Coordinate-Independent Dyadic Formulation of the Dispersion Relation for Bianisotropic Media

Eng Leong Tan and Soon Yim Tan

***Abstract*—This paper presents a coordinate-independent dyadic formulation of the dispersion relation for general bianisotropic media. The dispersion equation is expanded with the aid of dyadic operators including double-dot, double-cross and dot-cross or cross-dot products. From the dispersion relation, the Booker quartic equation is derived in a form well-suited for studying multilayered structures. Several deductions are made in conjunction with the bianisotropic media satisfying reciprocity and losslessness conditions. In particular, for reciprocal bianisotropic media, the dispersion equation is biquadratic in wave vector while for lossless bianisotropic media, all dispersion coefficients are of real values. As an application example, the dispersion equation for gyrotropic bianisotropic media is considered in detail.**

***Index Terms*—Bianisotropic media, dispersion relation.**

I. Introduction

In the study of plane wave propagation and interaction with various media, the dispersion relation plays a fundamentally important role. Since the past few decades, many papers have been devoted to the derivation and utilization of the dispersion relations for several complex media. In [1], the dispersion relation for both electrically and magnetically anisotropic media has been developed along with the Booker quartic equation for the refractive-index vector component normal to a given plane. In [2] and [3], the dispersion relations for lossless positive bianisotropic media and general anisotropic media follow readily from the wave normal and ray surface equations derived using coordinate-independent dyadic formulation. In [4], the dispersion equation and Booker quartic equation obtained in coordinate-free forms have been applied to solve the problem of wave reflection from an anisotropic medium. A more comprehensive treatment on the coordinate-free approach to wave propagation and reflection from anisotropic media can be found in [5]. For arbitrary general bianisotropic media, a detailed derivation of the dispersion relation has been carried out in Cartesian coordinates in [6]. Although the coefficients of the dispersion equation have been reduced largely by symmetry considerations, their remaining expressions are still fairly lengthy and cumbersome. These lengthy expressions tend to obscure the insights into the properties of the dispersion relation, for instance, when reciprocity and losslessness, which represent two important medium conditions, are under consideration [7].

Manuscript received July 21, 1998; revised September 28, 1999.

The authors are with the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore, 639798 Republic of Singapore.

Publisher Item Identifier S 0018-926X(99)09984-6.

In this paper, based on the coordinate-independent dyadic formulation, we present the dispersion relation for plane wave propagation in general bianisotropic media. The dispersion equation is expanded as a full quartic equation utilizing various dyadic operations such as the double-dot and double-cross products defined in [8]. Moreover, the less commonly used dot-cross or cross-dot operators are also exploited and incorporated into the dispersion equation to extract the antisymmetric vectors of dyadics. Some of the identities associated with these operators are listed in Appendix A. From the dispersion relation, the Booker quartic equation is derived in coordinate-free form as well. Several deductions are made in conjunction with the bianisotropic media satisfying reciprocity and losslessness conditions. To demonstrate the application of general dispersion equation, the gyrotropic bianisotropic media which comprise many recently proposed materials are considered in particular [9]–[14]. The explicit expressions of the dispersion coefficients are given in detail in Appendix B.

II. Formulation

A. Dispersion Equation

A homogeneous linear bianisotropic medium can be characterized by the constitutive relations of the form [7]

$$\overline{D} = \overline{\overline{\epsilon}} \cdot \overline{E} + \overline{\overline{\xi}} \cdot \overline{H} \tag{1}$$

$$\overline{B} = \overline{\overline{\zeta}} \cdot \overline{E} + \overline{\overline{\mu}} \cdot \overline{H} \tag{2}$$

where $\overline{\overline{\epsilon}}$ and $\overline{\overline{\mu}}$ are, respectively, the permittivity and permeability dyadics, while $\overline{\overline{\xi}}$ and $\overline{\overline{\zeta}}$ are the magneto-electric pseudodyadics. Substituting (1)–(2) into the source-free Maxwell equations and considering a plane wave with space-time dependence factor of $e^{i\overline{k}\cdot\overline{r}}e^{-i\omega t}$, we find the dispersion relation from the conditions for nontrivial solutions of electromagnetic fields as

$$\begin{aligned} D = {} & \omega^4 \det\overline{\overline{\epsilon}} \det\overline{\overline{\mu}} \det[\overline{\overline{\epsilon}}^{-1} \cdot (\tilde{k} \times \overline{\overline{I}} + \overline{\overline{\xi}}) \\ & \cdot \overline{\overline{\mu}}^{-1} \cdot (\tilde{k} \times \overline{\overline{I}} - \overline{\overline{\zeta}}) + \overline{\overline{I}}] = 0 \end{aligned} \tag{3}$$

where $\tilde{k} = (\overline{k}/\omega)$ and $\overline{\overline{I}}$ is the identity dyadic. The factor $\omega^4 \det\overline{\overline{\epsilon}} \det\overline{\overline{\mu}}$ has been introduced to normalize the dispersion equation and these determinant terms also make the resulting equation applicable in the limit of singular $\overline{\overline{\epsilon}}$ or $\overline{\overline{\mu}}$ [6]. Equation (3) relates the wave vector $\overline{k}$ and the angular frequency ω in the most compact form. In practical computation, it is sometimes more convenient to expand (3) into an algebraic equation in terms of $\overline{k}$ or its components. For this purpose, one may resort to the dyadic algebra which facilitates coordinate-independent formulation. Moreover, the dyadic notations also help to provide much insights into the functional dependence of the dispersion equation on various constitutive parameters.

We first note the expansion of the following determinant function [8]:

$$\det(\overline{\overline{A}}\cdot\overline{\overline{B}}+\overline{\overline{I}}) = \det\overline{\overline{A}}\det\overline{\overline{B}} + (\overline{\overline{A}}\cdot\overline{\overline{B}})^{(2)} : \overline{\overline{I}} + (\overline{\overline{A}}\cdot\overline{\overline{B}}) : \overline{\overline{I}} + 1 \tag{4}$$

which applies to the third det term in (3). Here, the superscript (2) denotes cross product square operation as $\overline{\overline{A}}^{(2)} = \frac{1}{2}\overline{\overline{A}} \overset{\times}{\times} \overline{\overline{A}}$ and the symbols ":" and "$\overset{\times}{\times}$" are the double-dot and double-cross product operators defined in [8]. For convenience of reference, we list some of their elementary identities and properties in Appendix A. In the language of matrix algebra, the double-dot operation gives the trace of a matrix as

$$\overline{\overline{A}} : \overline{\overline{I}} = \mathrm{tr}\,\overline{\overline{A}} \tag{5}$$

while the cross product square and double-cross operators are related to the transpose (superscript T) of matrix adjoints as

$$\overline{\overline{A}}^{(2)} = \mathrm{adj}\,\overline{\overline{A}} \tag{6}$$

$$(\overline{\overline{A}} \overset{\times}{\times} \overline{\overline{B}})^T = \mathrm{adj}\,(\overline{\overline{A}}+\overline{\overline{B}}) - (\mathrm{adj}\,\overline{\overline{A}} + \mathrm{adj}\,\overline{\overline{B}}). \tag{7}$$

In view of (7) [or (A9)] and the fact that $(\overline{k}\times\overline{\overline{I}})^{(2)} = \overline{kk}$, we have

$$(\overline{k}\times\overline{\overline{I}} \pm \overline{\overline{A}})^{(2)} = \overline{kk} + \overline{\overline{A}}^{(2)} \pm (\overline{k}\times\overline{\overline{I}}) \overset{\times}{\times} \overline{\overline{A}}. \tag{8}$$

Applying this relation along with the identities in Appendix A into (4), we obtain

$$D = d_4 + \omega d_3 + \omega^2 d_2 + \omega^3 d_1 + \omega^4 d_0 \tag{9}$$

where

$$d_4 = (\overline{\overline{\epsilon}} : \overline{kk})(\overline{\overline{\mu}} : \overline{kk}) - (\overline{\overline{\xi}} : \overline{kk})(\overline{\overline{\zeta}} : \overline{kk}) \tag{10}$$

$$d_3 = \overline{k}\times\overline{\overline{I}} : [(\overline{\overline{\zeta}} : \overline{kk})\overline{\overline{\xi}}^{(2)T} - (\overline{\overline{\xi}} : \overline{kk})\overline{\overline{\zeta}}^{(2)T} + \overline{\overline{\xi}} \overset{\times}{\times} (\overline{\overline{\epsilon}}\cdot\overline{kk}\cdot\overline{\overline{\mu}}) - \overline{\overline{\zeta}} \overset{\times}{\times} (\overline{\overline{\mu}}\cdot\overline{kk}\cdot\overline{\overline{\epsilon}})] \tag{11}$$

$$\begin{aligned} d_2 = \overline{kk} : [&\overline{\overline{\epsilon}}\cdot\overline{\overline{\zeta}}^{(2)T}\cdot\overline{\overline{\mu}} + \overline{\overline{\mu}}\cdot\overline{\overline{\xi}}^{(2)T}\cdot\overline{\overline{\epsilon}} \\ &- (\det\overline{\overline{\zeta}})\overline{\overline{\xi}} - (\det\overline{\overline{\xi}})\overline{\overline{\zeta}} - \overline{\overline{\epsilon}}^{(2)} \overset{\times}{\times} \overline{\overline{\mu}}^{(2)T}] \\ &+ (\overline{k}\times\overline{\overline{I}} : \overline{\overline{\xi}}^{(2)T})(\overline{k}\times\overline{\overline{I}} : \overline{\overline{\zeta}}^{(2)T}) \\ &- [(\overline{k}\times\overline{\overline{I}}) \overset{\times}{\times} \overline{\overline{\xi}}^T]\cdot\overline{\overline{\epsilon}}\cdot[(\overline{k}\times\overline{\overline{I}}) \overset{\times}{\times} \overline{\overline{\zeta}}^T]\cdot\overline{\overline{\mu}} : \overline{\overline{I}} \end{aligned} \tag{12}$$

$$\begin{aligned} d_1 = \overline{k}\times\overline{\overline{I}} : [&(\det\overline{\overline{\zeta}})\overline{\overline{\xi}}^{(2)T} - (\det\overline{\overline{\xi}})\overline{\overline{\zeta}}^{(2)T} \\ &+ \overline{\overline{\mu}}^{(2)T}\cdot\overline{\overline{\zeta}}\cdot\overline{\overline{\epsilon}}^{(2)T} - \overline{\overline{\epsilon}}^{(2)T}\cdot\overline{\overline{\xi}}\cdot\overline{\overline{\mu}}^{(2)T} \\ &+ \overline{\overline{\xi}} \overset{\times}{\times} (\overline{\overline{\epsilon}}\cdot\overline{\overline{\zeta}}^{(2)T}\cdot\overline{\overline{\mu}}) - \overline{\overline{\zeta}} \overset{\times}{\times} (\overline{\overline{\mu}}\cdot\overline{\overline{\xi}}^{(2)T}\cdot\overline{\overline{\epsilon}})] \end{aligned} \tag{13}$$

$$\begin{aligned} d_0 = &\det\overline{\overline{\epsilon}}\det\overline{\overline{\mu}} - \det\overline{\overline{\xi}}\det\overline{\overline{\zeta}} \\ &+ (\overline{\overline{\xi}}^{(2)T}\cdot\overline{\overline{\epsilon}}\cdot\overline{\overline{\zeta}}^{(2)T}\cdot\overline{\overline{\mu}} - \overline{\overline{\xi}}\cdot\overline{\overline{\mu}}^{(2)T}\cdot\overline{\overline{\zeta}}\cdot\overline{\overline{\epsilon}}^{(2)T}) : \overline{\overline{I}}. \end{aligned} \tag{14}$$

Equations (9)–(14) constitute the expansion of the dispersion relation into a fourth-order algebraic equation in $\overline{k}$. By considering the actual matrix representations of $\overline{kk}$, $\overline{k}\times\overline{\overline{I}}$ and $(\overline{k}\times\overline{\overline{I}}) \overset{\times}{\times} \overline{\overline{A}}$ in (8), these equations can be shown to coincide with those derived in [6]. Moreover, they have been written in a manner which is well suited for studying special medium conditions, namely reciprocity and losslessness to be discussed later. In particular, we observe that except the last two terms in d_2, c.f. (12), all other terms involving $\overline{k}$ in the dispersion equation have been cast in the form (descending order) $(\overline{\overline{A}} : \overline{kk})(\overline{\overline{B}} : \overline{kk})$, $\overline{k}\times\overline{\overline{I}} : \overline{\overline{C}}(\overline{kk})$, $\overline{kk} : \overline{\overline{D}}$, and $\overline{k}\times\overline{\overline{I}} : \overline{\overline{E}}$. It thus remains to convert those terms into the form $\overline{kk} : \overline{\overline{F}}$ since this would facilitate subsequent analyzes on the dispersion equation, e.g., derivatives with respect to $\overline{k}$ [15].

The term $(\overline{k}\times\overline{\overline{I}} : \overline{\overline{\xi}}^{(2)T})(\overline{k}\times\overline{\overline{I}} : \overline{\overline{\zeta}}^{(2)T})$ can be dealt with in a facile manner by noting the fact that the trace of the product of a symmetric matrix and an antisymmetric matrix is zero [5]; that is

$$\overline{a}\times\overline{\overline{I}} : \overline{\overline{A}}_s = 0, \qquad \text{for } \overline{\overline{A}}_s \text{ symmetric.} \tag{15}$$

In other words, for double-dot operation with $\overline{a}\times\overline{\overline{I}}$, only the antisymmetric part needs to be taken into account. This antisymmetric part is characterized by a vector which can be extracted through the dot-cross $\overset{\cdot}{\times}$ or cross-dot $\overset{\times}{\cdot}$ operations [8]. As shown in Appendix A, these two operators are intimately related to each other and either one can be chosen at will. Adopting the dot-cross operator which results in the following explicit vector components:

$$[\overline{\overline{A}}]_{xyz} \overset{\cdot}{\times} \overline{\overline{I}} = \begin{bmatrix} A_{zy} - A_{yz} \\ A_{xz} - A_{zx} \\ A_{yx} - A_{xy} \end{bmatrix} \tag{16}$$

we have

$$(\overline{a}\times\overline{\overline{I}} : \overline{\overline{A}})(\overline{a}\times\overline{\overline{I}} : \overline{\overline{B}}) = \overline{aa} : (\overline{\overline{A}} \overset{\cdot}{\times} \overline{\overline{I}})(\overline{\overline{B}} \overset{\cdot}{\times} \overline{\overline{I}}) \tag{17}$$

written in dyad terms. We next move on to $[(\overline{k}\times\overline{\overline{I}}) \overset{\times}{\times} \overline{\overline{\xi}}^T]\cdot\overline{\overline{\epsilon}}\cdot[(\overline{k}\times\overline{\overline{I}}) \overset{\times}{\times} \overline{\overline{\zeta}}^T]\cdot\overline{\overline{\mu}} : \overline{\overline{I}}$. Applying (17) together with the identity

$$(\overline{\overline{A}} \overset{\times}{\times} \overline{\overline{B}}) \overset{\cdot}{\times} \overline{\overline{I}} = \overline{\overline{A}}\cdot(\overline{\overline{B}} \overset{\cdot}{\times} \overline{\overline{I}}) + \overline{\overline{B}}\cdot(\overline{\overline{A}} \overset{\cdot}{\times} \overline{\overline{I}}) \tag{18}$$

we arrive at

$$\begin{aligned} [(\overline{a}\times\overline{\overline{I}}) \overset{\times}{\times} \overline{\overline{A}}^T]\cdot\overline{\overline{B}}\cdot[(\overline{a}\times\overline{\overline{I}}) \overset{\times}{\times} \overline{\overline{C}}^T]\cdot\overline{\overline{D}} : \overline{\overline{I}} = \overline{aa} : [&-\overline{\overline{A}}^T\cdot(\overline{\overline{B}} \overset{\times}{\times} \overline{\overline{D}}^T)\cdot\overline{\overline{C}} \\ &+ [(\overline{\overline{A}} \overset{\cdot}{\times} \overline{\overline{I}})\cdot\overline{\overline{B}}][(\overline{\overline{C}} \overset{\cdot}{\times} \overline{\overline{I}})\cdot\overline{\overline{D}}] \\ &+ \overline{\overline{A}}^T\cdot([(\overline{\overline{C}} \overset{\cdot}{\times} \overline{\overline{I}})\cdot\overline{\overline{D}}]\times\overline{\overline{I}})\cdot\overline{\overline{B}} \\ &+ \overline{\overline{C}}^T\cdot([(\overline{\overline{A}} \overset{\cdot}{\times} \overline{\overline{I}})\cdot\overline{\overline{B}}]\times\overline{\overline{I}})\cdot\overline{\overline{D}}]. \end{aligned} \tag{19}$$

Hence, with (17) and (19) determined, we have all the terms in d_2 expressed in the form involving $\overline{kk}$ only.

Incorporating the dot-cross operator into the first (d_1) and third (d_3) order terms, one can simplify them on the basis of (15). Furthermore, the fairly expensive double-cross operations in the resultant expressions can be eliminated via (18) although we choose to retain the cross-product square since they bear the meaning of matrix adjoints. Then, the final dispersion equation reads

$$D = f_4(\overline{kk}, \overline{kk}) + f_3(\overline{k}, \overline{kk}) + f_2(\overline{kk}) + f_1(\overline{k}) + f_0 \tag{20}$$

where

$$f_4(\bar{\bar{A}}_s, \bar{\bar{B}}_s) = (\bar{\bar{\epsilon}} : \bar{\bar{A}}_s)(\bar{\bar{\mu}} : \bar{\bar{B}}_s) - (\bar{\bar{\xi}} : \bar{\bar{A}}_s)(\bar{\bar{\zeta}} : \bar{\bar{B}}_s) \quad (21)$$

$$\begin{aligned} f_3(\bar{a}, \overline{uv}) = \omega \bar{a} \cdot [&(\bar{\bar{\zeta}} : \overline{uv})\bar{\bar{\xi}}^{(2)T} \stackrel{\cdot}{\times} \bar{\bar{I}} - (\bar{\bar{\xi}} : \overline{uv})\bar{\bar{\zeta}}^{(2)T} \stackrel{\cdot}{\times} \bar{\bar{I}} \\ &+ \bar{\bar{\xi}} \cdot [(\bar{v} \cdot \bar{\bar{\mu}}) \times (\bar{\bar{\epsilon}} \cdot \bar{u})] - \bar{\bar{\zeta}} \cdot [(\bar{v} \cdot \bar{\bar{\epsilon}}) \times (\bar{\bar{\mu}} \cdot \bar{u})] \\ &+ (\bar{\bar{\xi}} \stackrel{\cdot}{\times} \bar{\bar{I}}) \cdot (\bar{v} \cdot \bar{\bar{\mu}})(\bar{\bar{\epsilon}} \cdot \bar{u}) \\ &- (\bar{\bar{\zeta}} \stackrel{\cdot}{\times} \bar{\bar{I}}) \cdot (\bar{v} \cdot \bar{\bar{\epsilon}})(\bar{\bar{\mu}} \cdot \bar{u})] \end{aligned} \quad (22)$$

$$\begin{aligned} f_2(\bar{\bar{A}}_s) = \omega^2 \bar{\bar{A}}_s : [&\bar{\bar{\epsilon}} \cdot \bar{\bar{\zeta}}^{(2)T} \cdot \bar{\bar{\mu}} + \bar{\bar{\mu}} \cdot \bar{\bar{\xi}}^{(2)T} \cdot \bar{\bar{\epsilon}} \\ &- (\det \bar{\bar{\zeta}})\bar{\bar{\xi}} - (\det \bar{\bar{\xi}})\bar{\bar{\zeta}} \\ &- \bar{\bar{\epsilon}}^{(2)} \stackrel{\times}{\times} \bar{\bar{\mu}}^{(2)T} + (\bar{\bar{\xi}}^{(2)T} \stackrel{\cdot}{\times} \bar{\bar{I}})(\bar{\bar{\zeta}}^{(2)T} \stackrel{\cdot}{\times} \bar{\bar{I}}) \\ &+ \bar{\bar{\xi}}^T \cdot (\bar{\bar{\epsilon}} \stackrel{\times}{\times} \bar{\bar{\mu}}^T) \cdot \bar{\bar{\zeta}} \\ &- [(\bar{\bar{\xi}} \stackrel{\cdot}{\times} \bar{\bar{I}}) \cdot \bar{\bar{\epsilon}}][(\bar{\bar{\zeta}} \stackrel{\cdot}{\times} \bar{\bar{I}}) \cdot \bar{\bar{\mu}}] \\ &- \bar{\bar{\xi}}^T \cdot ([(\bar{\bar{\zeta}} \stackrel{\cdot}{\times} \bar{\bar{I}}) \cdot \bar{\bar{\mu}}] \times \bar{\bar{I}}) \cdot \bar{\bar{\epsilon}} \\ &- \bar{\bar{\zeta}}^T \cdot ([(\bar{\bar{\xi}} \stackrel{\cdot}{\times} \bar{\bar{I}}) \cdot \bar{\bar{\epsilon}}] \times \bar{\bar{I}}) \cdot \bar{\bar{\mu}}] \end{aligned} \quad (23)$$

$$\begin{aligned} f_1(\bar{a}) = \omega^3 \bar{a} \cdot [&(\det \bar{\bar{\zeta}})\bar{\bar{\xi}}^{(2)T} \stackrel{\cdot}{\times} \bar{\bar{I}} - (\det \bar{\bar{\xi}})\bar{\bar{\zeta}}^{(2)T} \stackrel{\cdot}{\times} \bar{\bar{I}} \\ &+ (\bar{\bar{\mu}}^{(2)T} \cdot \bar{\bar{\zeta}} \cdot \bar{\bar{\epsilon}}^{(2)T}) \stackrel{\cdot}{\times} \bar{\bar{I}} \\ &- (\bar{\bar{\epsilon}}^{(2)T} \cdot \bar{\bar{\xi}} \cdot \bar{\bar{\mu}}^{(2)T}) \stackrel{\cdot}{\times} \bar{\bar{I}} \\ &+ \bar{\bar{\xi}} \cdot [(\bar{\bar{\epsilon}} \cdot \bar{\bar{\zeta}}^{(2)T} \cdot \bar{\bar{\mu}}) \stackrel{\cdot}{\times} \bar{\bar{I}}] \\ &- \bar{\bar{\zeta}} \cdot [(\bar{\bar{\mu}} \cdot \bar{\bar{\xi}}^{(2)T} \cdot \bar{\bar{\epsilon}}) \stackrel{\cdot}{\times} \bar{\bar{I}}] \\ &+ (\bar{\bar{\epsilon}} \cdot \bar{\bar{\zeta}}^{(2)T} \cdot \bar{\bar{\mu}}) \cdot (\bar{\bar{\xi}} \stackrel{\cdot}{\times} \bar{\bar{I}}) \\ &- (\bar{\bar{\mu}} \cdot \bar{\bar{\xi}}^{(2)T} \cdot \bar{\bar{\epsilon}}) \cdot (\bar{\bar{\zeta}} \stackrel{\cdot}{\times} \bar{\bar{I}})] \end{aligned} \quad (24)$$

$$f_0 = \omega^4 \, d_0. \quad (25)$$

The functions $f_4, f_3, \cdots, f_0$ are so defined to cater for subsequent development of Booker quartic equation. In the case of lossless positive symmetric media where $\bar{\bar{\epsilon}}^T = \bar{\bar{\epsilon}}$, $\bar{\bar{\mu}}^T = \bar{\bar{\mu}}$, $\bar{\bar{\xi}}^T = \bar{\bar{\xi}} = \bar{\bar{\zeta}}$, we find (22) and (23) coincide readily with those in [2] since $\bar{\bar{A}}_s \stackrel{\cdot}{\times} I = \bar{0}$ for $\bar{\bar{A}}_s$ symmetric. Moreover, based on the dual transformation of wave normal and ray vectors introduced in that paper, we can write the second- and first-order terms as

$$\begin{aligned} f_2(\bar{\bar{A}}_s) = \omega^2 \, d_0 \, \bar{\bar{A}}_s : [&\bar{\bar{\kappa}}_d^{(2)T} + \bar{\bar{\nu}}_d^{(2)T} - (\bar{\bar{\kappa}}_d \stackrel{\cdot}{\times} \bar{\bar{I}})(\bar{\bar{\nu}}_d \stackrel{\cdot}{\times} \bar{\bar{I}}) \\ &- (\bar{\bar{\mu}} - \bar{\bar{\zeta}} \cdot \bar{\bar{\epsilon}}^{-1} \cdot \bar{\bar{\xi}})^{-1} \stackrel{\times}{\times} (\bar{\bar{\epsilon}} - \bar{\bar{\xi}} \cdot \bar{\bar{\mu}}^{-1} \cdot \bar{\bar{\zeta}})^{-1T}] \end{aligned} \quad (26)$$

$$f_1(\bar{a}) = \omega^3 \, d_0 \, \bar{a} \cdot (\bar{\bar{\kappa}}_d - \bar{\bar{\nu}}_d) \stackrel{\cdot}{\times} \bar{\bar{I}} \quad (27)$$

where

$$\bar{\bar{\kappa}}_d = \bar{\bar{\mu}}^{-1} \cdot \bar{\bar{\zeta}} \cdot (\bar{\bar{\epsilon}} - \bar{\bar{\xi}} \cdot \bar{\bar{\mu}}^{-1} \cdot \bar{\bar{\zeta}})^{-1} \quad (28)$$

$$\bar{\bar{\nu}}_d = \bar{\bar{\epsilon}}^{-1} \cdot \bar{\bar{\xi}} \cdot (\bar{\bar{\mu}} - \bar{\bar{\zeta}} \cdot \bar{\bar{\epsilon}}^{-1} \cdot \bar{\bar{\xi}})^{-1} \quad (29)$$

$$\begin{aligned} d_0 &= \det \bar{\bar{\mu}} \det(\bar{\bar{\epsilon}} - \bar{\bar{\xi}} \cdot \bar{\bar{\mu}}^{-1} \cdot \bar{\bar{\zeta}}) \\ &= \det \bar{\bar{\epsilon}} \det(\bar{\bar{\mu}} - \bar{\bar{\zeta}} \cdot \bar{\bar{\epsilon}}^{-1} \cdot \bar{\bar{\xi}}). \end{aligned} \quad (30)$$

These expressions are seen to be in more compact form, but they require explicit computation of the inverse of dyadics.

B. Booker Quartic Equation

The dispersion equations (9)–(14) or (20)–(25) have been expressed in terms of total wave vector $\bar{k}$ and wave dyad $\overline{kk}$. For the study of multilayered structures, one is often interested in waves propagating along a preferred direction, say $\hat{p}$, normal to the structures. Decomposing the wave vector into its components longitudinal and transverse to $\hat{p}$ as

$$\bar{k} = k_p \hat{p} + \bar{k}_t, \quad \bar{k}_t = k_{t1}\hat{t}_1 + k_{t2}\hat{t}_2 \quad (31)$$

we have the symmetric dyad $\overline{kk}$ expanded in the form

$$\overline{kk} = k_p^2 \hat{p}\hat{p} + k_p \bar{\bar{S}} + \bar{k}_t \bar{k}_t, \quad \bar{\bar{S}} = \hat{p}\bar{k}_t + \bar{k}_t \hat{p}. \quad (32)$$

The phase-matching condition at the layer interface requires k_{t1} and k_{t2} to be the same throughout all layers and each may run from $-\infty$ to $+\infty$ to account for all incident directions or point source excitation [16]. Then, the dispersion equation yields a fourth-order algebraic equation for k_p in each layer as

$$B = b_4 k_p^4 + b_3 k_p^3 + b_2 k_p^2 + b_1 k_p + b_0 = 0 \quad (33)$$

where

$$b_4 = f_4(\hat{p}\hat{p}, \hat{p}\hat{p}) \quad (34)$$

$$b_3 = f_4(\hat{p}\hat{p}, \bar{\bar{S}}) + f_4(\bar{\bar{S}}, \hat{p}\hat{p}) + f_3(\hat{p}, \hat{p}\hat{p}) \quad (35)$$

$$\begin{aligned} b_2 = &f_4(\hat{p}\hat{p}, \bar{k}_t\bar{k}_t) + f_4(\bar{\bar{S}}, \bar{\bar{S}}) + f_4(\bar{k}_t\bar{k}_t, \hat{p}\hat{p}) \\ &+ f_3(\hat{p}, \bar{\bar{S}}) + f_3(\bar{k}_t, \hat{p}\hat{p}) + f_2(\hat{p}\hat{p}) \end{aligned} \quad (36)$$

$$\begin{aligned} b_1 = &f_4(\bar{\bar{S}}, \bar{k}_t\bar{k}_t) + f_4(\bar{k}_t\bar{k}_t, \bar{\bar{S}}) \\ &+ f_3(\hat{p}, \bar{k}_t\bar{k}_t) + f_3(\bar{k}_t, \bar{\bar{S}}) + f_2(\bar{\bar{S}}) + f_1(\hat{p}) \end{aligned} \quad (37)$$

$$\begin{aligned} b_0 = &f_4(\bar{k}_t\bar{k}_t, \bar{k}_t\bar{k}_t) + f_3(\bar{k}_t, \bar{k}_t\bar{k}_t) \\ &+ f_2(\bar{k}_t\bar{k}_t) + f_1(\bar{k}_t) + f_0. \end{aligned} \quad (38)$$

Equation (33), commonly known as Booker quartic [1], [4], allows us to determine k_p in terms of angular frequency, constitutive parameters and transverse wave numbers. This equation can be solved numerically or analytically and their solution characteristics are worthy of further investigation.

III. Special Medium Conditions

In this section, we shall consider more closely two important special cases for the medium conditions, choosing (10)–(14) as the basis of our study.

A. Reciprocity

For reciprocal bianisotropic media, the constitutive dyadics satisfy the reciprocity requirements [7]

$$\bar{\bar{\epsilon}} = \bar{\bar{\epsilon}}^T, \quad \bar{\bar{\mu}} = \bar{\bar{\mu}}^T, \quad \bar{\bar{\zeta}} = -\bar{\bar{\xi}}^T. \quad (39)$$

Substituting these conditions into (10)–(14), the even terms d_4, d_2, and d_0 can be simplified slightly by noting $\overline{kk} : \bar{\bar{A}}^T = \overline{kk} : \bar{\bar{A}}$. Moreover, the odd terms d_3 and d_1 are found to be of the form $\bar{k} \times \bar{\bar{I}} : (\bar{\bar{A}} + \bar{\bar{A}}^T)$ and, therefore, vanish due to (15). Then, we have the dispersion equation biquadratic in $\bar{k}$ or k (its magnitude). This implies that there exist two pairs of solutions corresponding to two waves propagating with the same velocity in opposite directions in each pair. Notice that similar deductions can be made for anisotropic (not necessarily reciprocal) media where $\overline{\overline{\xi}} = \overline{\overline{\zeta}} = \overline{\overline{0}}$ since all terms in d_3 and d_1 are dependent on them [3], [8]. However, it should be noted that while only the f_3 and f_1 functions in (21)–(25) are void, the Booker equation (33) may still constitute full quartic polynomial in k_p. In other words, more stringent conditions must be imposed to make the Booker equation biquadratic.

B. Losslessness

For lossless bianisotropic media, the constitutive dyadics satisfy the conditions [7]

$$\overline{\overline{\epsilon}}^T = \overline{\overline{\epsilon}}^*, \quad \overline{\overline{\mu}}^T = \overline{\overline{\mu}}^*, \quad \overline{\overline{\zeta}}^T = \overline{\overline{\xi}}^* \tag{40}$$

where $*$ indicates complex conjugation. Applying these conditions into (10)–(14) and assuming $\hat{k}$ is real with direction cosines, we find that all coefficients in the algebraic equation for k are of real values since they merely involve dyadics of the form $\overline{\overline{A}} + \overline{\overline{A}}^*$ or operations leading to $a = a^*$. Thus, the dispersion equation represents a real polynomial of k whose roots form in general complex conjugate pairs. By the same token, all b's in the Booker quartic equation (33) are found to be real for $\hat{p}$ and $\overline{k}_t$ being real. Their roots for k_p will then be complex conjugate pairs as well [17]. For slightly lossy media where the losslessness conditions (40) are hardly violated, these conjugate roots may serve as initial guess for the actual roots in the respective region above and below the real axis.

IV. Application to Gyrotropic Bianisotropic Media

As an application example for the general dispersion equation derived above, let us consider the gyrotropic bianisotropic media whose constitutive dyadics are of the form [8]

$$\overline{\overline{G}}(G_t, G_g, G_a) = G_t \overline{\overline{I}}_t + G_g \hat{g}\hat{g} + G_a \hat{g} \times \overline{\overline{I}} \tag{41}$$

for $G = \epsilon, \mu, \xi, \zeta$. Here, $\hat{g}$ is the gyrotropic axis which has been normalized to $\hat{g} \cdot \hat{g} = 1$ and $\overline{\overline{I}}_t$ is the transverse (to $\hat{g}$) part of identity dyadic, i.e., $\overline{\overline{I}}_t = \overline{\overline{I}} - \hat{g}\hat{g}$. The gyrotropic dyadic has its trace, determinant, and antisymmetric vector given by

$$\overline{\overline{G}} : \overline{\overline{I}} = 2G_t + G_g \tag{42}$$

$$\det \overline{\overline{G}} = G_g(G_t^2 + G_a^2) \tag{43}$$

$$\overline{\overline{G}} \,{}^{\cdot}_{\times}\, \overline{\overline{I}} = 2G_a \hat{g} \tag{44}$$

where the factor 2 in (44) is due to the dot-cross operation employed. For two gyrotropic dyadics, their dot and double-cross products are

$$\begin{aligned}\overline{\overline{G}}_1 \cdot \overline{\overline{G}}_2 = {} & \overline{\overline{G}}(G_{t1}G_{t2} - G_{a1}G_{a2}, G_{g1}G_{g2}, \\ & G_{t1}G_{a2} + G_{t2}G_{a1})\end{aligned} \tag{45}$$

$$\begin{aligned}\overline{\overline{G}}_1 \,{}^{\times}_{\times}\, \overline{\overline{G}}_2 = {} & \overline{\overline{G}}(G_{t1}G_{g2} + G_{t2}G_{g1}, \\ & 2(G_{t1}G_{t2} + G_{a1}G_{a2}), \\ & G_{g1}G_{a2} + G_{g2}G_{a1})\end{aligned} \tag{46}$$

while the adjoint follows as

$$\overline{\overline{G}}^{(2)T} = \overline{\overline{G}}(G_tG_g, G_t^2 + G_a^2, -G_gG_a). \tag{47}$$

For operations involving the wave vector $\overline{k} = k\hat{k}$ with $\hat{k} \cdot \hat{k} = 1$, we have

$$\hat{k}\hat{k} : \overline{\overline{G}} = G_g c^2 + G_t s^2 \tag{48}$$

where c and s are the cosine and sine of the angle between $\hat{k}$ and $\hat{g}$

$$c = \hat{k} \cdot \hat{g} \tag{49}$$

$$s^2 = (\hat{k} \times \hat{g}) \cdot (\hat{k} \times \hat{g}). \tag{50}$$

Using the results (42)–(48), we obtain the dispersion equation for gyrotropic bianisotropic media as

$$D = f_4(\hat{k}\hat{k}, \hat{k}\hat{k})k^4 + f_3(\hat{k}, \hat{k}\hat{k})k^3 + f_2(\hat{k}\hat{k})k^2 + f_1(\hat{k})k + f_0 \tag{51}$$

with the f's given explicitly in Appendix B.

Equation (51) reduces readily to the case of many recently proposed materials including chiral [9], Faraday chiral [10], uniaxial chiro-omega [11], [12], uniaxial bianisotropic [13], and transversely bianisotropic uniaxial [14] media. These media may find potential applications in wide range of areas such as antenna radomes, absorbers, integrated circuit technology, integrated optics, etc.

V. Conclusion

This paper has presented a coordinate-independent dyadic formulation of the dispersion relation for plane wave propagation in general bianisotropic media. The dispersion equation has been expanded as a fourth-order algebraic equation with the aid of dyadic operators including double-dot, double-cross and dot-cross or cross-dot products. From the dispersion relation, the Booker quartic equation has been derived in a form well-suited for studying multilayered structures. Two important medium conditions, namely reciprocity and losslessness, have been examined more closely. For reciprocal bianisotropic media, the dispersion equation is biquadratic in wave vector. For lossless bianisotropic media, all dispersion coefficients are of real values. As an application example, the dispersion equation for gyrotropic bianisotropic media has been considered in detail.

Appendix A

Dyadic identities [8]

$$\overline{\overline{A}} : \overline{\overline{B}} = \overline{\overline{B}} : \overline{\overline{A}} = \overline{\overline{A}} \cdot \overline{\overline{B}}^T : \overline{\overline{I}} = \overline{\overline{B}}^T \cdot \overline{\overline{A}} : \overline{\overline{I}} \tag{A.1}$$

$$\overline{\overline{A}} : \overline{a}\overline{b} = \overline{\overline{A}}^T : \overline{b}\overline{a} = \overline{a} \cdot \overline{\overline{A}} \cdot \overline{b} \tag{A.2}$$

$$\overline{\overline{A}} : \overline{a} \times \overline{\overline{I}} = -\overline{\overline{A}}^T : \overline{a} \times \overline{\overline{I}} \tag{A.3}$$

$$\overline{\overline{A}} \,{}^{\times}_{\times}\, \overline{\overline{B}} = \overline{\overline{B}} \,{}^{\times}_{\times}\, \overline{\overline{A}} \tag{A.4}$$

$$(\overline{\overline{A}} \,{}^{\times}_{\times}\, \overline{\overline{B}})^T = \overline{\overline{A}}^T \,{}^{\times}_{\times}\, \overline{\overline{B}}^T \tag{A.5}$$

$$\overline{\overline{A}} \,{}^{\times}_{\times}\, \overline{a}\overline{b} = -\overline{a} \times \overline{\overline{I}} \cdot \overline{\overline{A}} \cdot \overline{b} \times \overline{\overline{I}} \tag{A.6}$$

$$\overline{\overline{A}} \,{}^{\times}_{\times}\, \overline{\overline{B}} : \overline{\overline{C}} = \overline{\overline{A}} : \overline{\overline{B}} \,{}^{\times}_{\times}\, \overline{\overline{C}} \tag{A.7}$$

$$(\overline{\overline{A}} \cdot \overline{\overline{B}})^{(2)} = \overline{\overline{A}}^{(2)} \cdot \overline{\overline{B}}^{(2)} \tag{A.8}$$

$$(\overline{\overline{A}} \pm \overline{\overline{B}})^{(2)} = \overline{\overline{A}}^{(2)} + \overline{\overline{B}}^{(2)} \pm \overline{\overline{A}} \,{}^{\times}_{\times}\, \overline{\overline{B}} \tag{A.9}$$

$$\overline{\overline{A}}^{-1} = \frac{\overline{\overline{A}}^{(2)T}}{\det \overline{\overline{A}}} \tag{A.10}$$

$$\overline{\overline{A}} \,{}^{\cdot}_{\times}\, \overline{\overline{I}} = \overline{\overline{I}} \,{}^{\times}_{\cdot}\, \overline{\overline{A}} \tag{A.11}$$

$$\overline{\overline{A}} \,{}^{\cdot}_{\times}\, \overline{\overline{B}} = -\overline{\overline{B}} \,{}^{\cdot}_{\times}\, \overline{\overline{A}} = (\overline{\overline{B}}^{T} \cdot \overline{\overline{A}}) \,{}^{\cdot}_{\times}\, \overline{\overline{I}} \tag{A.12}$$

$$\overline{\overline{A}}^{T} \,{}^{\cdot}_{\times}\, \overline{\overline{I}} = -\overline{\overline{A}} \,{}^{\cdot}_{\times}\, \overline{\overline{I}} \tag{A.13}$$

$$\overline{a}\overline{b} \,{}^{\cdot}_{\times}\, \overline{\overline{I}} = \overline{b} \times \overline{a} \tag{A.14}$$

$$(\overline{a} \times \overline{\overline{I}}) \,{}^{\cdot}_{\times}\, \overline{\overline{I}} = 2\overline{a} \tag{A.15}$$

$$(\overline{\overline{A}} \,{}^{\cdot}_{\times}\, \overline{\overline{I}}) \times \overline{\overline{I}} = \overline{\overline{A}} - \overline{\overline{A}}^{T}. \tag{A.16}$$

APPENDIX B

For the gyrotropic bianisotropic media described in Section IV, the f coefficients in (51) are

$$\begin{aligned} f_4(\hat{k}\hat{k}, \hat{k}\hat{k}) = {} & (\mu_t\epsilon_t - \xi_t\zeta_t)s^4 + (\mu_g\epsilon_g - \xi_g\zeta_g)c^4 \\ & + (\mu_t\epsilon_g + \epsilon_t\mu_g - \xi_t\zeta_g - \zeta_t\xi_g)s^2c^2 \end{aligned} \tag{B.1}$$

$$\begin{aligned} f_3(\hat{k}, \hat{k}\hat{k}) = {} & \omega c\{[(\xi_a - \zeta_a)(\mu_t\epsilon_g + \epsilon_t\mu_g) \\ & + (\xi_g - \zeta_g)(\mu_t\epsilon_a + \epsilon_t\mu_a) \\ & - (\xi_t - \zeta_t)(\mu_g\epsilon_a + \epsilon_g\mu_a) \\ & + 2(\xi_t\zeta_g\zeta_a - \zeta_t\xi_g\xi_a)]s^2 \\ & + 2(\xi_a - \zeta_a)(\mu_g\epsilon_g - \xi_g\zeta_g)c^2\} \end{aligned} \tag{B.2}$$

$$\begin{aligned} f_2(\hat{k}\hat{k}) = {} & \omega^2\{[(\xi_t\xi_g + \zeta_t\zeta_g)(\mu_t\epsilon_t - \mu_a\epsilon_a) \\ & + (\xi_g\xi_a + \zeta_g\zeta_a)(\mu_t\epsilon_a + \epsilon_t\mu_a) \\ & + (\xi_t\zeta_t - \xi_a\zeta_a)(\mu_t\epsilon_g + \epsilon_t\mu_g) \\ & + (\xi_t\zeta_a + \zeta_t\xi_a)(\mu_g\epsilon_a + \epsilon_g\mu_a) \\ & - \xi_t\zeta_g(\zeta_t^2 + \zeta_a^2) - \zeta_t\xi_g(\xi_t^2 + \xi_a^2) \\ & - \mu_t\mu_g(\epsilon_t^2 + \epsilon_a^2) - \epsilon_t\epsilon_g(\mu_t^2 + \mu_a^2)]s^2 \\ & + (\mu_g\epsilon_g - \xi_g\zeta_g)[\xi_t^2 + \xi_a^2 + \zeta_t^2 + \zeta_a^2 \\ & - 2(\mu_t\epsilon_t - \mu_a\epsilon_a) - 4\xi_a\zeta_a]c^2\} \end{aligned} \tag{B.3}$$

$$\begin{aligned} f_1(\hat{k}) = {} & 2\omega^3 c(\mu_g\epsilon_g - \xi_g\zeta_g)[\xi_a(\zeta_t^2 + \zeta_a^2) - \zeta_a(\xi_t^2 + \xi_a^2) \\ & + (\xi_t - \zeta_t)(\mu_t\epsilon_a + \epsilon_t\mu_a) \\ & - (\xi_a - \zeta_a)(\mu_t\epsilon_t - \mu_a\epsilon_a)] \end{aligned} \tag{B.4}$$

$$\begin{aligned} f_0 = {} & \omega^4(\mu_g\epsilon_g - \xi_g\zeta_g)[(\mu_t^2 + \mu_a^2)(\epsilon_t^2 + \epsilon_a^2) \\ & + (\xi_t^2 + \xi_a^2)(\zeta_t^2 + \zeta_a^2) \\ & - 2(\xi_t\zeta_t - \xi_a\zeta_a)(\mu_t\epsilon_t - \mu_a\epsilon_a) \\ & - 2(\xi_t\zeta_a + \zeta_t\xi_a)(\mu_t\epsilon_a + \epsilon_t\mu_a)]. \end{aligned} \tag{B.5}$$

ACKNOWLEDGMENT

The authors would like to thank Prof. I. V. Lindell of Helsinki University of Technology, Finland, for providing a copy of the paper [2].

REFERENCES

[1] T. W. Johnston, “Plane-wave dispersion in gyrotropic media,” *Radio Sci.*, vol. 4, no. 8, pp. 729–732, 1969.

[2] I. V. Lindell, “Wave normal and ray propagation in lossless positive bianisotropic media,” *Acta Polytechnica Scandinavica*, Elect. Eng. Ser. El31, Helsinki, Finland, 1972.

[3] ——, “Coordinate independent dyadic formulation of wave normal and ray surfaces of general anisotropic media,” *J. Math Phys.*, vol. 14, no. 1, pp. 65–67, 1973.

[4] H. C. Chen, “A coordinate-free approach to wave reflection from an anisotropic medium,” *Radio Sci.*, vol. 16, no. 6, pp. 1213–1215, 1981.

[5] ——, *Theory of Electromagnetic Waves—A Coordinate-Free Approach*. New York: McGraw-Hill, 1983.

[6] R. D. Graglia, P. L. E. Uslenghi, and R. E. Zich, “Dispersion relation for bianisotropic materials and its symmetry properties,” *IEEE Trans. Antennas Propagat.*, vol. 39, pp. 83–90, Jan. 1991.

[7] J. A. Kong, *Electromagnetic Wave Theory*. New York: Wiley, 1990.

[8] I. V. Lindell, *Methods for Electromagnetic Field Analysis*. Oxford, U.K.: Clarendon, 1992.

[9] I. V. Lindell, A. H. Sihvola, S. A. Tretyakov, and A. J. Viitanen, *Electromagnetic Waves in Chiral and Bi-Isotropic Media*. London, U.K.: Artech House, 1994.

[10] N. Engheta, D. L. Jaggard, and M. W. Kowarz, “Electromagnetic waves in Faraday chiral media,” *IEEE Trans. Antennas Propagat.*, vol. 40, pp. 367–374, Apr. 1992.

[11] I. V. Lindell, S. A. Tretyakov, and A. J. Viitanen, “Plane-wave propagation in a uniaxial chiro-omega medium,” *Microwave Opt. Technol. Lett.*, vol. 6, no. 9, pp. 517–520, 1993.

[12] S. A. Tretyakov and A. A. Sochava, “Eigenwaves in uniaxial chiral omega media,” *Microwave Opt. Technol. Lett.*, vol. 6, no. 12, pp. 701–703, 1993.

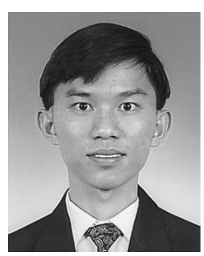
[13] I. V. Lindell and A. J. Viitanen, “Plane wave propagation in uniaxial bianisotropic medium,” *Electron. Lett.*, vol. 29, no. 2, pp. 150–151, 1993.

[14] I. V. Lindell, A. J. Viitanen, and P. K. Koivisto, “Plane-wave propagation in transversely bianisotropic uniaxial medium,” *Microwave Opt. Technol. Lett.*, vol. 6, no. 8, pp. 478–481, 1993.

[15] P. G. Cottis, “Radiation in an unbounded gyroelectric medium,” *J. Electromagn. Waves Applicat.*, vol. 12, no. 2, pp. 249–268, 1998.

[16] E. L. Tan and S. Y. Tan, “On the eigenfunction expansions of the dyadic Green’s functions for bianisotropic media,” *Progress Electromagn. Res.*, PIER 20, pp. 227–247, 1998.

[17] J. L. Tsalamengas, “Electromagnetic fields of elementary dipole antennas embedded in stratified general gyrotropic media,” *IEEE Trans. Antennas Propagat.*, vol. 37, pp. 399–403, Mar. 1989.



Eng Leong Tan received the B.Eng. degree (first class honors) in electrical engineering from the University of Malaya, Malaysia, in 1996.

He is presently a graduate student at the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore. His current research interests are in electromagnetic scattering and multilayered complex media.

Soon Yim Tan received the B.Eng. degree (first class honors) in electrical engineering from the University of Malaya, Malaysia, in 1991, and the Ph.D. degree from the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore, in 1996.

Currently, he is an Assistant Professor at the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore. His current research interests are in electromagnetic scattering, propagation models, and measurements for microcellular communication systems.