The Foster Reactance Theorem for Antennas and Radiation Q

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***Abstract*—The calculation of antenna Q has been an interesting and a controversial topic for years. In this paper, we first give a rigorous study of antenna Q by introducing a complete description of the complex power balance relation for an antenna system. Using the complex Poynting theorem, we have shown that the antenna is essentially equivalent to a one port lossy network. The Foster reactance theorem is usually stated for a lossless network. The main purpose of this paper is to determine whether the Foster reactance theorem holds for antennas. By making use of a complex frequency domain version of the Poynting theorem, we have shown that the Foster reactance theorem is valid for an antenna. Finally, the Foster reactance theorem for the antenna has been applied to demonstrate the widely held assumption $Q \approx 1/B$, provided $Q \gg 1$, where B stands for the fractional bandwidth of an arbitrary antenna.**

***Index Terms*—Antenna input impedance, energy storage, Q factor.**

I. Introduction

THE *quality factor* or $\boldsymbol{Q}$ is used to describe the high-input reactance and narrow bandwidth of small antennas. The Q of a system is usually defined as [1]

$$Q = \frac{\omega \tilde{W}}{P} \tag{1}$$

where $\tilde{W}$ is the time average energy stored in the system and P is the average dissipated power. In most applications of this definition, the Q is evaluated at the resonant frequency. In this case the Q can be expressed as

$$Q = \frac{2\omega \tilde{W}_i}{P} \tag{2}$$

where $\tilde{W}_i$ is either the average stored electric or magnetic energy. For a nonresonant antenna, it is tacitly assumed that the antenna system should be tuned to resonance. Resonance is achieved by adding a capacitive or inductive energy storage element depending on whether the stored energy is predominantly magnetic or electric. In this case, $\tilde{W}_i$ is chosen as average stored magnetic energy $\tilde{W}_m$ or the average stored electric energy $\tilde{W}_e$ in the near-field zone around the antenna, whichever is larger, and P is the total radiated power. The parameter defined by (2) has been referred to as antenna Q.

The evaluation of antenna Q can be traced back to the classical work of Chu [2] who derived the theoretical value of Q for an ideal antenna enclosed in an imaginary sphere. Chu's analysis is based on the spherical TE and TM mode expansions produced by sources distributed on the surface of the imaginary sphere. The major shortcomings of the Chu method are that it is restricted to spherical modes and requires several approximations. Collin and Rothschild also presented a method for evaluating antenna Q. Their method is based on recognizing that the total stored reactive energy can be calculated by subtracting the radiated field energy away from the total energy in the field. Such method has been successfully used by Fante and reexamined by McLean to study antenna Q [5], [6].

Although the concept of antenna Q was proposed many years ago, its definition, calculation, and relation to system bandwidth have long been controversial topics [7]–[14]. Some authors even questioned the applicability of the complex Poynting theorem in the calculation of antenna Q [12]–[14].

In this paper, we try to give a more rigorous discussion of antenna Q. In Section II, we present a complete description of the complex power balance relation for electromagnetic fields, hopefully removing the misunderstanding associated with the complex Poynting theorem. To complete this paper we give a derivation of the RLC equivalent circuit of an antenna directly from the complex Poynting theorem. It is well known that the Foster reactance theorem holds only for a lossless network. From the point of view of circuit theory, an antenna is a one-port network with loss. The loss represents the radiated power from the antenna and this loss prohibits the direct use of the Foster reactance theorem. A formula for the frequency derivative of the input reactance of any linear, passive, time-invariant electromagnetic system has been derived by Rhodes [8], [9]. The formula consists of five terms and has several frequency derivatives of field quantities. So it is very difficult to figure out how the frequency derivative of the input reactance changes with the frequency. In Section IV, we make use of a complex frequency domain approach to demonstrate that the Foster reactance theorem (i.e., the frequency derivative of the input reactance of the antenna is always greater than zero) still holds for an arbitrary antenna system. Antenna engineers have used the fact that the antenna fractional bandwidth B is approximately equal to the reciprocal of antenna Q for a high Q system, but no explicit demonstration has been given. Wheeler has pointed out that the common expression for bandwidth in terms of $1/Q$ is neither logical nor helpful in clear exposition [15]. It is believed that such a relationship must exist for any complete radiating system, just as surely as it does for a nonradiating system. Another common belief is that the relation cannot be proven.[1] Rhodes has shown that the reciprocal relationship between Q and B does hold for a planar dipole [7]. Fante has tried to give

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[1] D. R. Rhodes, "Author's reply," *IEEE Trans. Antennas Propagat.*, vol. 15, pp. 568–569, July 1967.

a study of antenna Q of general ideal antenna based on the frequency derivatives of Maxwell's equations, which has been the standard way of deriving the Foster reactance theorem. However, such a study, if applied to an antenna system, may yield a surface integral of the product between field and its frequency derivative and require an assumption that either the terminal voltage or the current is a constant [5]. An exact analysis for such an integral is essentially difficult. In Section V, we apply the Foster theorem for antenna system to give a simple demonstration of the widely held assumption $Q \approx 1/B$.

II. Poynting Theorem and Evaluation of Q Using Field Equations

The differential form of the complex Poynting theorem for time harmonic field in an isotropic medium is given by

$$\nabla \cdot \vec{S} = -\tfrac{1}{2}\,\vec{J}^* \cdot \vec{E} - j2\omega(\langle w_m\rangle - \langle w_e\rangle) \tag{3}$$

where $\vec{S} = \frac{1}{2}\,\vec{E} \times \vec{H}^*$ is the complex Poynting vector $\langle w_m\rangle = \mu\vec{H}\cdot\vec{H}^*/4$ and $\langle w_e\rangle = \varepsilon\vec{E}\cdot\vec{E}^*/4$ are the magnetic and electric field energy densities.

Let V_0 be the volume occupied by the electric current source $\vec{J}$ and ∂V_0 be the surface surrounding V_0, take the integration of the imaginary part of (3) over a volume V containing V_0, as shown in Fig. 1, we obtain

$$\mathrm{Im}\int_{\partial V} \vec{n}\cdot\vec{S}\,ds = \mathrm{Im}\int_{V_0} -\tfrac{1}{2}\,\vec{J}^*\cdot\vec{E}\,dv - 2\omega\int_V (\langle w_m\rangle - \langle w_e\rangle)\,dv \tag{4}$$

where ∂V is the boundary of V. Choosing $V = V_0$, we have

$$\mathrm{Im}\int_{\partial V_0} \vec{n}\cdot\vec{S}\,ds = \mathrm{Im}\int_{V_0} -\frac{1}{2}\,\vec{J}^*\cdot\vec{E}\,dv - 2\omega\int_{V_0} (\langle w_m\rangle - \langle w_e\rangle)\,dv \tag{5}$$

and if we choose $V = V_\infty$, where V_∞ is the region enclosed by a sphere with radius r_∞, where r_∞ is sufficiently large so that it lies in the far field region of the antenna system, we get

$$\mathrm{Im}\int_{\partial V_\infty} \vec{n}\cdot\vec{S}\,ds = \mathrm{Im}\int_{V_0} -\tfrac{1}{2}\,\vec{J}^*\cdot\vec{E}\,dv - 2\omega\int_{V_\infty} (\langle w_m\rangle - \langle w_e\rangle)\,dv. \tag{6}$$

Since $\vec{S}$ is a real vector in the far-field region, we have

$$\mathrm{Im}\int_{V_0} -\tfrac{1}{2}\,\vec{J}^*\cdot\vec{E}\,dv = 2\omega\int_{V_\infty} (\langle w_m\rangle - \langle w_e\rangle)\,dv. \tag{7}$$

From (4), (5), and (7), we obtain

$$\mathrm{Im}\int_{\partial V_0} \vec{n}\cdot\vec{S}\,ds = 2\omega\int_{V_\infty - V_0} (\langle w_m\rangle - \langle w_e\rangle)\,dv \tag{8}$$

$$\mathrm{Im}\int_{\partial V} \vec{n}\cdot\vec{S}\,ds = 2\omega\int_{V_\infty - V} (\langle w_m\rangle - \langle w_e\rangle)\,dv. \tag{9}$$

Take the integration of the real part of (3) over the volume V containing the source region V_0, we obtain the radiated power

$$P^{\mathrm{rad}} = \mathrm{Re}\int_{\partial V} \vec{n}\cdot\vec{S}\,ds = \mathrm{Re}\int_{V_0} -\tfrac{1}{2}\,\vec{J}^*\cdot\vec{E}\,dv. \tag{10}$$

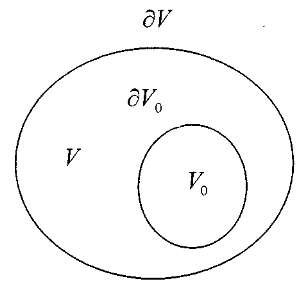

Fig. 1. A volume V containing source region V_0.

As mentioned by McLean [6] and Grimes [13], some authors implicitly define reactive power to be the imaginary part of the surface integral of the complex Poynting vector. This is a misleading definition. Actually (10) shows that the surface integral of the real part of the Poynting vector is independent of the surface ∂V as long as it encloses the source region V_0. Equations (8) and (9) show that the surface integral of the imaginary part of the Poynting vector depends on the integration surface ∂V in the near field region (in the far field region it becomes zero).

Using (4), (10), and (7) we get

$$\begin{aligned}\int_{\partial V} \vec{n}\cdot\vec{S}\,ds &= \int_{V_0} -\tfrac{1}{2}\,\vec{J}^*\cdot\vec{E}\,dv - j2\omega\int_V (\langle w_m\rangle - \langle w_e\rangle)\,dv\\ &= P^{\mathrm{rad}} + j\,\mathrm{Im}\int_{V_0} -\frac{1}{2}\,\vec{J}^*\cdot\vec{E}\,dv - j2\omega\int_V (\langle w_m\rangle - \langle w_e\rangle)\,dv\\ &= P^{\mathrm{rad}} + j2\omega\int_{V_\infty - V} (\langle w_m\rangle - \langle w_e\rangle)\,dv.\end{aligned} \tag{11}$$

The above relation indicates that the complex power flowing out of ∂V is equal to the radiation power plus the reactive power outside ∂V. This expression seems to be the most general form for the Poynting theorem. Most of the writers directly explain the third term on the second line of (11) as the reactive energy stored around the antenna. In Collin's and Rhode's work [3], [7], the second term of the right-hand side of (11) was missed. Collin and Fante's work [4], which is based on Collin's method, have been questioned by other authors [12]–[14]. These authors even proposed that the complex Poynting theorem is intrinsically an insufficient basis for a full description of power in a radiated field and resorted to the time dependent Poynting theorem. Such a misunderstanding is actually caused by an improper explanation of the power balance relation (11).

To clarify this point, let $\langle\tilde{w}_e\rangle(\langle w_e^{\mathrm{rad}}\rangle)$ and $\langle\tilde{w}_m\rangle$ $(\langle w_m^{\mathrm{rad}}\rangle)$ denote the stored (radiated) electric field and magnetic field energy densities, respectively, we can then define

$$\begin{aligned}\langle\tilde{w}_m\rangle &= \langle w_m\rangle - \langle w_m^{\mathrm{rad}}\rangle\\ \langle\tilde{w}_e\rangle &= \langle w_e\rangle - \langle w_e^{\mathrm{rad}}\rangle.\end{aligned} \tag{12}$$

These calculations are physically appropriate since density is a summable quantity. It is readily seen from (9) that $\langle w_m\rangle = \langle w_e\rangle$ in the far field zone since the complex Poynting vector becomes real as V approaches V_∞. This observation indicates that the electromagnetic field energy and the magnetic field energy for the radiated field are identical everywhere. To be explicit, we cite the far-field expression generated by an arbitrary current distribution as follows:

$$\vec{E}^{\rm rad}(\vec{r}_\infty) = -\frac{j\omega\mu}{4\pi r_\infty}e^{-jkr_\infty}\int_{V_0}[\vec{J}-(\vec{J}\cdot\hat{r}_\infty)]e^{jk\hat{r}_\infty\cdot\vec{r}'}\,dv$$
$$\vec{H}^{\rm rad}(\vec{r}_\infty) = -\eta\frac{j\omega\varepsilon}{4\pi r_\infty}e^{-jkr_\infty}\int_{V_0}[\hat{r}_\infty\times\vec{J}]e^{jk\hat{r}_\infty\cdot\vec{r}'}\,dv \tag{13}$$

where $\vec{r}_\infty$ is an observation point in the far-field region and $\hat{r}$ is a unit vector along $\vec{r}$. From the above expression, we easily obtain

$$\vec{E}^{\rm rad}(\vec{r}_\infty) = \eta\vec{H}^{\rm rad}(\vec{r}_\infty)\times\hat{r}_\infty. \tag{14}$$

Hence, we have

$$\langle w_e^{\rm rad}\rangle = \tfrac{1}{4}\varepsilon\vec{E}^{\rm rad}\cdot\vec{E}^{\rm rad*} = \tfrac{1}{4}\mu\vec{H}^{\rm rad}\cdot\vec{H}^{\rm rad*} = \langle w_m^{\rm rad}\rangle \tag{15}$$

and the total energy of the radiated fields is simply twice the electric or magnetic energy density of the radiated fields. Mathematically, (15) holds everywhere. Therefore we obtain

$$\begin{aligned}\tilde{W}_m - \tilde{W}_e &= \int_{V_\infty - V_0}[\langle\tilde{w}_m\rangle - \langle\tilde{w}_e\rangle]\,dv\\ &= \int_{V_\infty - V_0}[\langle w_m\rangle - \langle w_e\rangle]\,dv\\ &= \frac{1}{2\omega}\,{\rm Im}\int_{\partial V_0}\vec{S}\cdot\vec{n}\,ds\end{aligned} \tag{16}$$

from (8), (9), and (12). Here, $\tilde{W}_m$ and $\tilde{W}_e$ stand for the total stored magnetic and electric energy in the volume surrounding the radiator. From (12) the total stored reactive energy can be expressed as

$$\begin{aligned}&\tilde{W}_e + \tilde{W}_m\\ &= \int_{V_\infty - V_0}(\langle w_e\rangle - \langle w_e^{\rm rad}\rangle + \langle w_m\rangle - \langle w_m^{\rm rad}\rangle)\,dv\\ &= \left\{\int_{V_\infty - V_0}(\langle w_e\rangle + \langle w_m\rangle)\,dv - \int_{V_\infty - V_0}(\langle w_e^{\rm rad}\rangle\right.\\ &\qquad \left. + \langle w_m^{\rm rad}\rangle)\,dv\right\}\\ &= \left\{\int_{V_\infty - V_0}(\langle w_e\rangle + \langle w_m\rangle)\,dv - \frac{r_\infty}{c}\,{\rm Re}\int_{\partial V_0}\vec{S}\cdot\vec{n}\,ds\right\}\end{aligned} \tag{17}$$

where r_∞ is the radius of the sphere ∂V_∞, and c is the wave velocity. Both terms in the curved bracket are divergent as $r_\infty \to \infty$, but it can be shown that the net term in the curved bracket is convergent [5]. The following has been used in deriving (17)

$${\rm Re}\int_{\partial V_0}\vec{S}\cdot\vec{n}\,ds = {\rm Re}\int_{\partial V_\infty}\vec{S}\cdot\vec{n}\,ds.$$

So the stored electric and magnetic field energy will be given by

$$\begin{aligned}\tilde{W}_m &= \tfrac{1}{2}\left(\left\{\int_{V_\infty - V_0}(\langle w_e\rangle + \langle w_m\rangle)\,dv - \frac{r_\infty}{c}\,{\rm Re}\int_{\partial V_0}\vec{S}\cdot\vec{n}\,ds\right\} + \frac{1}{2\omega}\,{\rm Im}\int_{\partial V_0}\vec{S}\cdot\vec{n}\,ds\right)\\ \tilde{W}_e &= \tfrac{1}{2}\left(\left\{\int_{V_\infty - V_0}(\langle w_e\rangle + \langle w_m\rangle)\,dv - \frac{r_\infty}{c}\,{\rm Re}\int_{\partial V_0}\vec{S}\cdot\vec{n}\,ds\right\} - \frac{1}{2\omega}\,{\rm Im}\int_{\partial V_0}\vec{S}\cdot\vec{n}\,ds\right).\end{aligned} \tag{18}$$

The antenna Q can be calculated as

$$\begin{cases} Q = \dfrac{2\omega\tilde{W}_m}{P^{\rm rad}}, & \text{if } \tilde{W}_m > \tilde{W}_e\\ Q = \dfrac{2\omega\tilde{W}_e}{P^{\rm rad}}, & \text{if } \tilde{W}_e > \tilde{W}_m.\end{cases} \tag{19}$$

The above calculation has been used by McLean to calculate Q directly from the fields of the TM_{01} spherical mode. This approach is actually equivalent to Collin's method.

III. The Antenna as a One-Port Device

In the following, we choose V_0 in such a way that ∂V_0 is coincident with the antenna surface (except for a portion A where ∂V_0 crosses the antenna feed plane), as shown in Fig. 2.

Taking the integration of (3) and using the divergence theorem over the region $V_\infty - V_0$ we get

$$\begin{aligned}&\tfrac{1}{2}\int_{\partial V_\infty}(\vec{E}\times\vec{H}^*)\cdot d\vec{s} + \tfrac{1}{2}\int_{\partial V_0}(\vec{E}\times\vec{H}^*)\cdot d\vec{s}\\ &\quad = -j2\omega\int_{V_\infty - V_0}(\langle w_m\rangle - \langle w_e\rangle)\,dv\\ &\qquad - \tfrac{1}{2}\int_{V_\infty - V_0}\sigma\vec{E}\cdot\vec{E}^*\,dv\end{aligned} \tag{20}$$

where σ is the conductivity of the medium in $V_\infty - V_0$. If we assume the antenna surface is perfectly conducting, $\vec{E}\times\vec{H}^*$ vanishes everywhere on ∂V_0 except over the input terminal A. For a single-mode transmission line we have

$$\tfrac{1}{2}\int_{\partial V_0}(\vec{E}\times\vec{H}^*)\cdot d\vec{s} = \tfrac{1}{2}\int_A(\vec{E}\times\vec{H}^*)\cdot d\vec{s} = -\tfrac{1}{2}VI^* \tag{21}$$

where V and I are equivalent voltage and current at the feeding plane respectively. Introducing (21) into (20) and using the fact that $P^{\rm rad} = \frac{1}{2}\int_{\partial V_\infty}(\vec{E}\times\vec{H}^*)\cdot d\vec{s}$, we find

$$\tfrac{1}{2}VI^* = P^{\rm rad} + P^{\rm loss} + j2\omega\int_{V_\infty - V_0}(\langle w_m\rangle - \langle w_e\rangle)\,dv \tag{22}$$

where

$$P^{\rm loss} = \frac{1}{2}\int_{V_\infty - V_0}\sigma\vec{E}\cdot\vec{E}^*\,dv \tag{23}$$

is the power loss.

The antenna impedance Z_A is defined by

$$Z_A = \frac{V}{I} \tag{24}$$

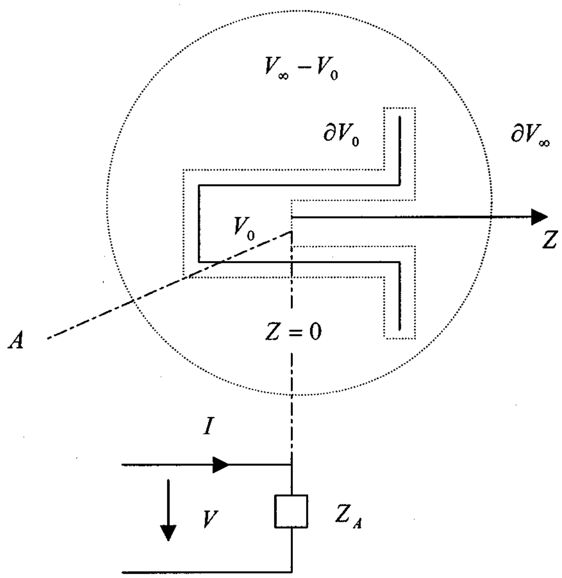

Fig. 2. An arbitrary antenna and its equivalent circuit.

and by using (16), we can rewrite (22) as

$$\begin{aligned} Z_A &= \frac{2P^{\rm rad}}{|I|^2} + \frac{2P^{\rm loss}}{|I|^2} + j\,\frac{4\omega(\tilde{W}_m - \tilde{W}_e)}{|I|^2} \\ &= R_A^{\rm rad} + R_A^{\rm loss} + jX_A \\ &= R_A^{\rm rad} + R_A^{\rm loss} + j\left(\omega L_A - \frac{1}{\omega C_A}\right) \end{aligned} \tag{25}$$

where

$$\begin{aligned} R_A^{\rm rad} &= \frac{2P^{\rm rad}}{|I|^2}, \quad R_A^{\rm loss} = \frac{2P^{\rm loss}}{|I|^2} \\ X_A &= \omega L_A - \frac{1}{\omega C_A} = \frac{4\omega(\tilde{W}_m - \tilde{W}_e)}{|I|^2} \\ L_A &= \frac{4\tilde{W}_m}{|I|^2}, \quad C_A = \frac{|I|^2}{4\omega^2 \tilde{W}_e}. \end{aligned} \tag{26}$$

Hence, the equivalent circuit for the antenna is shown in Fig. 3.

IV. Foster Reactance Theorem for Antennas and Evaluation of Antenna Q Using Equivalent Circuits

The Foster reactance theorem is a very important tool for the synthesis of networks. In standard textbooks, the Foster reactance theorem is usually stated for a loss-free network. It has been shown that an antenna system is equivalent to a one port lossy network ($R_A^{\rm rad} \neq 0$). So the question may be raised whether the Foster reactance theorem could apply for an antenna system. In the following section, we will attempt to answer this question. First, we introduce a complex frequency $\tilde{s} = \alpha + j\omega$ and all calculations will be confined to the complex frequency plane. For clarity, all quantities in the complex frequency plane will be embellished using this symbol $\breve{}$ to distinguish them from the corresponding quantities in the frequency domain. Taking the Laplace transform of the time-domain Maxwell's equations in a lossless medium we have

$$\nabla \times \breve{\vec{E}}(\vec{r}, \tilde{s}) = -\tilde{s}\mu\breve{\vec{H}}$$

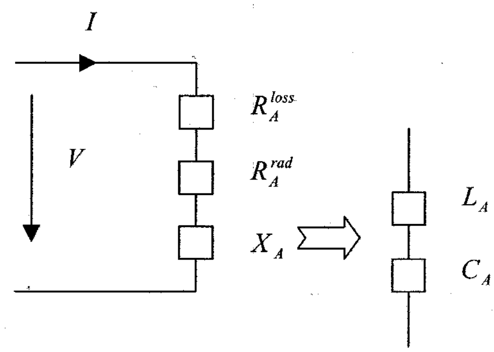

Fig. 3. Impedance of an arbitrary antenna.

$$\nabla \times \breve{\vec{H}}(\vec{r}, \tilde{s}) = \tilde{s}\varepsilon\breve{\vec{E}}. \tag{27}$$

All the frequency-domain quantities can be recovered by letting $\alpha = 0$ in (27). From (27) a relation similar to (3) can be obtained in the region outside V_0

$$\begin{aligned} &\nabla \cdot \left(\tfrac{1}{2}\,\breve{\vec{E}}(\vec{r}, \tilde{s}) \times \breve{\vec{H}}^*(\vec{r}, \tilde{s})\right) \\ &\quad = -\tfrac{1}{2}\,\alpha(\mu|\breve{\vec{H}}(\vec{r}, \tilde{s})|^2 + \varepsilon|\breve{\vec{E}}(\vec{r}, \tilde{s})|^2) \\ &\qquad - j\,\tfrac{1}{2}\,\omega(\mu|\breve{\vec{H}}|^2 - \varepsilon|\breve{\vec{E}}|^2). \end{aligned} \tag{28}$$

If we take the integration of (28) over the connected region $V_\infty - V_0$, as shown in Fig. 2, we find the relation

$$\begin{aligned} &\int_{\partial V_0 + \partial V_\infty} \tfrac{1}{2}\,(\breve{\vec{E}}(\vec{r}, \tilde{s}) \times \breve{\vec{H}}^*(\vec{r}, \tilde{s})) \cdot d\vec{s} \\ &\quad = -2\alpha(\breve{W}_m + \breve{W}_e) - 2j\omega(\breve{W}_m - \breve{W}_e) \end{aligned} \tag{29}$$

where

$$\begin{aligned} \breve{W}_m &= \tfrac{1}{4} \int_{V_\infty - V_0} \mu|\breve{\vec{H}}|^2\, dv \geq 0 \\ \breve{W}_e &= \tfrac{1}{4} \int_{V_\infty - V_0} \varepsilon|\breve{\vec{E}}|^2\, dv \geq 0. \end{aligned} \tag{30}$$

The above quantities have no physical interpretations except when $\alpha = 0$. Following a discussion similar to the previous section, we have

$$\begin{aligned} &\tfrac{1}{2} \int_{\partial V_0} (\breve{\vec{E}}(\vec{r}, \tilde{s}) \times \breve{\vec{H}}^*(\vec{r}, \tilde{s})) \cdot d\vec{s} \\ &\quad = \tfrac{1}{2} \int_A (\breve{\vec{E}}(\vec{r}, \tilde{s}) \times \breve{\vec{H}}^*(\vec{r}, \tilde{s})) \cdot d\vec{s} \\ &\quad = -\tfrac{1}{2}\,\breve{V}(\tilde{s})\breve{I}^*(\tilde{s}) \end{aligned} \tag{31}$$

for a single-mode transmission line. Define $\breve{P}^{\rm rad} = \frac{1}{2}\int_{\partial V_\infty} (\breve{\vec{E}}(\vec{r}_\infty, \tilde{s}) \times \breve{\vec{H}}^*(\vec{r}_\infty, \tilde{s})) \cdot d\vec{s}$ and substitute it into (29) to get

$$\tfrac{1}{2}\,\breve{V}(\tilde{s})\breve{I}^*(\tilde{s}) = \breve{P}^{\rm rad} + 2\alpha(\breve{W}_m + \breve{W}_e) + 2j\omega(\breve{W}_m - \breve{W}_e). \tag{32}$$

If α is sufficiently small so that $\alpha \ll c/r_\infty$ we can make a first order approximation $e^{-\alpha r_\infty/c} \approx 1 - \alpha r_\infty/c$ and derive directly from the Maxwell equations, defined in the complex plane as shown in (33) at the bottom of the page. So we have

$$\breve{P}^{\rm rad} = P^{\rm rad}\left(1 - r_\infty \frac{\alpha}{c}\right)^2 \approx \left(1 - r_\infty \frac{2\alpha}{c}\right) P^{\rm rad} \tag{34}$$

where $P^{\rm rad}$, previously defined in (10), is radiated power in the frequency domain which is independent of α. Substituting (34) into (32) we obtain

$$\begin{aligned}\tfrac{1}{2}\breve{V}(\tilde{s})\breve{I}^*(\tilde{s}) = &P^{\rm rad} + 2\alpha\left[\breve{W}_m + \breve{W}_e - \frac{r_\infty}{c}P^{\rm rad}\right]\\ &+ 2j\omega(\breve{W}_m - \breve{W}_e).\end{aligned} \tag{35}$$

The impedance in the complex frequency plane can then be expressed as

$$\begin{aligned}\breve{Z}_A(\tilde{s}) = &\frac{2P^{\rm rad}}{|\breve{I}(\tilde{s})|^2} + \frac{4\alpha\left[\breve{W}_m + \breve{W}_e - \frac{r_\infty}{c}P^{\rm rad}\right]}{|\breve{I}(\tilde{s})|^2}\\ &+ \frac{4j\omega(\breve{W}_m - \breve{W}_e)}{|\breve{I}(\tilde{s})|^2}.\end{aligned} \tag{36}$$

If α is sufficiently small, we can make the first order approximation $\breve{I}(s) \approx I$ which is independent of α. So (36) can be rewritten as

$$\begin{aligned}\breve{Z}_A(\tilde{s}) = &\frac{2P^{\rm rad}}{|I|^2} + \frac{4\alpha\left[\breve{W}_m + \breve{W}_e - \frac{r_\infty}{c}P^{\rm rad}\right]}{|I|^2}\\ &+ \frac{4j\omega(\breve{W}_m - \breve{W}_e)}{|I|^2}.\end{aligned} \tag{37}$$

Now we decompose the complex impedance $\breve{Z}(\tilde{s})$ into the sum of its real and imaginary parts, i.e.,

$$\breve{Z}_A(\tilde{s}) = \breve{R}_A(\alpha, \omega) + j\breve{X}_A(\alpha, \omega) \tag{38}$$

where

$$\breve{R}_A(\alpha, \omega) = \frac{2P^{\rm rad}}{|I|^2} + \frac{4\alpha\left[\breve{W}_m + \breve{W}_e - \frac{r_\infty}{c}P^{\rm rad}\right]}{|I|^2} \tag{39}$$

$$\breve{X}_A(\alpha, \omega) = \frac{4\omega(\breve{W}_m - \breve{W}_e)}{|I|^2} = \omega\breve{L}_A(\alpha, \omega) - \frac{1}{\omega\breve{C}_A(\alpha, \omega)} \tag{40}$$

and the definitions of $\breve{L}_A$ and $\breve{C}_A$ are similar to L_A and C_A, respectively. Since $\breve{Z}(\tilde{s})$ is an analytic function its real and imaginary parts satisfy the Cauchy–Riemann conditions, i.e.,

$$\frac{\partial \breve{R}_A(\alpha, \omega)}{\partial \alpha} = \frac{\partial \breve{X}_A(\alpha, \omega)}{\partial \omega} \tag{41a}$$

$$\frac{\partial \breve{R}_A(\alpha, \omega)}{\partial \omega} = -\frac{\partial \breve{X}_A(\alpha, \omega)}{\partial \alpha}. \tag{41b}$$

By direct calculation we have

$$\begin{aligned}\left.\frac{\partial \breve{R}_A(\alpha, \omega)}{\partial \alpha}\right|_{\alpha=0} &= \frac{4\left(W_m + W_e - \frac{r_\infty}{c}P^{\rm rad}\right)}{|I|^2}\\ &= \frac{4(\tilde{W}_m + \tilde{W}_e)}{|I|^2}\end{aligned} \tag{42}$$

where use of (17) has been made. From (41a) and (42) we obtain

$$\frac{\partial X_A}{\partial \omega} = \left.\frac{\partial \breve{X}_A}{\partial \omega}\right|_{\alpha=0} = \frac{4(\tilde{W}_m + \tilde{W}_e)}{|I|^2} > 0 \tag{43}$$

which is the Foster reactance theorem for an antenna system.

A formula for the frequency derivative the of input reactance of any linear, passive, time-invariant electromagnetic system has been derived by Rhodes [8], [9]. This formula contains four integrals of frequency derivatives of field quantities, making it difficult to obtain any useful information from it.

The above discussion can be easily generalized to an antenna array system with N input ports. In this case (35) should be changed to

$$\begin{aligned}\tfrac{1}{2}[\breve{I}^*(\tilde{s})]^t[\breve{V}(\tilde{s})] &= \tfrac{1}{2}\sum_{i=1}^{N}\breve{V}_i(\tilde{s})\breve{I}_i^*(\tilde{s})\\ &= P^{\rm rad} + 2\alpha\left[\breve{W}_m + \breve{W}_e - \frac{r_\infty}{c}P^{\rm rad}\right]\\ &\quad + 2j\omega(\breve{W}_m - \breve{W}_e)\end{aligned} \tag{44}$$

where $[\tilde{V}(s)]$ and $[\tilde{I}(s)]$ are the voltage and current vectors at the antenna array terminals. Introducing the impedance matrix of the antenna array

$$[\breve{Z}_A(s)] = [\breve{R}_A(\alpha, \omega)] + j[\breve{X}_A(\alpha, \omega)] \tag{45}$$

$$\begin{aligned}\vec{\breve{E}}^{\rm rad}(\vec{r}_\infty, \tilde{s}) &\approx -\left(1 - r_\infty\frac{\alpha}{c}\right)\frac{j\omega\mu}{4\pi r_\infty}e^{-jkr_\infty}\int_{V_0}[\vec{J} - (\vec{J}\cdot\hat{r}_\infty)]e^{-jk\hat{r}_\infty\cdot\vec{r}'}\,dv\\ \vec{\breve{H}}^{\rm rad}(\vec{r}_\infty, \tilde{s}) &\approx -\eta\left(1 - r_\infty\frac{\alpha}{c}\right)\frac{j\omega\varepsilon}{4\pi r_\infty}e^{-jkr_\infty}\int_{V_0}[\hat{r}_\infty \times \vec{J}]e^{-jk\hat{r}_\infty\cdot\vec{r}'}\,dv\end{aligned} \qquad \vec{r}_\infty \in \partial V \tag{33}$$

and following a similar discussion we have

$$[I^*]^t[\breve{R}_A(\alpha,\omega)][I] = 2P^{\rm rad} + 4\alpha\left[\breve{W}_m + \breve{W}_e - \frac{r_\infty}{c}P^{\rm rad}\right] \tag{46}$$

$$[I^*]^t[\breve{X}_A(\alpha,\omega)][I] = 4\omega(\breve{W}_m - \breve{W}_e). \tag{47}$$

Instead of (42) we have

$$[I^*]^t\left[\frac{\partial \breve{R}_A(\alpha,\omega)}{\partial\alpha}\right]\Bigg|_{\alpha=0}[I] = 4\left(W_m + W_e - \frac{r_\infty}{c}P^{\rm rad}\right) = 4(\tilde{W}_m + \tilde{W}_e) \tag{48}$$

for a lossless system. So the Foster reactance theorem for the antenna array will take the form

$$[I^*]^t\left[\frac{\partial X_A}{\partial\omega}\right][I] = 4(\tilde{W}_m + \tilde{W}_e) > 0. \tag{49}$$

The above derivation is also applicable to a bounded one port lossless microwave system where $P^{\rm rad} = 0$ [17]. The advantage of such an approach is that we do not need to assume that either the terminal voltage or the current is a constant, as most of the textbooks do to get the same result as (43) or (49).

Now the antenna Q can be calculated from the corresponding equivalent circuit using the Foster reactance theorem. For simplicity, we assume that the antenna system is lossless ($R_A^{\rm loss} = 0$). The impedance of the antenna array for this case will be

$$[Z_A] = [R_A^{\rm rad}] + j[X_A]. \tag{50}$$

From (47) and (49) we obtain the stored magnetic and electric field energy as follows

$$\tilde{W}_m = \frac{1}{8}[I^*]^t\left(\frac{d[X_A]}{d\omega} + \frac{[X_A]}{\omega}\right)[I]$$
$$\tilde{W}_e = \frac{1}{8}[I^*]^t\left(\frac{d[X_A]}{d\omega} - \frac{[X_A]}{\omega}\right)[I]. \tag{51}$$

The radiated power of the antenna array is

$$\begin{aligned}P^{\rm rad} &= \tfrac{1}{2}\,{\rm Re}[I^*]^t[V]\\ &= \tfrac{1}{2}\,{\rm Re}[I^*]^t[Z_A][I]\\ &= \tfrac{1}{4}\,[I^*][[Z_A] + [Z_A^*]^t][I].\end{aligned} \tag{52}$$

So the antenna Q will be given by

$$Q = \frac{2\omega\tilde{W}}{P^{\rm rad}} \tag{53}$$

where $\tilde{W} = \tilde{W}_m$ or $\tilde{W}_e$ whichever is larger. Substituting (51) and (52) into (53) we have

$$Q = \frac{[I^*]^t\left[\omega\,\dfrac{d[X_A]}{d\omega} \pm [X_A]\right][I]}{[I^*]^t[[Z_A] + [Z_A^*]^t][I]} \tag{54}$$

where either the $+$ or $-$ value is chosen to give the higher Q. For a single antenna system (54) is simplified to

$$Q = \frac{\omega\,\dfrac{dX_A}{d\omega} \pm X_A}{2R_A^{\rm rad}}. \tag{55}$$

The term obtained from $\omega(dX_A/d\omega)$ is usually larger than that obtained from X_A for a high Q system [19]. If Q is large, it is assumed that it is related to the frequency bandwidth of the antenna as follows. Consider the antenna to be resonated with a suitable reactive network at the frequency of interest ω_r. The frequency at which the normalized input $|Z_A|$ is equal to $\sqrt{2}$ times its value at resonance, is the half-power point. At this point the average power delivered from the antenna is one-half that delivered at resonance. Let $\Delta\omega$ be defined as the frequency increment, $\Delta\omega$ is the distance in frequency between the half-power points (3-dB points) of the antenna. The fractional bandwidth B is then defined as

$$B = \frac{\Delta\omega}{\omega_r}. \tag{56}$$

If the Q is high the following relationship, which holds for a nonradiating system, is assumed

$$Q \approx \frac{1}{B}. \tag{57}$$

This widely held assumption has never been explicitly proven for distributed systems. As pointed by Wheeler [15], the common expression for the bandwidth in terms of $1/Q$ is neither logical nor helpful in clear exposition. Most writers state that if Q is large compared with unity, it is equal to B^{-1}. Rhodes demonstrated that this relation does hold for the planar dipole by computing Q and bandwidth independently [7] and he firmly believed that such a relation must exist for any complete radiating system, just as surely as it does for nonradiating systems.

A discussion of the relation between B and Q was given by Fante. His approach was based on the frequency derivatives of Maxwell's equations, which has been the standard way of deriving the Foster reactance theorem. For an antenna system, however, a surface integral between the product between field and its frequency derivative must appear in the derivation and an assumption that either the terminal voltage and the current is a constant must be made. An exact analysis for such an integral is essentially difficult [4]. In the next section, we will give a simple demonstration of this observation using the Foster reactance theorem for an antenna system.

V. Antenna Q and Antenna Bandwidth

In this section, we consider a high Q system. Let ω_r denote one of the resonant frequencies of a single antenna system; then by definition we have

$$X_A(\omega_r) = 0. \tag{58}$$

We rewrite (40) at the resonant frequency ω_r as follows:

$$\begin{aligned}\breve{X}_A(\alpha, \omega_r) &= \frac{4\omega_r(\breve{W}_m - \breve{W}_e)}{|I|^2} \\ &= \omega_r \breve{L}_A(\alpha, \omega_r) - \frac{1}{\omega \breve{C}_A(\alpha, \omega_r)} \\ &\approx \omega_r L_A - \frac{1}{\omega_r C_A} = X_A(\omega_r) = 0. \end{aligned}\tag{59}$$

Here we assume that α is sufficiently small so we can make the first-order approximation

$$\begin{aligned}\breve{L}_A(\alpha, \omega_r) &= \breve{L}_A(0, \omega_r) = L_A \\ \breve{C}_A(\alpha, \omega_r) &= \breve{C}_A(0, \omega_r) = C_A.\end{aligned}$$

By (41b) we have

$$\left.\frac{dR_A^{\mathrm{rad}}}{d\omega}\right|_{\omega_r} = -\left.\frac{\partial \tilde{X}_A(\alpha, \omega_r)}{\partial \alpha}\right|_{\alpha=0} \approx 0. \tag{60}$$

Thus, as one moves off resonance, Z_A can be written as

$$Z_A \approx R_A^{\mathrm{rad}} + j(\omega - \omega_r)\left.\frac{dX_A}{d\omega}\right|_{\omega_r} + \cdots. \tag{61}$$

The half-power points occur when

$$R_A^{\mathrm{rad}} = \left|(\omega - \omega_r)\left.\frac{dX_A}{d\omega}\right|_{\omega_r}\right| \tag{62}$$

so that the fractional bandwidth B can be written

$$B = \frac{\Delta\omega}{\omega_r} \approx \frac{2R_A^{\mathrm{rad}}}{\omega_r\left|\left.\frac{dX_A}{d\omega}\right|_{\omega_r}\right|} = \frac{4P^{\mathrm{rad}}}{\omega_r |I|^2\left|\left.\frac{dX_A}{d\omega}\right|_{\omega_r}\right|}. \tag{63}$$

So the problem is to evaluate $dX_A/d\omega$. To do this, Fante employed the frequency derivative of Maxwell's equations and extended the treatment of Harrington's work [19, pp. 394–396]. As a result, he obtained an integral defined on an infinitely large surface for which an asymptotic analysis is very difficult. In addition, he had to assume that either the terminal voltage or the current is a constant. In fact, this can be easily done using the Foster reactance theorem. From (51) we get

$$\begin{aligned}\frac{dX_A}{d\omega} &= \frac{8\tilde{W}_e}{|I|^2} + \frac{X_A}{\omega} \\ \frac{dX_A}{d\omega} &= \frac{8\tilde{W}_m}{|I|^2} - \frac{X_A}{\omega}.\end{aligned}\tag{64}$$

Introducing the above into (63) and using (58), we find

$$\begin{aligned}B &= \frac{4P^{\mathrm{rad}}}{\omega_r |I|^2 \left|\left.\frac{dX_A}{d\omega}\right|_{\omega_r}\right|} \\ &= \frac{4P^{\mathrm{rad}}}{|8\omega_r \tilde{W} \pm |I|^2 X_A|_{\omega_r}} \\ &= \frac{4P^{\mathrm{rad}}}{|8\omega_r \tilde{W}|_{\omega_r}} \\ &= \frac{1}{Q}.\end{aligned}\tag{65}$$

Here $\tilde{W} = \tilde{W}_m$ or $\tilde{W}_e$, whichever is larger.

VI. Conclusion

Although the concept of antenna Q was proposed many years ago, its definition, calculation and relationship to system bandwidth have long been controversial topics. A recent discussion focuses on the applicability of the well-known complex Poynting theorem to the calculation of antenna Q [1], [12]–[14]. In this paper, we have shown that the complex Poynting theorem can be used in the calculation of antenna Q and one does not need to revert to the time domain Poynting theorem as suggested by references [12]–[14].

The central theme of this paper has been to show that the traditional Foster reactance theorem holds for an arbitrary antenna system using the Poynting theorem in the complex frequency domain. The Foster reactance theorem has then been used to demonstrate the widely held assumption $Q \approx 1/B$ is valid for an arbitrary antenna system, provided $Q \gg 1$.

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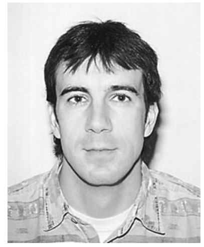

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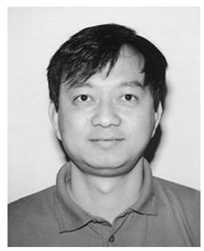

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