Solution of Large Dense Complex Matrix Equations Utilizing Wavelet-Like Transforms

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***Abstract*— This paper presents the wavelet-like transforms, which are quite different from the wavelet transform for the solution of large dense complex matrix equations. From a purely numerical standpoint, these wavelet-like transforms are not true orthogonal transforms as the condition number of the resulting matrix changes after the thresholding. These effects are illustrated through examples.**

***Index Terms*—Method of moments, wavelet transforms.**

I. Introduction

CONSIDER the solution of a dense complex matrix equation of size Q. A wavelet transform [1], [2] takes a large dense matrix and compresses it to a sparse matrix. Specifically, if the matrix is generated from a kernel of the form $1/|x-y|^{\alpha}$, where the elements decay in magnitude from the diagonal, it is possible to compress the Q^2 elements to a sparse matrix containing $\theta(Q)$ elements where θ denotes of the order of Q. However, this orthogonal transform cannot be used for solution of matrix equations. For these problems a different transform is used. This has been depicted in [3] with different qualifiers. It is also an orthogonal transformation. Hence, from a theoretical point of view, the condition number of the matrix is preserved even though it gets sparse. This implies that if an iterative method is used—like, say, the conjugate gradient—it will take the same number of iterations to arrive at the solution for both the cases of the full and the sparse matrix. However, an advantage for the sparse system is that only $\theta(Q)$ multiplications are required to obtain a matrix vector product in contrast to Q^2 that is necessary to obtain a matrix-vector product for a dense matrix. It is important to note that the subtle distinction between the wavelet and wavelet like transforms are often overlooked. The two procedures are depicted in Figs. 1 and 2 for the first case and Figs. 3 and 4 for the wavelet-like transform utilizing the terminology of [9], [12]. The subtleties in their differences are now more revealing. However, there are many questions that remain unanswered [4]–[8] (e.g., the following).

1) In electromagnetic problems, the kernel is not of the form $1/|x-y|^{\alpha}$. Then when one utilizes a wavelet-like transform [3] does one still obtain $\theta(Q)$ elements?
2) It is true that the wavelet-like transform is an orthogonal matrix transformation. However, when one utilizes a thresholding operation, the condition number of the matrix is sure to change! What happens to the condition number?
3) What is the difference between the phrases-wavelet [1], [2] and "wavelet-like" [3] transformation?

We address these issues by considering a large dense complex matrices of size 2048 $\times$ 2048 that arises in the solution of electromagnetic scattering from wire structures.

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II. Solution of Large Matrix Equations by the Wavelet-Like Transform

For the application of wavelet-like techniques Q has to be an integer power of two. If the original matrix is not of size 2^m (for an integer m), then the matrix $[A]$ can be augmented by a diagonal identity matrix to make it 2^m [9]. To implement and carry out the discrete wavelet transform it is not even necessary to introduce the concept of scaling functions and wavelets. The basic principles, as outlined by Beylkin *et al.* [1] based on [10], are that typically, one would have only $10Q\log_{10}(1/\varepsilon)$ elements in the sparse system where ε is the truncation level, i.e., elements of the resultant matrix whose absolute value is less than ε will be discarded. Hence, it is useful to study such transformations. An extension of the one-dimensional (1-D) wavelet transform was made by Mallat [2] to extend the 1-D discrete wavelet transform to the two-dimensional (2-D) discrete wavelet transform as illustrated by Figs. 1 and 2. This algorithm was later used by Beylkin *et al.* in [1] to illustrate how the fast wavelet transform can be utilized to compress matrix. They illustrated that the 2-D discrete wavelet transform is computationally efficient and it is possible to obtain a very sparse matrix with $\theta(Q)$ elements out of Q^2 elements. Most of the estimates of convergence have been developed for this approach. However, this form cannot be used for solution of matrix equations as pointed by Alpert *et al.* [3] in their second paper. The transformations for the wavelet-like transforms are given by Figs. 3 and 4. We call this procedure a "wavelet-like" transform [3] in contrast to Mallat's algorithm, which is termed as a true "wavelet" transform [1].

To carry out the 2-D wavelet (orthogonal) transform of a matrix $[A]$ utilizing a finite impulse response filter (FIR) of length L, one requires $4Q^2L^2+2QL$ operations which has at most $\theta(Q)$ elements. Since whatever product operations we carry out are essentially convolutions one can utilize fast Fourier transform (FFT) to reduce the computation further and this then shows up not as $4Q^2L^2$, but rather as $L^2\log^2(L)$ as the dominant term. However, this result is an asymptotic

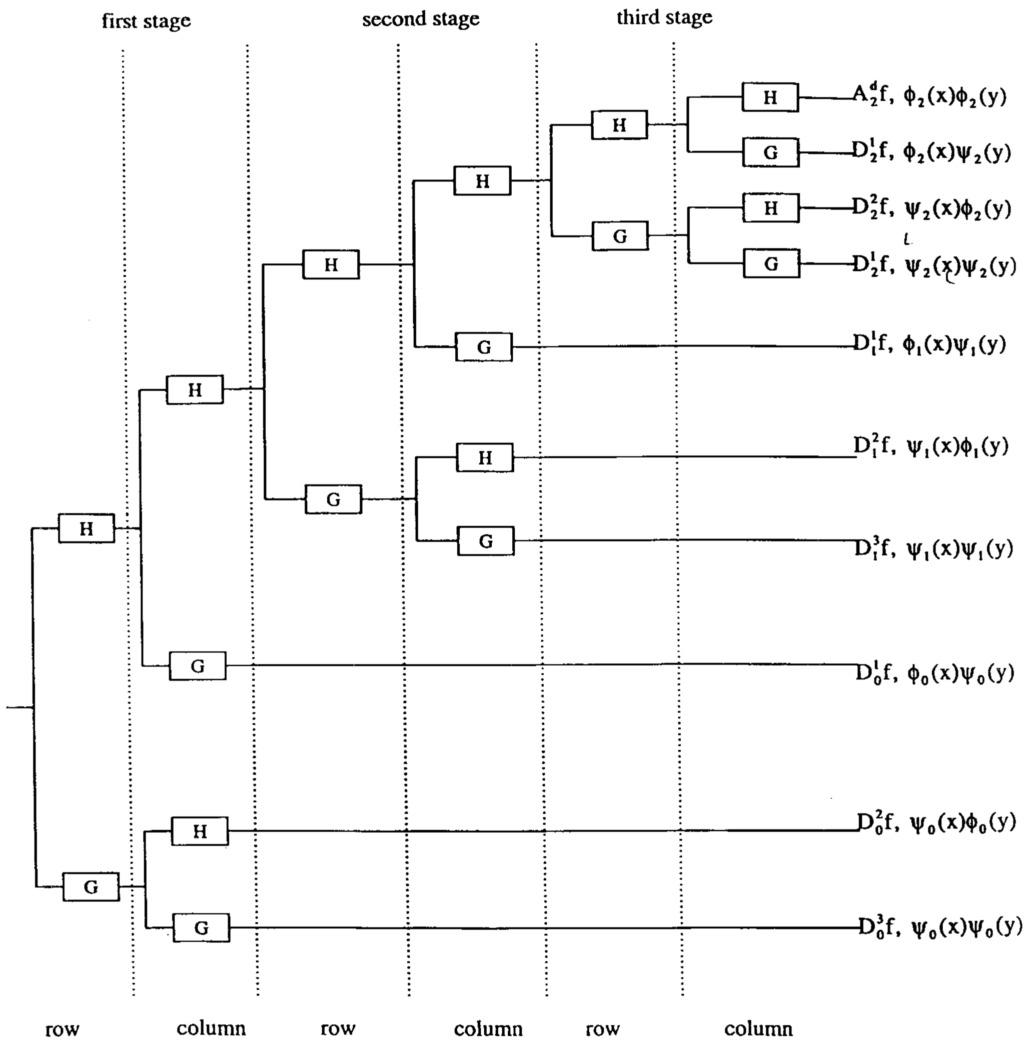

Fig. 1. Filter representation for the wavelet transform.

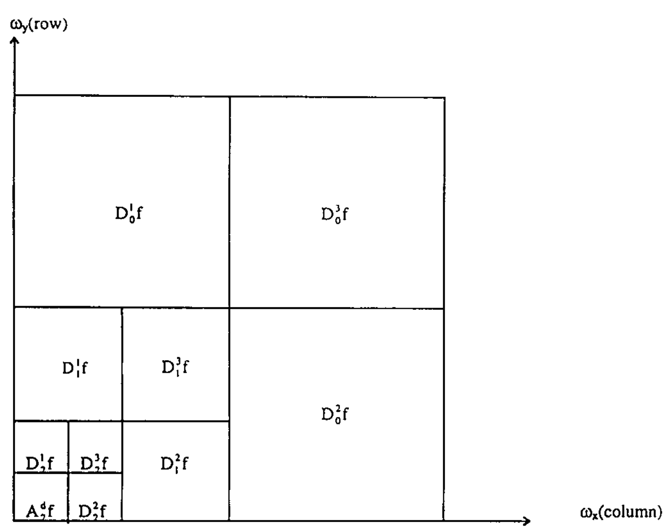

Fig. 2. Transform-domain decomposition for the D wavelet transform.

one and one does not get these efficient numbers as filters of order higher than 32 are never used for computational reasons. Therefore, these computational counts are never reached in practice!

The results shown in this paper are as follows.

1) For a large dense matrix, one observes that due to round-off errors, the transformed matrix has a different condition number and, therefore, orthogonality is not strictly maintained from a purely numerical standpoint.
2) The transformed matrix does not reach the asymptotic $\theta(N)$ nonzero elements. The number of nonzero elements are still quite large. Also, in order to achieve such results, we observed a diagonal preprocessing is absolutely necessary, which is not related to the wavelet transform.
3) The operation count of $\theta[L^2 \log^2(L)]$ is seldom reached as the order of the filter never goes beyond 32 for practical reasons and, therefore, a normal matrix-vector product is computationally more efficient than carrying out a FFT.

III. Example

Consider the solution of electromagnetic scattering from an array of randomly spaced wires by utilizing the method of moments (MoM). There are 56 thin wire antennas. Six of them are 2.7λ long and of radius 0.005λ. The remaining 50 wires are

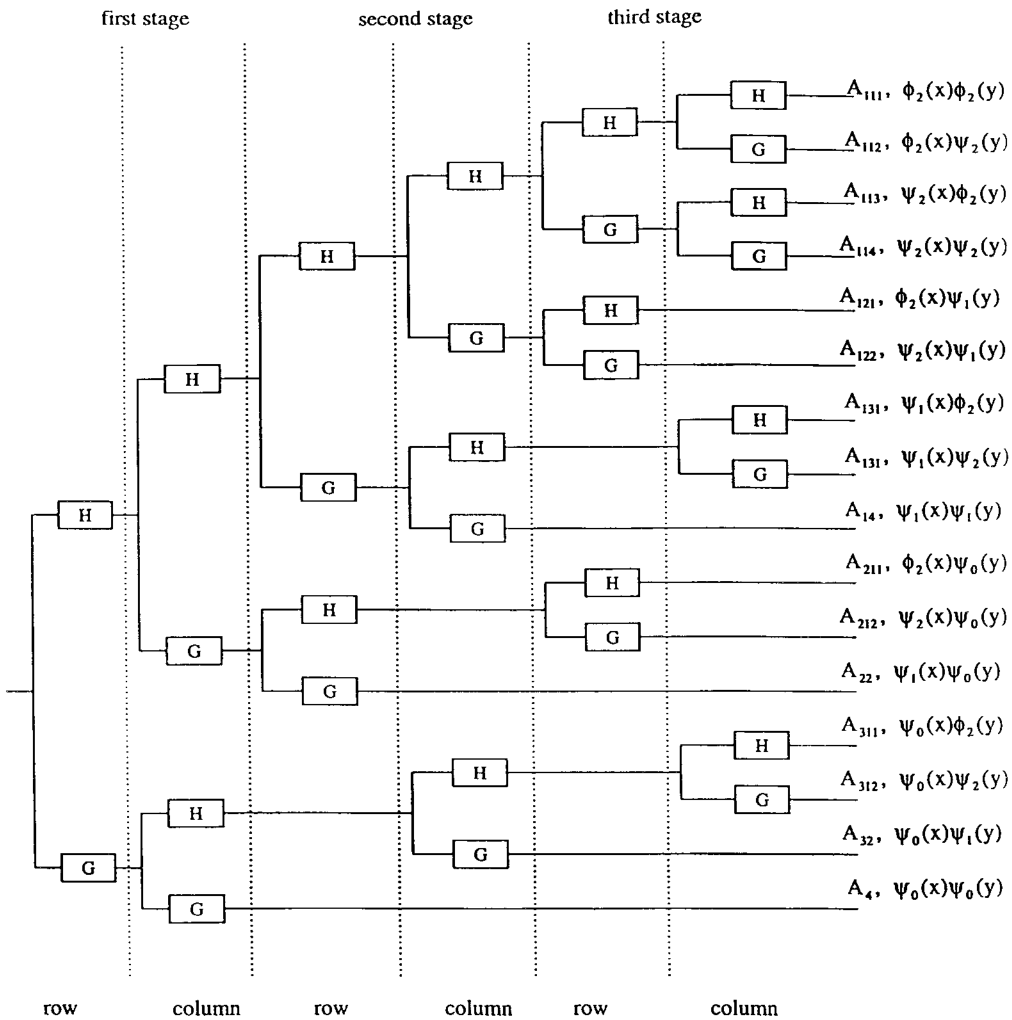

Fig. 3. Filter representation for the wavelet-like transforms.

3.0λ with the same radius. The 56 wires are randomly spaced inside a cube of dimension 27 $\lambda\times$ 25 $\lambda\times$ 21 λ. The impedance matrix that results from this structure is $Q = 2048$ and is a full complex matrix. Before we apply the wavelet transform, we rescale the entire matrix elements by the square root of the diagonal elements. This transformation is necessary otherwise the results obtained from the wavelet applications is not good! Without this transformation, it has been our experience that one obtains only 30–35% compression of the matrix instead of >90% in order to make this procedure cost effective. For these classes of problems dealing with arbitrary oriented wire antennas, diagonal preprocessing is not only necessary but mandatory.

We apply the wavelet-like transform to the rescaled matrix and the thresholding is applied separately to the real and the imaginary parts of A as the wavelet transform of a real function is real. This is another disturbing feature of the "wavelet-like" transform. The point here is that one can apply the wavelet-like transform to the elements of the complex matrix directly or apply separately to the real and the imaginary parts of A. If the transform is applied directly to the complex matrix A, the degree of compression is not good and the results are erratic. This is because there is a large difference in the magnitudes of the real and the imaginary parts of the matrix (typically the real part is 10^{-3} times the imaginary part). Hence, to get reasonable degrees of compression the transform must be applied to the real and the imaginary parts separately along with the diagonal scaling introduced above. For the filter (as outlined in [9]) we choose different orders of Daubechies filters, namely, 4, 8, 16, and 32. The goal is to see for this 2048 $\times$ 2048 complex matrix A what order of filter provide the maximally sparse matrix. The reconstruction error is the difference between the actual elements of the matrix and the one reconstructed from the thresholded wavelet coefficients. The results of this numerical experiment are summarized in Table I.

For a 2048 $\times$ 2048 matrix there are 41 94 304 elements. To obtain a sparse matrix $\theta(Q)$ elements, one should obtain a sparsity of the order of 99.95%. However, for an extremely large threshold of 0.1, we observe that $\theta(22.5Q)$ are nonzero and for the threshold of 0.001 we get the best compression when we use an eighth-order filter and it is $\theta(235.2Q)$.

We take the nonzero elements of the matrix and observe how accurately one can reconstruct the original matrix. The average mean squared reconstruction error for the real and the imaginary parts of the matrices are evaluated separately from the thresholded elements and are given in the table. The

TABLE I
RESULTS OF NUMERICAL EXPERIMENTATION WITH A 2048 × 2048 COMPLEX MATRIX

% of Elements that are zero

	order of filter	4	8	16	32
real part	ε=0.1 ε=0.01 ε=0.001	98.96 96.38 86.11	98.94 96.28 88.55	98.84 95.64 87.35	98.46 94.35 84.49
imag part	ε=0.1 ε=0.01 ε=0.001	98.93 96.34 86.08	98.89 96.20 88.50	98.75 95.56 87.30	98.32 94.25 84.35

Reconstruction error $X10^{-3}$

	order of filter	4	8	16	32
real part	ε=0.1 ε=0.01 ε=0.001	3.73 0.95 0.15	3.80 0.87 0.12	4.28 0.88 0.12	4.81 0.94 0.13
imag part	ε=0.1 ε=0.01 ε=0.001	3.77 0.95 0.15	3.86 0.87 0.12	4.34 0.88 0.12	4.90 0.94 0.13

Condition number of the original matrix = 121.9

	order of filter	4	8	16	32
after the similarity transform		188.8	198.4	247.4	288.6
after threshold	ε=0.1 ε=0.01 ε=0.001	201.0 184.4 129.2	197.2 192.2 133.0	242.8 246.9 151.1	281.1 298.6 159.9

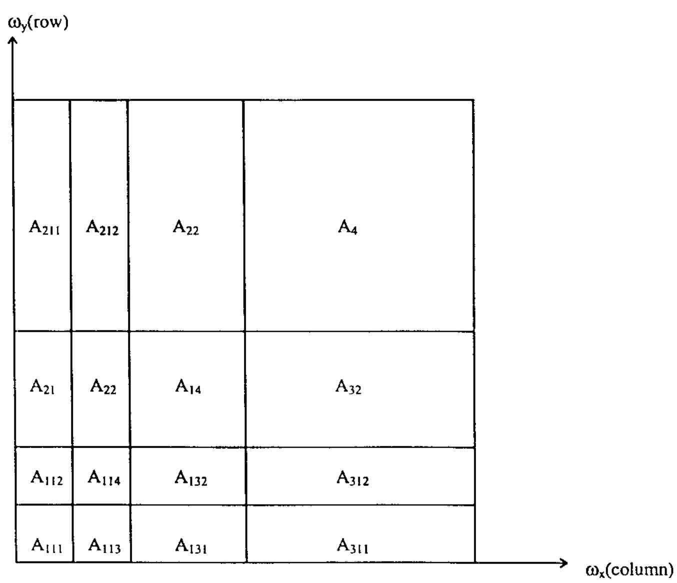

Fig. 4. Transform domain decomposition for the wavelet-like transforms.

reconstruction error both in the real and in the imaginary parts of the matrix do not show any pattern with the increase of the order of the filter for the matrix. For example, when the threshold is 0.001, the reconstruction error for $Q = 2048$ decreases with the increase of the order of the filter and then increases when the filter order becomes 32. Here, also there is no systematic rule as to what is the best possible strategy! However, the average error of the reconstruction is always invariably less than ε, the level of the threshold. In fact, in all our numerical experiments that we have conducted, we never found a situation where the average reconstruction error was not significantly less than ε.

Finally, we present the results of solving the sparse thresholded wavelet-like transformed complex matrix by the iterative conjugate gradient method to observe how efficient is the solution procedure. The iteration was stopped when the error in the residuals was ε. The conjugate gradient method took 22 iterations in all cases whether the matrix was full or sparse. This is to be expected. However, for the scaled wavelet-like transformed problem, we are performing only $0.020Q^2$ multiplications per iteration as opposed to Q^2 for the unscaled problem. Also, for the scaled problem, since 98% of the matrix is zero, we can store the entire matrix in the main memory and thereby not incur any page faults at all. This significantly reduces the computation time as far as the solution of the matrix equations are concerned.

IV. CONCLUSION

Even though the wavelet-like transform does not exactly produce sparse matrices with $\theta(Q)$ elements instead of the original Q^2 elements, it still can compress Q^2 elements significantly with proper additional mandatory diagonal preprocessing. Also when an order $\theta(Q)$ type of result is specified, one should always ask the question, what is the multiplicative constant in the front. This is because the true scientific answer may be shocking! For wire antennas, the results have been of the order of >98%. However, such compression can only be achieved, if the compression is applied to the real and the imaginary parts of the matrix separately and an appropriate diagonal preprocessing is done before hand. This is achieved at a low computational cost of $\theta(Q^2)$ operations. Hence, it is possible to store the entire original large matrix into the main memory of the computer and so fast computations can be carried out without any page faulting, which generally consumes an enormous amount of central processing unit (CPU) time. This can be a significant contribution of the wavelet-like transform for solution of large dense complex matrix equations. Even though for our examples we never observed $\theta(Q)$ nonzero elements of the compressed matrix, yet >98% compression can reduce the computational cost. The most disturbing factor being that even though a set of orthogonal transformations are used to compress the matrix, our numerical results show a significant change of the condition number of the original matrix! Also, what order of filter will be used to process the data needs to be determined through numerical experimentation for a particular matrix at hand. Hence, even though many theoretical bounds have been derived for the wavelet transform, they really do not hold numerically for the wavelet-like transforms.

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Kyungjung Kim, photograph and biography not available at the time of publication.