jump relations yield $\mathbf{J} = \mathbf{M} = \mathbf{0}$. Equation (26) then permits us to conclude that $\mathbf{L} = \mathbf{0}$.

Remark It worth noting that when η is a varying (or even is an operator valued) function, the previous well-posedness continues to hold if η vanishes in a strict part of the boundary Γ, as long as it remains strictly absorbing in its complement.

III. Boundary Finite-Element Method

A. Approximation of the Currents

As stated above, an essential requirement for a finite-element approximation of the currents is to ensure the conservation of charges. Hence, since the method is intended to deal with surfaces of arbitrary shape, the only suitable procedure is to use a flux finite-element method. Let us sketch the main features of this method (see [25] for more details).

The surface scatterer is meshed by a collection of planar triangles satisfying the general overlapping conditions for a finite-element method (e.g., [5]).

The degrees of freedom of the surface currents $\mathbf{J}$ are their fluxes across each edge of the mesh, assuming that a positive counting for the fluxes has been fixed there. This is done by labeling the edges and introducing a signed table of connectivity linking the degrees of freedom to the currents flowing out of each of the three edges of any triangle as follows:

- Let K designate any triangle of the mesh; $\mathbf{a}_j^K$, $j=1,2,3$ are its vertexes (assumed to have been counted in the counterclockwise direction in the plane of K determined by the unit normal $\mathbf{n}$, which is pointing to the exterior of the scatterer); K_l' is the edge connecting $\mathbf{a}_l^K$ to $\mathbf{a}_{l+1}^K$ (making the usual convention of a circular permutation of the indexes when the index is greater than 3).
- The table of connectivity of the degrees of freedom is given by a signed integer $n_l^K, l=1,2,3$ for each triangle K so that $|n_l^K|$ gives the number of the edge K_l'. An orientation is fixed over each edge of the mesh so that n_l^K is positive when this orientation is compatible with that one relative to the above numbering of the vertices of K and negative otherwise.
- Let N_e be the number of edges of the mesh. The currents $\mathbf{J}$ are recovered from the column-wise vector $\{J\} := \{J_j\}_{j=1}^{j=N_e}$ of the degrees of freedom by
$$\mathbf{J}(x)=\sum_{l=1}^{3}\varepsilon_l^K J_{|n_l^K|}\mathbf{B}_l^K(x), \qquad \text{for } x \text{ in } K \tag{27}$$
with $\varepsilon_l^K = n_l^K/|n_l^K|$ and $\mathbf{B}_l^K$ is the basis function (see [25]) given by
$$\mathbf{B}_l^K(x)=\frac{1}{2|K|}\left(x-\mathbf{a}_{l+2}^K\right)$$
where $|K|$ designates the area of triangle K.
- In this way, charges do not accumulate on either the edges or the vertexes and their expression may be obtained inside each triangle K by
$$\nabla_\Gamma\cdot\mathbf{J}(x)=\frac{1}{|K|}\sum_{l=1}^{3}\varepsilon_l^K J_{|n_l^K|}, \qquad \text{for each } x \text{ in } K.$$

B. Approximation of the Lagrange Multipliers

There is once more one degree of freedom per edge for the approximation of the Lagrange multiplier $\mathbf{L}$. But now, the latter is chosen to be continuous at each mid-point of any edge of the mesh and tangential there to that edge. More precisely, each Lagrange multiplier $\mathbf{L}$ is characterized by a column-wise vector $\{L\} := \{L_j\}_{j=1}^{j=N_e}$ giving its degrees of freedom as follows:
$$\mathbf{L}(x)=\sum_{l=1}^{3}\varepsilon_l^K L_{|n_l^K|}\left(1-2\lambda_{l+2}^K(x)\right)\left(\mathbf{a}_{l+1}^K-\mathbf{a}_l^K\right), \quad \text{for each } x \text{ in } K \tag{28}$$
where $\lambda_l^K(x)$ designates the lth barycentric coordinate of x relative to triangle K (also called area coordinates, see e.g., [18]).

The above form for the multiplier has the following important consequence. Since $\mathbf{L}'(x)\cdot\mathbf{n}\times\mathbf{M}(x)$ is a quadratic polynomial in the interior of K, it can be integrated exactly over K using the three edge mid-points formula
$$\int_K \mathbf{L}'(x)\cdot\mathbf{n}\times\mathbf{M}(x)\,dK = \frac{|K|}{3}\sum_{l=1}^{3}\mathbf{L}'\left(\mathbf{a}_{l+1/2}^K\right)\cdot\mathbf{n}\times\mathbf{M}\left(\mathbf{a}_{l+1/2}^K\right)$$
where $\mathbf{a}_{l+1/2}^K := (\mathbf{a}_l^K+\mathbf{a}_{l+1}^K)/2$ is the mid-point of edge K_l' (cf. [11]). But, from the form of the currents (27) and of the multipliers (28) over each triangle, in the above expression only the degrees of freedom which are relative to the same edge are coupled. More precisely, we have
$$\begin{aligned}\mathbf{L}'\left(\mathbf{a}_{l+1/2}^K\right)&\cdot\mathbf{n}\times\mathbf{M}\left(\mathbf{a}_{l+1/2}^K\right)\\ &=\varepsilon_l^K L'_{|n_l^K|}\left(\mathbf{a}_{l+1}^K-\mathbf{a}_l^K\right)\times\mathbf{n}\cdot\mathbf{M}\left(\mathbf{a}_{l+1/2}^K\right)\\ &=\varepsilon_l^K|K_l'|L'_{|n_l^K|}\mathbf{M}\left(\mathbf{a}_{l+1/2}^K\right)\cdot\boldsymbol{\nu}_l^K\end{aligned}$$
where $|K_l'|$ is the length of the edge K_l' and $\boldsymbol{\nu}_l^K$ is the unit normal to this edge in the plane of K pointing to the exterior of K. From the very definition (27) of the degrees of freedom characterizing the currents, we get
$$\int_K \mathbf{L}'(x)\cdot\mathbf{n}\times\mathbf{M}(x)\,dK=\frac{|K|}{3}\sum_{l=1}^{3}L'_{|n_l^K|}M_{|n_l^K|} \tag{29}$$
where vector $\{M\} := \{M_j\}_{j=1}^{j=N_e}$ gives the degrees of freedom characterizing the magnetic currents as above for the electric currents.

C. Elimination of the Magnetic Currents and the Lagrange Multipliers

Formula (29) makes it possible to readily express the degrees of freedom of the magnetic currents $\mathbf{M}$ from that of the electric currents $\mathbf{J}$. To be specific, pick any edge K' of the mesh. Let K^+ and K^- be the two triangles sharing this edge and assume that $n=n_l^{K^+}=-n_m^{K^-}$ gives the number of K'. Let $\mathbf{L}'$ be the Lagrange multiplier having all of its degrees of freedom equal to zero except that relative to K' which is taken equal to one. Since $\mathbf{L}'$ vanishes on any triangle except K^+ and K^-, we can write

$$\int_\Gamma \mathbf{L}'\cdot\mathbf{n}\times\mathbf{M}\,d\Gamma = \int_{K^+}\mathbf{L}'\cdot\mathbf{n}\times\mathbf{M}\,dK^+ + \int_{K^-}\mathbf{L}'\cdot\mathbf{n}\times\mathbf{M}\,dK^-.$$

Formula (29) then yields

$$\frac{|K^+|+|K^-|}{3}M_n = ikZ\left(\int_{K^+}\eta\mathbf{L}'\cdot\mathbf{J}\,dK^+ + \int_{K^-}\eta\mathbf{L}'\cdot\mathbf{J}\,dK^-\right)$$

thus permitting us to express M_n with only the degrees of freedom relative to triangles K^+ and K^-

$$M_n = \frac{3ikZ}{|K^+|+|K^-|}\sum_{l=1}^{3}\left\{\varepsilon_l^{K^+}J_{|n_l^{K^+}|}\int_{K^+}\eta\mathbf{L}'\cdot\mathbf{B}_l^{K^+}\,dK^+ + \varepsilon_l^{K^-}J_{|n_l^{K^-}|}\int_{K^-}\eta\mathbf{L}'\cdot\mathbf{B}_l^{K^-}\,dK^-\right\}. \tag{30}$$

The above expression enables us to explicitly write out the degrees of freedom of the magnetic currents from that of the electric currents. Now, for testing currents $\mathbf{J}'$ and $\mathbf{M}'$ having their degrees of freedom related by (30), the integral involving the Lagrangian multiplier cancels, namely

$$\int_\Gamma \mathbf{L}\cdot(\mathbf{n}\times\mathbf{M}' - ikZ\eta\mathbf{J}')\,d\Gamma = 0$$

and, henceforth, the magnetic currents as well as the Lagrangian multipliers are completely removed from the formulation.

Remark: Note that the above procedure remains valid for a varying (or even an operator valued) function η.

In view of (30), which explicitly expresses $\mathbf{M} = D\mathbf{J}$ and $\mathbf{M}' = D\mathbf{J}'$ from, respectively, $\mathbf{J}$ and $\mathbf{J}'$ at the element level, the variational system (23) becomes

$$(A\{\mathbf{J}, D\mathbf{J}\}, \{\mathbf{J}', D\mathbf{J}'\}) = V\{\mathbf{J}', D\mathbf{J}'\}, \qquad \text{for each } \mathbf{J}'. \tag{31}$$

The symmetry properties of the kernel $G(x,y)$ lead to the following relation:

$$(A\{\mathbf{J}, D\mathbf{J}\}, \{\mathbf{J}', D\mathbf{J}'\}) = (A\{\mathbf{J}', D\mathbf{J}'\}, \{\mathbf{J}, D\mathbf{J}\})$$

which indicates that in fact system (31) can be reduced to a linear system

$$\{Z\}\{J\} = \{V\} \tag{32}$$

with a symmetric matrix $\{Z\}$.

Clearly, if η vanishes, the expressions $D\mathbf{J}$ and $D\mathbf{J}'$ given by (30) for, respectively, $\mathbf{M}$ and $\mathbf{M}'$ vanish as well and the linear system is nothing else but the one corresponding to a perfect conductor boundary condition on the scatterer.

D. Assembly Process

We only sketch the derivation of matrices $\{Z\}$ and $\{V\}$. The interested reader can consult [4] for the details.

Since matrix $\{Z\}$ is symmetric, only its upper triangular part is stored. It is obtained by assembling the contribution of the elementary matrix $Z^{K,L}$ that describes the interaction of the currents on the pair of triangles K and L. It is defined by the relation

$$\begin{aligned}\{J_K'\}^\top Z^{K,L}\{J_L\} &= ikZ\int_K\int_L G(x,y)\left(\mathbf{J}(y)\cdot\mathbf{J}'(x) - \frac{1}{k^2}\nabla_\Gamma\mathbf{J}(y)\nabla_\Gamma\cdot\mathbf{J}'(x)\right)dL(y)\,dK(x)\\ &\quad - ikZ^{-1}\int_K\int_L G(x,y)\left(\mathbf{M}(y)\cdot\mathbf{M}'(x) - \frac{1}{k^2}\nabla_\Gamma\cdot\mathbf{M}(y)\nabla_\Gamma\cdot\mathbf{M}'(x)\right)dL(y)\,dK(x)\\ &\quad + \int_K\int_L \nabla_y G(x,y)\times\mathbf{J}(y)\cdot\mathbf{M}'(x)\,dL(y)\,dK(x)\\ &\quad + \int_K\int_L \nabla_y G(x,y)\times\mathbf{J}'(y)\cdot\mathbf{M}(x)\,dL(y)\,dK(x).\end{aligned}$$

In the previous formula, $\mathbf{M} = D\mathbf{J}$ and $\mathbf{M}' = D\mathbf{J}'$ according to (30). Thus, the involved degrees of freedom $\{J_L\}$ and $\{J_K'\}$ are those relative to the triangles, respectively, sharing with L and K a common edge. Matrix $\{J_K'\}^\top$ is the transpose of matrix $\{J_K'\}$. Matrix $Z^{K,L}$ is a square matrix of order nine. In fact, the elimination of the magnetic currents and the Lagrangian multipliers results in a kind of nonlocalized finite-element method where every triangle L (respectively, K), in addition to its own degrees of freedom, also involves those of its adjacent triangles.

Essentially, matrix $Z^{K,L}$ is determined from the two following matrices defined by

$$\begin{aligned}[J_1' \quad J_2' \quad J_3']S^{K,L}\begin{bmatrix}J_1\\J_2\\J_3\end{bmatrix} &= \int_K\int_L G(x,y)\left(\mathbf{J}(y)\cdot\mathbf{J}'(x) - \frac{1}{k^2}\nabla_\Gamma\cdot\mathbf{J}(y)\nabla_\Gamma\cdot\mathbf{J}'(x)\right)dL(y)\,dK(x)\\ [M_1' \quad M_2' \quad M_3']D^{K,L}\begin{bmatrix}J_1\\J_2\\J_3\end{bmatrix} &= \int_K\int_L \nabla_y G(x,y)\times\mathbf{J}(y)\cdot\mathbf{M}'(x)\,dL(y)\,dK(x)\end{aligned}$$

with $\mathbf{J}(x) = \sum_{l=1}^3 J_l\mathbf{B}_l^L(x)$, $\mathbf{J}'(x) = \sum_{l=1}^3 J_l'\mathbf{B}_l^K(x)$, and $\mathbf{M}'(x) = \sum_{l=1}^3 M_l'\mathbf{B}_l^K(x)$.

Clearly, since $S^{K,L} = (S^{L,K})^\top$ and $D^{K,L} = (D^{L,K})^\top$, only one half of these matrices needs to be computed. The outer integral is determined by the three-points Gauss quadrature formula. Inner integrals are determined in the same way if the triangles K and L are not adjacent. Otherwise, the

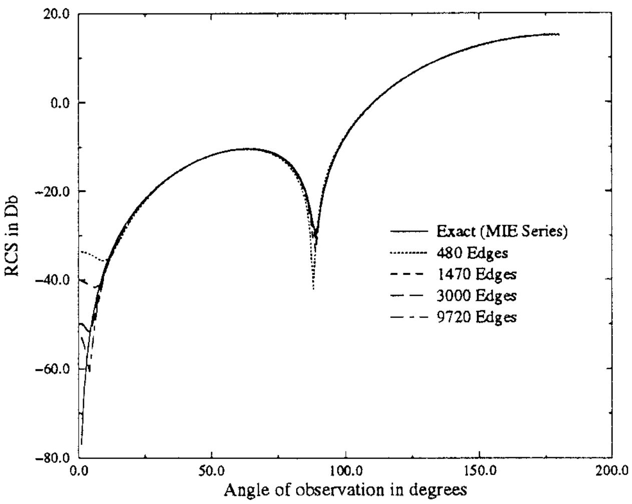

Fig. 1. Bistatic RCS for $k = 2$ and $\tilde{\eta} = 1$.

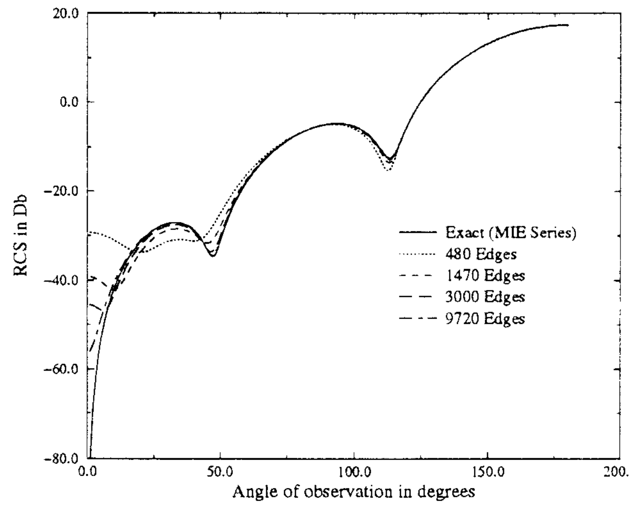

Fig. 2. Bistatic RCS for $k = 2.76$ and $\tilde{\eta} = 1$.

singular part corresponding to the following decomposition of the kernel:

$$G(x,y) = \frac{1}{4\pi|x-y|} + \frac{e^{ik|x-y|}-1}{4\pi|x-y|}$$

is performed exactly using the analytical expressions for single- and double-layer electrostatic potentials created by linear distributions over planar triangles in the space (e.g., [26]), the regular one being computed by the previous quadrature formula.

Matrix $\{V\}$ is assembled from the elementary matrices defined by

$$[J_1' \quad J_2' \quad J_3'] \begin{bmatrix} VE_1^K \\ VE_2^K \\ VE_3^K \end{bmatrix} = \int_K \mathbf{E}^{\mathrm{inc}}(y) \cdot \mathbf{J}'(x)\, dK(x)$$

$$[M_1' \quad M_2' \quad M_3'] \begin{bmatrix} VH_1^K \\ VH_2^K \\ VH_3^K \end{bmatrix} = \int_K \mathbf{H}^{\mathrm{inc}}(y) \cdot \mathbf{M}'(x)\, dK(x).$$

The above integrals are computed using the previous quadrature formula.

IV. Illustrative Results

To illustrate the present approach, we compute the RCS of a unit sphere with material properties described by a scalar constant impedance η. We follow the notation in [27] (from which we have borrowed the cases considered) and set $\tilde{\eta} := ik\eta$.

Every mesh of the sphere is quasiuniform; that is, all the triangles of the mesh are approximately of the same size and form, and is characterized by the number N of its edges which is the size of the final linear system to be solved.

The first example is that of an absorbing sphere with $\tilde{\eta} = 1$. It is a challenge for the accuracy of a numerical method since there is then a complete extinction of the backscattering RCS. Figs. 1 and 2 report the bistatic RCS determined numerically and exactly by a Mie series expansion for, respectively, a wave number $k = 2$ and $k = 2.76$. The directions of observation are in the plane of incidence. Angle 0 corresponds to the backscattering RCS. The latter wave number is chosen to bring out the effect of an eigenfrequency for the interior perfect conductor cavity on the stability of the method. As expected, the results confirm the theoretical predictions. The values obtained for the bistatic RCS are in good agreement with the exact ones for any direction except those very close to the backscattering direction, even for coarse meshes (see Fig. 3). No special feature distinguishes the results corresponding to the regular frequency for $k = 2$ from those relative to $k = 2.76$. The behavior of the curves obtained numerically near the extinction direction, in a way, show the stability of the method and the accuracy which can be reached. Table I gives the upper bounds of the size of an edge in wavelengths for the used meshes.

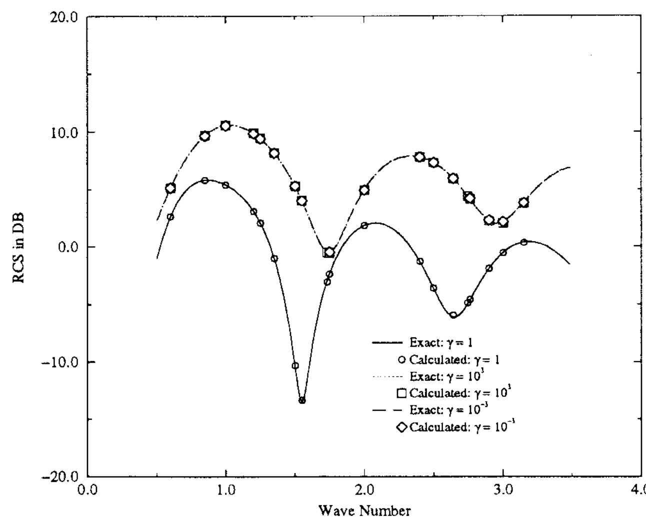

Fig. 3. Backward RCS for various surface impedances.

Fig. 3 reports the results obtained for the monostatic RCS for $\tilde{\eta} = \gamma(1+i)$ with the indicated values of γ. It has been obtained by a mesh with 4320 edges. The upper bound of the size of an edge in wavelengths is varying between 0.052 and 0.009. These results show good stability relative to the magnitude of the surface impedance. Particularly, one

TABLE I
UPPER BOUNDS FOR THE SIZE OF AN EDGE IN WAVELENGTHS

Number of Edges	$k = 2$	$k = 2.76$
480	0.103	0.075
1470	0.060	0.043
3000	0.042	0.030
9720	0.023	0.017

TABLE II
CPU TIMES ON WORSTATION IBM RISC 6000

Number of Edges	480		1080		1920		3000	
Problem	PC	IBC	PC	IBC	PC	IBC	PC	IBC
Assembly Process	13	57	66	292	207	916	505	2220
Solution	5	5	51	51	285	285	1105	1105
Total	18	62	117	343	492	1201	1520	3324
Ratio (IBC/PC)		3.44		2.93		2.44		2.18

can observe that even for a surface impedance of very small magnitude, for which the RCS cannot be distinguished from that of a perfectly conducting scatterer, the method continues to give accurate results even at the interior cavity frequency.

Finally, Table II compares the CPU time in seconds used for solving the boundary integral equation for an IBC and a perfectly conducting (PC) sphere on an IBM RISC 6000 workstation.

This table shows that for moderate meshes, solving the impedance scattering problem takes twice as long as for solving the perfect conductor case. The table also clearly indicates that the ratio of the former to the latter time is decreasing as the mesh size increases. Since the CPU time used for the assembly process and for the linear system solution is respectively a quadratic function and a cubic function of the number of edges N, this ratio must be nearly one for meshes of large sizes. However in this case, due to the related huge CPU time and memory which are required, only an implementation of the method on parallel machines has proved to be practicable. The results have been obtained on the CRAY supercomputers T3D and T3E. The elapsed CPU time and the repetition of some calculations to efficiently distribute the amount of the computation on all the processors makes comparison of the related CPU times meaningless. Thus, we do not give the above comparison for this case.

ACKNOWLEDGMENT

The authors would like to thank the first reviewer for his thorough reading. His suggestions have contributed significantly to improve the final appeareance of this paper.

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