A Numerical Approach for the Diffraction of a Gaussian Beam from a Perfectly Conducting Wedge

Giuseppe Pelosi, *Senior Member, IEEE*, and Stefano Selleri, *Member, IEEE*

***Abstract*— A finite difference (FD) solution to the problem of high-frequency scattering from a perfectly conducting wedge of arbitrary external angle illuminated by a Gaussian beam is presented. The solution is obtained through the application of the parabolic equation method. The solution is compared with the analytical asymptotic solution available in the literature.**

***Index Terms*— Electromagnetic scattering, Gaussian beams, wedges.**

I. Introduction

FINITE-DIFFERENCE (FD) techniques applied to the parabolic equation method have been used for the determination of the field scattered from isotropic [1] and anisotropic [2] impedance wedges in the high-frequency approximation. Recently, this method has also been extended to cover the problem of inhomogeneous plane wave scattering from an impedance wedge [3]. In this paper, the same method is applied to analyze the high-frequency scattering from a perfectly conducting wedge illuminated by a Gaussian beam. Gaussian beams are useful wave objects for charting the transition from an extended aperture excitation to the far field and they also are advantageous in studying the passage through complex propagation environments. These attributes have contributed to make Gaussian beams really attractive as base elements in field representations. Although more than one definition of Gaussian beam exists, the one reported in [4] is the more appealing for its properties and is thus used in this study. Since in [4] a nonuniform solution is shown, a validation of the present solution is sought by comparison with the uniform solutions presented in [5] and [6].

The formulation of the problem is presented in Section II. Section III discusses the parabolic approach. Finally, some numerical results for the scattering of Gaussian beams are presented and compared with the results obtainable analytically for the simpler problem of the half-plane in the context of the uniform geometrical theory of diffraction (UTD) [5]–[7].

II. Formulation

The geometry for the two-dimensional (2-D) scattering at the edge of a perfectly conducting wedge is depicted in Fig. 1. The exterior angle of the wedge is $n\pi$. The wedge is illuminated by a 2-D Gaussian beam with the electric field parallel (TM_z $(e), H_z = 0$) to the edge of the wedge. An harmonic time dependence is implicitly assumed. The TE_z $(h)(E_z = 0)$ case can, of course, be treated with a similar approach.

Manuscript received September 26, 1997; revised July 20, 1999.

The authors are with the Department of Electronic Engineering, University of Florence, Florence, I50134 Italy.

Publisher Item Identifier S 0018-926X(99)09847-6.

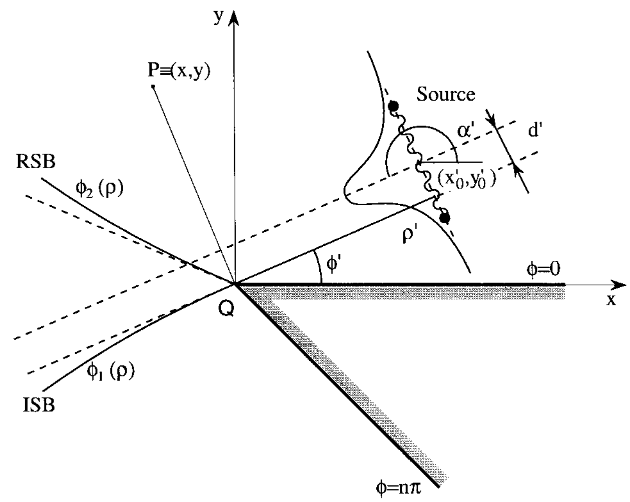


Fig. 1. Geometry of the wedge problem.

According to the definition provided in [4], the field E_z^i of a Gaussian beam in free-space can be seen as the field radiated by a line source parallel to the z direction and located in the complex space in point $(x' = -jb', y' = 0)$. This corresponds to a Gaussian beam being generated at the origin of an xy coordinate system and focused in the positive direction of the x axis (Fig. 2). A more general formula can be obtained by an appropriate coordinate transformation aimed at focusing the beam in a generic direction α' and at having the beam generated at an arbitrary point (x_0', y_0') of the plane (Fig. 1). This corresponds to placing the source in point $S' \equiv (x', y') \equiv (-jb' \cos\alpha' + x_0', -jb' \sin\alpha' + y_0')$. The relative field incident on the wedge, is

$$E_z^i(x, y; x', y') = -\frac{j}{4} H_0^{(2)}(kR) \tag{1}$$

where $H_0^{(2)}(kR)$ is the zero-order Hankel function of the second kind, k is the free-space propagation constant, and R is the distance in the complex plane between the source in (x', y') and the observation point $P \equiv (x, y)$. The complex displacement R has two branch point singularities described in [4] that can be regarded, for this generalized case of source placement and orientation, as corresponding to the points $(\mp b' \sin\alpha' + x_0', \pm b' \cos\alpha' + y_0')$. In order for R and, hence, the incident field to be single valued, it is necessary to connect the branch points with a cut and one can imagine the beam

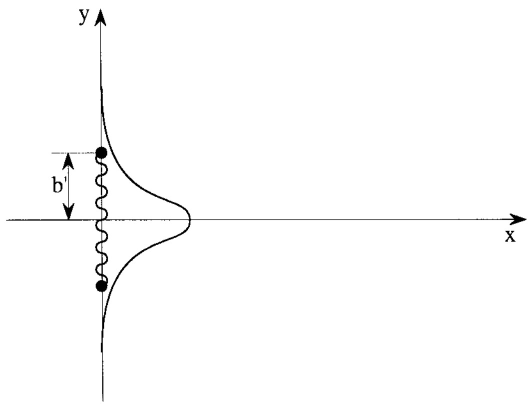


Fig. 2. Isotropic line source in the complex plane, generating a directive beam source in the real plane.

as generated by an appropriate set of real equivalent sources along such segment. In Fig. 1 and subsequently, the branch points are indicated by bullets and the branch cut connecting them is shown by the wavy line. In Fig. 1, the angle α' gives the direction of maximum radiation of the beam. To describe the incident beam, besides parameter b', three quantities have been chosen in place of x_0' and y_0' for an easier handling of the beam in polar coordinates. These quantities are d' ρ' and ϕ'. Angle ϕ' exhibits an immediate link with the beam focusing direction $\alpha' = \pi + \phi'$. ρ' is the distance between the edge of the wedge and the line perpendicular to the beam focusing direction; that is, the line passing for the branch points. Finally, d' is the distance along this latter line between the origin of the Gaussian beam itself and segment ρ'. This parameter represent an offset in the beam focusing. If $d' = 0$, then the beam is focused exactly on the edge of the wedge (see Fig. 3).

The total field is, of course, the solution of Helmholtz equation with Dirichlet boundary conditions on the wedge faces ($\phi = 0, \phi = n\pi$); in our approach, the total field is written as the sum of the geometrical optics (GO) field E^{go} (incident plus reflected) and the diffracted field E^d. The former can be seen as a superposition of an incident and of a reflected beam [4, eq. (15)]

$$E_z^{go} \simeq u(\phi - \phi_1(\rho))\left(-\frac{j}{4}H_0^{(2)}(kR_1)\right) - u(\phi - \phi_2(\rho))\left(-\frac{j}{4}H_0^{(2)}(kR_2)\right) \tag{2}$$

where $u(t)$ is the unit step function. R_1 and R_2 are the complex distances from the source and from the relative image source, respectively, the latter being characterized by $\alpha_r = 2\pi - \alpha'$, $\rho_r = \rho'$, $\phi_r = -\phi'$, $d_r = -d'$, and by the same parameter b' (Fig. 3). The first term in (2) represents the incident field and the second the reflected field. Finally, $\phi_1(\rho)$ and $\phi_2(\rho)$ are the equations of the incident and reflected shadow boundaries (ISB and RSB), respectively. In particular, $\phi_{1,2}(\rho)$ are the solution of $\mathrm{Re}(W_{1,2}) = 0$ for which $\mathrm{Im}(W_{1,2}) < 0$, being

$$W_{1,2} = 2e^{j(\pi/4)}\sqrt{\frac{k\rho\rho'}{\rho + \rho' + R_{1,2}}}\begin{Bmatrix} -\sin\frac{1}{2}(\phi - \phi') \\ \cos\frac{1}{2}(\phi + \phi') \end{Bmatrix}. \tag{3}$$

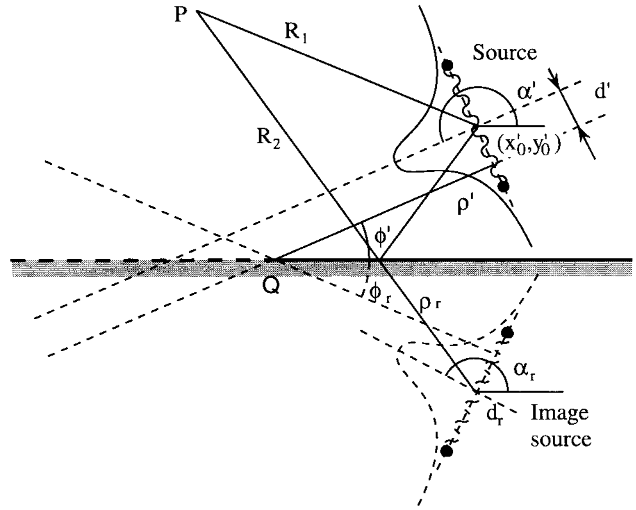


Fig. 3. Incident and reflected Gaussian beams for a perfectly conducting plane.

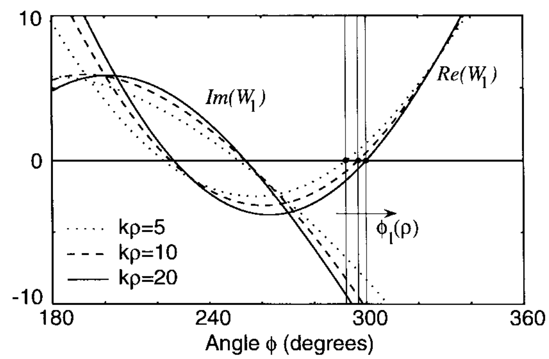


Fig. 4. Behavior of $\mathrm{Re}(W_1)$ and $\mathrm{Im}(W_1)$ as a function of for different values of distance ρ. The zeroes of $\mathrm{Re}(W_1)$ for which $\mathrm{Im}(W_1) < 0$ gives the location $\phi_1(\rho)$ of the ISB.

For a Gaussian beam, the ISB and RSB are generally curved and, thus, a function of the distance from the edge ρ. Only for the case $d' = 0$ are these boundaries straight lines as in standard ray theory.

To show the typical behavior of such boundaries, the function W_1 is represented in Fig. 4 as a function of the observation angle ϕ for different values of the observation distance ρ. The beam parameters are $\rho' = 2, \phi' = 30°, b = 3, d' = 3$. As may be noted, the angle ϕ_1, which would have been $266°$ regardless of ρ for $b = 0$, is a function of the distance. The displacement from $266°$ and the relatively large variation of $\phi_1(\rho)$ are due to the small value of ρ' and to the large value of $b = 3$.

III. Parabolic Approach

By introducing one scalar auxiliary function $U(\rho, \phi)$ so that $E_z^d(\rho, \phi) = U(\rho, \phi)\exp(-jk\rho)$ and by assuming $k\rho \gg 1$, one can neglect the term involving the second derivative with respect to ρ in the Helmholtz equation in cylindrical coordinates. This process transforms the elliptic Helmholtz equation into the parabolic equation

$$\left\{-2jk\frac{\partial}{\partial\rho} - \frac{jk}{\rho} + \frac{1}{\rho^2}\,\frac{\partial^2}{\partial\phi^2}\right\}U(\rho, \phi) = 0. \tag{4}$$

If the value of the auxiliary function $U(\rho,\phi)$ or an approximation of it is known at $\rho = 0$, (4) can be solved on an open domain by a marching in space FD procedure [1]–[3]. Suitable conditions at the ISB and RSB and on both faces of the perfectly conducting wedge have to be established for the auxiliary function $U(\rho,\phi)$. This because the total field must be continuous, while the GO solution is not. Consequently, the diffracted field, of which $U(\rho,\phi)$ is the slowly-varying component, must be discontinuous too and its discontinuities must cancel out those of the GO field. We will refer to these conditions in the following as the compatibility conditions.

The solution of (4) is generally different from that of the Helmholtz equation, but it is a good approximation of the latter for $k\rho \gg 1$. It is important to note the difference between our procedure, where only the phase is extracted, from the procedure in [8] for the same problem but with a source in the real space, thus, a cylindrical wave illumination; there, not only the phase term $\exp(-jk\rho)$ is extracted, but also the amplitude factor $1/\sqrt{k\rho}$. Reference [8] is anyway extremely interesting since it contains the demonstration that the asymptotic solution of the parabolic equation coincides with the asymptotic solution of Helmholtz equation.

The singularity of (4) for $\rho \to 0$ does not upset the convergence or the stability of the FD solution as has been shown in early papers on the parabolic equation [9], [10].

By assuming the nonrestrictive hypothesis that only the face $\phi = 0$ of the wedge is illuminated, i.e., $0 \le \phi' \le (n-1)\pi$, the following compatibility conditions are obtained:

$$U(\rho,\phi_{1,2}^{+}(\rho)) - U(\rho,\phi_{1,2}^{-}(\rho)) = \gamma_{1,2}(\rho) \tag{5}$$

$$\frac{1}{\rho}\left.\frac{\partial U(\rho,\phi(\rho))}{\partial\phi}\right|_{\phi=\phi_{1,2}^{+}(\rho)} - \frac{1}{\rho}\left.\frac{\partial U(\rho,\phi(\rho))}{\partial\phi}\right|_{\phi=\phi_{1,2}^{-}(\rho)} = \hat{\gamma}_{1,2}(\rho) \tag{6}$$

where both the discontinuity of the GO field and of its derivative have been taken into account. Furthermore, at every given ρ of the FD space iteration process, the location of the discontinuities $\phi_{1,2}(\rho)$ must be computed by solving (3) numerically. The actual values of the discontinuities of (5) and (6) on the basis of the GO field [4]–[6] are

$$\gamma_1(\rho) = H_0^{(2)}(kR_1)|_{\rho,\phi_1(\rho)} \tag{7}$$

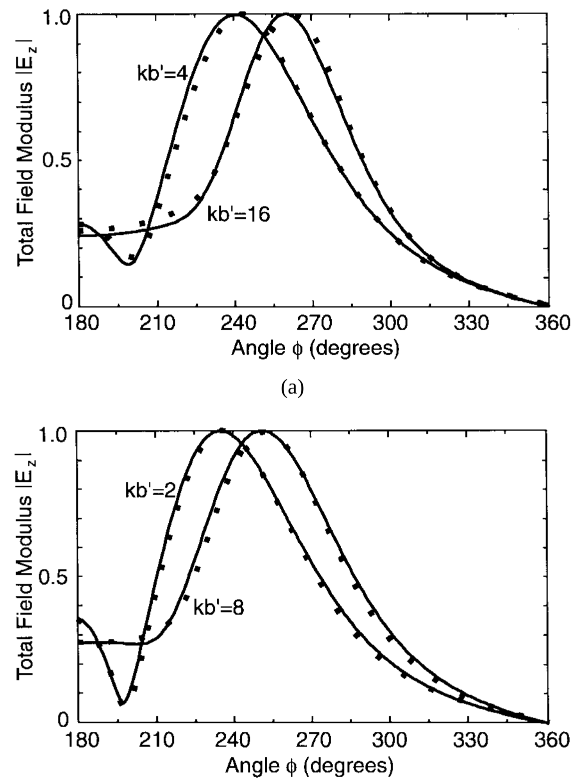
$$\gamma_2(\rho) = H_0^{(2)}(kR_1)|_{\rho,\phi_2(\rho)} - H_0^{(2)}(kR_2)|_{\rho,\phi_2(\rho)} \tag{8}$$

$$\hat{\gamma}_1(\rho) = \left.\frac{\partial H_0^{(2)}(kR_1)}{\partial\phi}\right|_{\rho,\phi_1(\rho)} \tag{9}$$

$$\hat{\gamma}_2(\rho) = \left.\frac{\partial H_0^{(2)}(kRk_1)}{\partial\phi}\right|_{\rho,\phi_2(\rho)} - \left.\frac{\partial H_0^{(2)}(kR_2)}{\partial\phi}\right|_{\rho,\phi_2(\rho)} \tag{10}$$

being $\hat{\gamma}_{1,2}(\rho) = 0$ only if $d' = 0$; that is, if the beam is focused on the edge of the wedge.

As previously stated, to start the FD solution process of (4), the value of $U(\rho,\phi)$ for $\rho = 0$ is needed. This is done by expanding the solution in a Taylor series $U(\rho,\phi) = \Sigma_{n=0}^{\infty}\, u_n(\phi)\rho^n$. Substituting this series into (4), a recursive set of equation is obtained [9]



(b)

Fig. 5. Comparison between the solution provided by the parabolic equation method (———) and the analitycal uniform solution in [5] ($\cdots$) for a beam focused on the edge of a perfectly conducting half plane.

$$\frac{d^2u_0(\phi)}{d\phi^2} = 0$$

$$\frac{d^2u_n(\phi)}{d\phi^2} = jk(2n-1)u_{n-1}(\phi). \tag{11}$$

To have $U(0,\phi)$, it is sufficient to compute the solution of the first equation of the set, which, taking into account the boundary at the perfect conducting faces and conditions (7)–(10), gives

$$U(0,\phi) = u_0(\phi) = \begin{cases} B & 0<\phi<\phi_1 \\ B+\gamma_1(0) & \phi_1<\phi<\phi_2 \\ B+\gamma_1(0)+\gamma_2(0) & \phi_2<\phi<n\pi \end{cases} \tag{12}$$

where

$$B = -\frac{\gamma_1(0)+\gamma_2(0)}{2}. \tag{13}$$

This holds even if $d' \neq 0$ since the presence of the term $1/\rho$ for $\rho \to 0$ annihilates the contribution of $\hat{\gamma}_{1,2}(0)$ coming from (9) and (10). This is actually the exact solution of (4) for $\rho = 0$ and not an approximate one.

IV. Numerical Results

In a first series of results, a comparison has been carried out with respect to the uniform solution presented in [5] (see Fig. 5). In this case, the beam is characterized by $\phi' = 90°$,

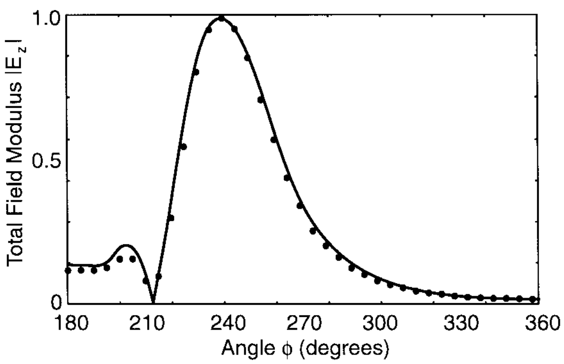


Fig. 6. Comparison between the solution provided by the parabolic equation method (_____) and the uniform solution in [6] $(\cdots)$ for a beam focused away from the edge of a perfectly conducting half-plane.

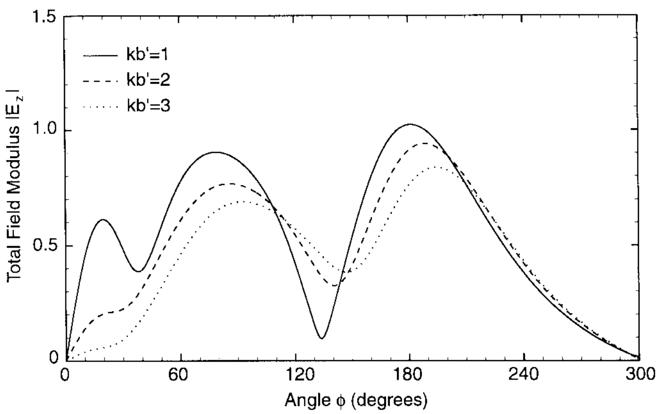


Fig. 7. Total field computed at $k\rho = 20$ for a $n = 5/3$ perfectly conducting wedge illuminated by a Gaussian beam with $\phi' = 60°$, $k\rho' = 5$, $kd' = 0$, and different values of kb'.

$k\rho' = 8$, $kd' = 0$, and various values of the beam width kb. In particular, Fig. 5 reports the normalized values of $|E_z|$ for $kb = 2, 4, 8, 16$. The results in [5] are far-field diffraction patterns; that is, for $\rho \gg \rho'$ since the technique exploited there is asymptotic. In the present work, a distance $k\rho = 80$ has been chosen to have $k\rho = 10k\rho'$. The agreement is good and is better for low kb values since this parameter is linked to an equivalent aperture of the source in the real plane and this parameter too should be kept much smaller than the distance of the observation point in an asymptotic approximation.

A second test has been carried out on the more critical case of an offsetted beam over a half plane, a problem for which exists a uniform solution [6]. The chosen configuration has $\phi' = 75°$, $k\rho' = 16$, $kd' = 4.14$, and beam width $kb = 2$. These values corresponds to the first offsetted case presented in [6]. The results, after taking into account the differences between the reference systems used in this paper and in [6] are reported in Fig. 6, where the normalized $|E_z|$ is reported.

A third series of numerical result is presented in Fig. 7, where a TM_z $(H_z = 0)$ Gaussian beam impinges on an $n = \frac{5}{3}$ perfectly conducting wedge. Various coefficients of beam width b are considered, in particular, $kb' = 1, 2, 3$. The other geometrical parameters are $\phi' = 60° k\rho' = 5$ and $kd' = 0$, thus, a beam focused on the edge itself. The figure reports the modulus of the total field at a distance $k\rho = 20$ from the edge of the wedge.

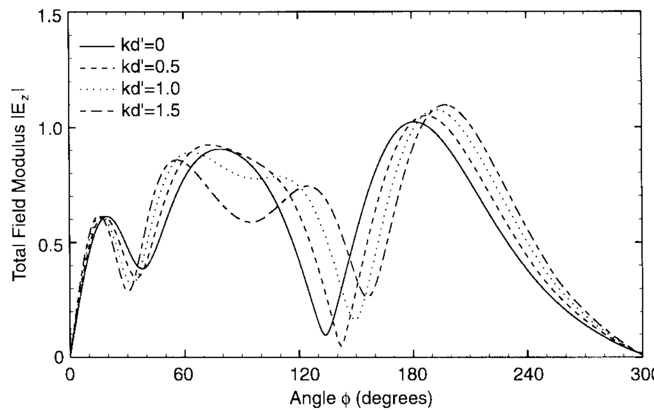


Fig. 8. Total field computed at $k\rho = 20$ for a $n = 5/3$ perfectly conducting wedge illuminated by a Gaussian beam with $\phi' = 60°$, $k\rho' = 5$, $kb' = 1$, and different values of kd'.

Fig. 8, on the other hand, presents the results for the much more complex problem of an offsetted beam. The case is again TM_z $(H_z = 0)$ and the wedge configuration is the same, but the beam width is fixed at $kb' = 1$ and the offset varies assuming the values $kd' = 0, 0.5, 1, 1.5$. We note that the direction of maximum forward radiation for the beam gets closer to 240° as the offset increases, being 240° the direction of the maximum if no wedge is present. Even if the distance remains the same, the contribution due to edge diffraction decreases since the edge illumination decreases as the offset increases.

As a last example, Fig. 9 reports the contours at equal amplitude of the GO field [Fig. 9(a)] and of the total field [Fig. 9(b)] for the same wedge configuration in the $kb' = 2$, $kd' = 1$, $\phi' = 60°$, and $k\rho' = 5$ case. In Fig. 9(a), three discontinuities can be noted: the ISB and RSB, which, being $kd' \neq 0$, are curved and the branch cut (BC) itself. In Fig. 9(b), on the other hand, the GO discontinuities at the ISB and RSB are completely canceled out by the diffracted field while the BC discontinuity still holds. This is due to the fact that the equivalent real sources generating the beam are located along this line and the total field is thus physically discontinuous there. The directivity of the source itself can, of course, be noted in both Fig. 9(a) and (b), since the contours shows how the field is much stronger in the direction pointing toward the edge.

Although both plots in Fig. 9 present data in the $k\rho \in [0, 20]$ range, it has been stated before that the solution of the parabolic equation is a good approximation of the solution of the original elliptic problem for $k\rho \gg 1$. Previous investigations showed [1]–[3] that the agreement is already good for $k\rho$ as small as four. Thus, in Fig. 9(b), to point out the area of lower accuracy, the disk at $k\rho < 4$ has been grayed out.

V. Conclusions

A complete numerical approach in the framework of the parabolic equation method to the problem of electromagnetic scattering from a perfectly conducting wedge of arbitrary

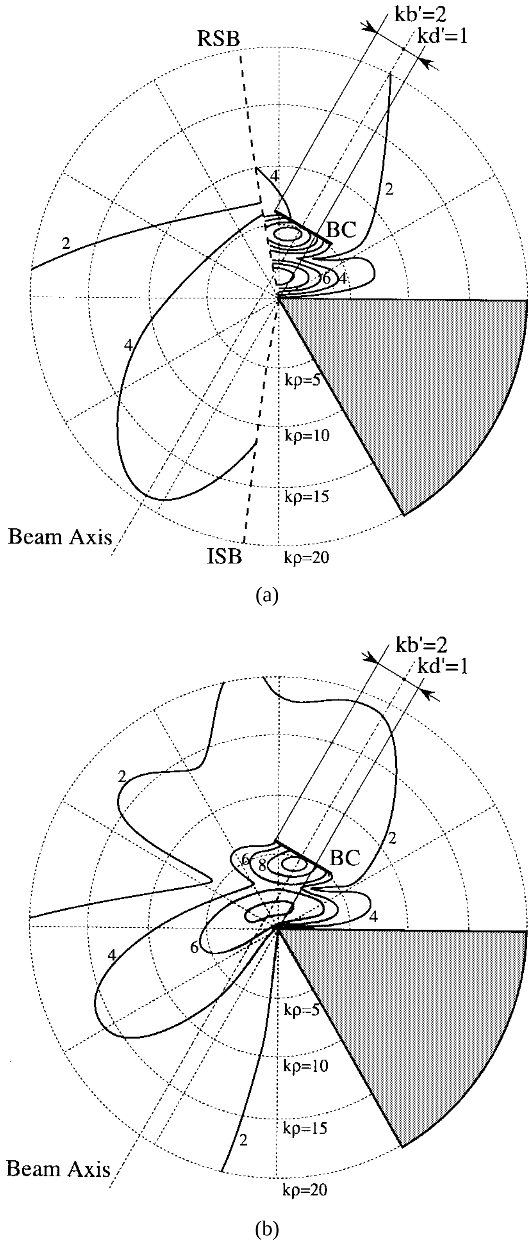


Fig. 9. Field map representing the modulus of the field from $k\rho \in [0, 20]$ for a $n = 5/6$ perfectly conducting wedge illuminated by a Gaussian beam with $\phi' = 60°$, $k\rho' = 5$, $kd' = 1$, and $kb' = 2$. (a) Only the GO field. (b) Total field.

internal angle illuminated by a Gaussian beam has been presented.

The solution can then be further developed since the parabolic equation method, suitable to treat impedance wedges [1]–[3], may be applied in this case to the scattering from an impedance wedge illuminated by a Gaussian beam once a suitable description for the GO field is available.

REFERENCES

[1] G. Pelosi, S. Selleri, and R. D. Graglia, "The parabolic equation model for the numerical analysis of the diffraction at an impedance wedge: Skew incidence case," *IEEE Trans. Antennas Propagat.*, vol. 44, pp. 267–268, Feb. 1996.

[2] ——, "Numerical analysis of the diffraction at an anisotropic impedance wedge," *IEEE Trans. Antennas Propagat.*, vol. 45, pp. 767–771, May 1997.

[3] G. Pelosi, S. Selleri, and G. Manara, "FD solution for the diffraction of an inhomogeneous plane wave at an impedance wedge," *Microwave Opt. Technol. Lett.*, vol. 14, no. 4, pp. 207–210, Mar. 1997.

[4] A. C. Green, H. L. Bertoni, and L. B. Felsen, "Properties of the shadow cast by a half-screen when illuminated by a Gaussian beam," *J. Opt. Soc. Amer.*, vol. 69, no. 11, pp. 1503–1508, Nov. 1979.

[5] G. A. Suedan and E. V. Jull, "Two-dimensional beam diffraction by a half-plane and wide slit," *IEEE Trans. Antennas Propagat.*, vol. AP-35, pp. 1077–1083, Sept. 1987.

[6] ——, "Beam diffraction by half planes and wedges: Uniform and asymptotic solutions," *J. Electromagn. Waves Applicat.*, vol. 3, no. 1, pp. 17–26, 1989.

[7] R. G. Kouyoumjian and P. H. Pathak, "A uniform geometrical theory of diffraction for an edge in a perfectly conducting surface," *Proc. IEEE*, vol. 62, pp. 1448–1461, 1974.

[8] P. Ya. Ufimtsev, "Transverse diffusion for diffraction by a wedge," *Radio Eng. Electron. Phys. (English ed. Radiotekhnika i Elektronika)*, no. 6, pp. 866–875, 1965.

[9] A. V. Popov, "Numerical solution of the wedge diffraction problem by the transverse diffusion method," *Soviet Phys.—Acoust.*, vol. 15, no. 2, pp. 226–233, Oct./Dec. 1969.

[10] ——, "Solution of the parabolic equation of diffraction theory by the method of finite differences," *USSR Comp. Math. Phys.*, vol. 8, no. 5, pp. 282–288, 1968.

Giuseppe Pelosi (M'88–SM'91) was born in Pisa, Italy. He received the Laurea (Doctor) degree in physics (*summa cum laude*) from the University of Florence, Italy, in 1976.

Since 1979, he has been with the Department of Electrical Engineering, University of Florence, where he is currently an Associate Professor. He has been a Visiting Scientist at McGill University, Montreal, Canada, from 1993 to 1995. He is coauthor of *Finite Elements for Wave Electromagnetics* (Piscataway, NJ: IEEE Press, 1994), *Finite Element Software for Microwave Engineering* (New York: Wiley, 1996), and *Quick Finite Elements for Electromagnetic Waves* (Norwood, MA: Artech House, 1998). He has mainly been involved in research in the field of numerical and asymptotic techniques for applied electromagnetics. His research interests included extensions and applications of the geometrical theory of diffraction as well as methods for radar cross section analysis of complex targets. His current research activity is mainly devoted to the development of numerical procedures in the context of the finite-element method, with particular emphasis on radiation and scattering problems.

Dr. Pelosi is a member of the Applied Computational Electromagnetics Society.

Stefano Selleri (M'91) was born in Viareggio, Italy, on December 9, 1968. He received the Laurea degree (*cum laude*), in electronic engineering and the Ph.D. degree in computer science and telecommunications, both from the University of Florence, Italy, in 1992 and 1997, respectively.

In 1992, he was a Visiting Scholar at the University of Michigan, Ann Arbor, MI, and in 1994 at the McGill University, Montreal, Canada, in 1997 at the Laboratoire d'Electroniquem University of Nice–Sophia Antipolis, France. In the first half of 1998 he was a Research Engineer at the CNET-France Telecom, La Turbie. He collaborated on the development of numerical methods applied to canonical scattering problems, scattering from complex objects and to the optimization of devices via genetic algorithms and artificial neural networks. He is coauthor of *Quick Finite Elements for Electromagnetic Waves* (Norwood, MA: Artech House, 1998). He is currently a Postdoctoral Researcher at the University of Florence. His research interests are mainly in the field of analytical and numerical techniques for applied electromagnetics, with emphasis on the application of the finite differences and finite-element method both in frequency and time domain to antenna and scattering problems.