Sparsity and Conditioning of Impedance Matrices Obtained with Semi-Orthogonal and Bi-Orthogonal Wavelet Bases

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***Abstract*—Wavelet and wavelet packet transforms are often used to sparsify dense matrices arising in discretization of CEM integral equations. This paper compares orthogonal, semi-orthogonal, and bi-orthogonal wavelet and wavelet packet transforms with respect to the condition numbers, matrix sparsity, and number of iterations for the transformed systems. The best overall results are obtained with the orthogonal wavelet packet transforms that produce highly sparse matrices requiring fewest iterations. Among wavelet transforms the semi-orthogonal wavelet transforms lead to sparsest matrices, but require too many iterations due to high-condition numbers. The bi-orthogonal wavelets produce very poor sparsity and require many iterations and should not be used in these applications.**

***Index Terms*—Electromagnetic integral equations, electromagnetic scattering, integral equations, sparse matrices, wavelet packets, wavelet transformations.**

I. Introduction

DISCRETIZATION of electromagnetic integral equations via the method of moments produces dense matrix equations. For electrically large objects the systems become so large that iterative solvers must be used. In such cases, an approximation to the solution is obtained in $O(pN^2)$ flops, where p is the number of iterations and N the size of the system, in contrast to $O(N^3)$ flops used by the direct solver. For very large systems the complexity of $O(pN^2)$ flops is still unacceptable (even assuming that p is of reasonable size) and special techniques must be devised to reduce the N^2 cost of a single matrix-vector multiplication. Such approaches include the fast multipole method [1], [2], the adaptive integral method [3], the impedance matrix localization method [4], and the fast Fourier transform (FFT) and wavelet based methods [5]–[7].

The wavelet based methods have attracted considerable attention lately. They can be applied either directly (using wavelets as test and trial functions) or indirectly (transforming the impedance matrix with wavelet transformations). The direct approach might be more attractive in principle, because it allows the *a priori* location of small matrix elements and replaces them with zeros thus avoiding the evaluation of the full impedance matrix. An analysis of such an approach for nonoscillatory kernels can be found, for example, in [8]. Similar analysis for oscillatory kernels and wavelet packets has not been performed.

The indirect approach starts out with the matrix equation

$$ZJ = E, z1 \tag{1}$$

obtained from an electromagnetic integral equation by the method of moments with standard basis functions. Here Z represents the complex, non-Hermitian, dense impedance matrix of size $N \times N$, J is the surface current vector, and E is the excitation vector. A wavelet transform matrix T is then applied to the system yielding the new system

$$Z'J' = E' \tag{2}$$

where $Z' = TZT^*$, $E' = TE$, and T^* denotes the complex conjugate transpose of T. Once J' is found, current vector J can computed from $J = T^*J'$. Matrix Z' is closely related to the impedance matrix obtained directly by discretizing integral equations with wavelet basis functions.

It can be shown that Z' contains many negligible elements which can be set to zero without adversely affecting the quality of the solution approximation. Matrix Z' is usually computed by repeated application of low-pass and high-pass filter coefficients which define transformation T. It is important that such filters are as short as possible yet still capable of producing high sparsity of Z'. These two requirements are somewhat contradictory and a compromise must be sought.

Popular candidates for such filters are the compactly supported orthogonal wavelets of Daubechies [5]–[7], [9], the B-spline semi-orthogonal wavelets of Chui [10], and the B-spline bi-orthogonal wavelets of Cohen, Daubechies, and Feauveau [11]. In most cases the scattering surfaces allowed the use of periodic wavelets, but the interval wavelets have been tried as well [12], [13].

Sparsity of Z' is only one of the important parameters in the iterative solution of linear systems. Another is the number of iterations necessary to reduce a measure of the solution error to a given tolerance. This depends on the distribution of singular values of the system matrix. A widely used iterative method for non-Hermitian systems is the conjugate gradient for the normal equations (CGNE) solver. For this method the Euclidean norm of the residual vector $r_n = E - ZJ_n$ monotonically decreases with the number of iterations n and is bounded by

$$\frac{||r_n||}{||r_0||} = \inf_{p \in P_n} \frac{||p(Z^*Z)r_0||}{||r_0||} \leq \inf_{p \in P_n} \max_{\sigma \in S(Z)} |p(\sigma)| \tag{3}$$

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where P_n is the space of polynomials of degree up to n, Z^* is the Hermitian transpose of Z, and $S(Z)$ denotes the set of singular values of Z [14].

In the case of periodic orthogonal wavelets, T^* is the inverse of T, so that matrix Z' is orthogonally similar to Z. This implies that Z and Z' have identical singular values. The convergence behavior of the CGNE solver for the systems (1) and (2) will be very similar.

The situation is quite different in the case of periodic semi-orthogonal, periodic bi-orthogonal, or the interval wavelets. In this case, the transformation T changes singular values of Z (the singular spectra of Z and Z' are different). A convenient measure of this is the spectral condition number κ defined as the ratio of the largest and the smallest singular values of a given matrix. Since we have

$$\inf_{p\in P_n} \max_{\sigma\in S(Z)} |p(\sigma)| \le 2\left(\frac{\kappa(Z)-1}{\kappa(Z)+1}\right)^n$$

and for large $\kappa(Z)$, the right side of this inequality is approximately equal to $2(1-2/\kappa(Z))^n$; it follows that for large $\kappa(Z)$ convergence to specified, tolerance can be expected in $O(\kappa(Z))$ iterations. Of course, the convergence may be faster if the singular values are clustered [14].

Recently it has been demonstrated that the semi-orthogonal (SWT) or bi-orthogonal wavelet transforms (BWT) produce higher matrix sparsities than the orthogonal wavelet transforms (OWT) at the same levels of accuracy [10], [15]. However, since the SWT's and BWT's are nonorthogonal it is likely that their application will *increase* the condition numbers of transformed matrices leading to an increased number of iterations. The exact relation between the impedance matrix conditioning and the number of iterations on one hand and the sparsity and the accuracy of the solution on the other is very difficult to quantify. In this paper, we seek to compare the nonorthogonal and orthogonal wavelet transforms with respect to these relationships. To our knowledge, such a study of several classes of wavelet transforms of many orders with respect to their overall effectiveness in numerical electromagnetic integral equations has not been performed before. Its findings should be of use to those in the computational electromagnetics community with an interest in wavelet applications.

In Section II, we briefly present the discrete wavelet transforms and the discrete wavelet packet transforms based on compactly supported orthogonal, semi-orthogonal, and bi-orthogonal wavelets. In Section III, we compare the conditioning of the semi-orthogonal and bi-orthogonal discrete wavelet transforms as the function of the number of vanishing moments and the number of decomposition levels. Section IV investigates the conditioning of the semi-orthogonal and bi-orthogonal discrete wavelet packet transforms (the SWP's and BWP's). Section V presents the matrix sparsities obtained with various wavelet and wavelet packet transforms for systems obtained from discretization of the combined field integral equation defined on a circular contour. Section VI shows the condition numbers and number of iterations obtained with different wavelets for this numerical example. The results illustrate the limitations of nonorthogonal wavelet transforms and nonorthogonal wavelet packet transforms in numerical solution of electromagnetic integral equations.

II. Wavelets and Wavelet Packets

Detailed descriptions of compactly supported orthogonal, semi-orthogonal, and bi-orthogonal wavelets can be found in [16], [17], and [18], respectively. Section II introduces specific notation needed later in the paper. The scaling and wavelet functions ϕ and ψ satisfy two-scale relations

$$\begin{aligned}\phi(x) &= \sqrt{2}\sum_n h_n\phi(2x-n),\\ \psi(x) &= \sqrt{2}\sum_n g_n\phi(2x-n)\end{aligned} \tag{4}$$

where h_n and g_n denote decomposition filter coefficients. In the following, only finite-length filters will be considered. The wavelet ψ is said to have m vanishing moments if $\sum_n g_n n^k = 0$, $k = 0, \cdots, m-1$.

The filters for Daubechies orthogonal wavelets with m vanishing moments are each of length $2m+2$ (the total filter length is $4m+4$) and are related by the formula

$$g_k = (-1)^k h_{2m+1-k}, \qquad k = 0, \cdots, 2m+1. \tag{5}$$

There is no explicit formula for values of h_k, but they can be computed with a well-known algorithm [16]. For $m = 1$ the filters (given here with four significant digits) are

$$\begin{aligned}[h_0, h_1, h_2, h_3] &= [0.4830, 0.8365, 0.2241, -0.1294]\\ [g_0, g_1, g_2, g_3] &= [-0.1294, -0.2241, 0.8365, -0.4830].\end{aligned}$$

Filter coefficients for the dual semi-orthogonal wavelet transform (SWT) are given by the following formulas:

$$\begin{aligned}h_k &= 2^{-m-1/2}\binom{m+1}{k}, \qquad k = 0, \cdots, m+1\\ g_l &= (-1)^l 2^{-m-1/2}\sum_{j=0}^{m+1}\binom{m+1}{j}N_{2m+2}(l+1-j)\end{aligned} \tag{6}$$

where $l = 0, \cdots, 3m-2$ and N_{2m+2} denotes the cardinal B-spline of order $2m+2$ [17]. The SWT wavelet has m vanishing moments and the filter lengths are $m+2$ and $3m+2$, respectively, for the total filter length of $4m+4$. The reconstruction filters for SWT are of infinite length but they are not used in the impedance matrix applications. For $m = 1$ the filters are

$$\begin{aligned}[h_0, h_1, h_2] &= [0.3536, 0.7071, 0.3536],\\ [g_0, \cdots, g_4] &= [0.0589, -0.3536, 0.5893, -0.3536, 0.0589].\end{aligned}$$

In the case of dual B-spline bi-orthogonal wavelet transform (BWT) for each order of the spline scaling function there exists a whole family of wavelets with different filter lengths and different number of vanishing moments. Specifically, for order

$m+1$ B-splines the decomposition filter coefficients are given by the following formulas [18]:

$$h_k = 2^{-m-1/2}\binom{m+1}{k}, \qquad k=0,\cdots,m+1$$
$$g_k = (-1)^k \tilde{h}_{4K-m-2-k} \tag{7}$$

where K must satisfy $2K \geq m+1$ and

$$\sum \tilde{h}_k z^k = \sqrt{2}\left(\frac{1+z}{2}\right)^{2K-m-1} \cdot \sum_{j=0}^{K-1}\binom{K-1+j}{j}\left(\frac{1-z}{2i}\right)^{2j} z^{K-j}$$
$$\tilde{g}_k = (-1)^k h_{m-k-1}. \tag{8}$$

The wavelet transform defined by above filters will be called the (m, K)-BWT transform. Note that in contrast to the SWT case, the reconstruction filters given by $\tilde{h}_k, \tilde{g}_k$ are finite. It can be shown that the (m, K)-BWT wavelet has $2K-m-2$ vanishing moments and the filters lengths are $m+2$ and $4K-m-2$, respectively. If $K=m+1$, then the total filter length is $4m+4$ and the BWT has m vanishing moments. For $m=1$ and $K=2$ the filters are

$$[h_0, h_1, h_2] = [0.3536, 0.7071, 0.3536],$$
$$[g_0, \cdots, g_4] = [-0.1768, -0.3536, 1.0607, -0.3536, -0.1768].$$

The l-level wavelet transform T is defined by the product of one-level wavelet transforms W_k

$$T = W_{n-l+1}\cdots W_{n-1}W_n \tag{9}$$

$$W_{n-j} = \begin{bmatrix} \begin{bmatrix} H_{n-j} \\ G_{n-j} \end{bmatrix} & \\ & I_{N-N/2^j} \end{bmatrix} \tag{10}$$

where I_k denotes the identity matrix of rank k and H_{n-j}, G_{n-j} are low- and high-pass matrices of size $N/2^{j+1}$–by–$N/2^j$ defined by their filter coefficients $\{h_k, g_k\}$. For example, the low-pass matrix H_n for the periodic filter of length four is

$$H_n = \begin{bmatrix} h_1 & h_2 & h_3 & h_4 & 0 & & & \cdots & 0 \\ 0 & 0 & h_1 & h_2 & h_3 & h_4 & 0 & \cdots & 0 \\ & & & & \ddots & & & & \\ 0 & \cdots & & & 0 & h_1 & h_2 & h_3 & h_4 \\ h_3 & h_4 & 0 & & & \cdots & 0 & h_1 & h_2 \end{bmatrix}.$$

The high-pass matrix G_n has the same structure as H_n, but is defined in terms of wavelet coefficients g_k.

Equations (9) and (10) define the discrete wavelet transform. Of course, instead of recursive transformations of the low-pass signals only [see (10)], one can select to transform only the high-pass signal. At the second level this would result in the following definition of W_{n-1}:

$$W_{n-1} = \begin{bmatrix} I_{N-N/2^j} & \\ & \begin{bmatrix} H_{n-1} \\ G_{n-1} \end{bmatrix} \end{bmatrix}. \tag{11}$$

The other possibility at level 2 is to transform both the low- and the high-pass signals. The required transformation can be written now as

$$W_{n-1} = \begin{bmatrix} \begin{bmatrix} H_{n-1} \\ G_{n-1} \end{bmatrix} & \\ & \begin{bmatrix} H_{n-1} \\ G_{n-1} \end{bmatrix} \end{bmatrix}. \tag{12}$$

If this process is repeated recursively with l levels it results in a binary tree structure of more than 2^{2^l} possible discrete wavelet packet transformations [19], [20], [9].

TABLE I
CONDITION NUMBERS $\kappa(T)$ FOR ONE-LEVEL (m, K) BWT

K \ m	0	2	4	6	8
1	1.00				
3	1.42	4.00	16.0		
5	1.58	4.00	16.0	64.0	834.
7	1.68	4.00	16.0	64.0	259.
9	1.75	4.00	16.0	64.0	256.

III. CONDITIONING OF NONORTHOGONAL WAVELET TRANSFORMS

Define the condition number of a matrix T by $\kappa(T) = \|T\|_2\|T^{-1}\|_2$, where $\|\cdot\|_2$ denotes the matrix 2-norm. If T denotes a wavelet transform defined by (9), then its condition number depends on the kind of the wavelet selected. Due to their orthogonality, the orthogonal wavelet transforms have $\kappa(T) = 1$ irrespective of filter lengths and number of decomposition levels. The condition numbers for the BWT's and SWT's are larger and increase with filter lengths and decomposition levels, but do not depend on the size of matrix T. In this section, we study the dependence of $\kappa(T)$ first on the wavelet order for a single level transforms and second on the number of decomposition levels.

It can be shown that the condition numbers of one level BWT's are bounded as a function of K for a fixed m. For the fixed K the condition numbers increase with m at least as fast as 2^{m-1} (see Table I). This property and its harmful impact on sparsity and conditioning of the impedance matrices was investigated in [11]. However, it is possible to scale the BWT wavelet coefficients g_k and $\tilde{g}_k$ from (7) to mollify the quick growth of the condition numbers with m. Selecting $\alpha = 2^{-m/2}$ one can define the scaled bi-orthogonal wavelet transforms (SBWT) through the new coefficients computed from

$$g_k^{new} = \alpha g_k, \quad \tilde{g}_k^{new} = \tilde{g}_k/\alpha.$$

The optimal value of $\alpha = 2^{-m/2}$ slows the growth of the condition numbers to $2^{m/2}$ [21], as shown in Table II.

TABLE II
CONDITION NUMBERS $\kappa(T)$ FOR SCALED ONE-LEVEL (m, K)BWT

K \ m	0	2	4	6	8
1	1.00				
3	1.42	2.00	4.39		
5	1.57	2.00	4.00	12.2	38.1
7	1.66	2.00	4.00	8.00	16.1.
9	1.72	2.00	4.00	8.00	16.0

Table III compares $\kappa(T)$ for 1-level wavelet transforms (unscaled and scaled BWT's and SWT's) as functions of vanishing moments m. In order to make a viable comparison we set $K = m + 1$ for the BWT's, which results in the filters of the same (total) length as those for the SWT. In all three cases, the condition numbers increase with m and $\kappa(T)$ are smallest for the scaled BWT's.

It is well known that to obtain significant sparsification of impedance matrices it is necessary to perform several levels l of wavelet decomposition. Table IV shows the condition numbers of SWT's and scaled BWT's with different number of vanishing moments m as a function of number of levels l. We do not consider unscaled BWT's since their condition numbers (for level $l = 1$) were considerably larger than those for the SWT's and scaled BWT's. Note that, for a given m, the condition numbers for each transform increase with l, but appear to be bounded. In fact, it can be shown that (for a fixed m)

$$\lim_{l\to\infty} \kappa(T) = c_m < \infty$$

that is, the SWT's and SBWT's are *stable* in mathematical terms [22], [23]. In addition, note that the SWT condition numbers are smaller than scaled BWT condition numbers for sufficiently large number of levels l (for a fixed m). This indicates that the condition number of one-level transform is not a good indicator of conditioning for larger l.

However, large condition numbers of T with higher m might significantly slow down the iterative solvers. The transforms must be chosen carefully so that the gain in sparsity from selecting large m will not be lost in additional iterations due to a high condition number of the transformed system.

The condition numbers of the transformed impedance matrices Z' satisfy the inequality $\kappa(Z') \leq \kappa(Z)\kappa^2(T)$. However, although $\kappa(T)$ is only an indirect measure of conditioning of the impedance matrix Z', the results in Table IV do suggest to expect huge values of $\kappa(Z')$ for larger m and l.

IV. CONDITIONING OF NONORTHOGONAL WAVELET PACKET TRANSFORMS

Recently, it has been demonstrated that the orthogonal wavelet packet transforms yield much better sparsity than the classical wavelet transforms [20], [24], [9]. Due to the orthogonality of the transforms the condition numbers of the original impedance matrices were preserved. Numerical experiments illustrated that the number of significant elements in the impedance matrices grows as $O(N^p)$, where $p \approx 4/3$.

Wavelet packet transforms can also be constructed with semi- or bi-orthogonal wavelets. Since some nonorthogonal wavelet bases yield better impedance matrix sparsity than the orthogonal ones, it is natural to expect that the nonorthogonal wavelet packet transforms might improve sparsity even further. However, it has recently been shown that for any fixed m, the condition numbers of the bi-orthogonal wavelet packet (BWP) transforms are not bounded as a function of levels l [23]. Related results on the *instability* of wavelet packets for the BWP and semi-orthogonal wavelet packet (SWP) transforms can be found in [25] and [26]. These results suggest that the iterative solvers might be considerably slowed down if nonorthogonal wavelet packets were to be used.

Table V displays condition numbers $\kappa(T)$ of full tree wavelet packet transforms for the SWP and scaled BWP transforms with various numbers of vanishing moments m. The growth of condition numbers with level l and order m is very pronounced. Sur-

TABLE III
CONDITION NUMBERS $\kappa(T)$ FOR ONE-LEVEL BWT's AND SWT's WITH m VANISHING MOMENTS

m	1	2	3	4	5	6	7
BWT	2.0	4.0	8.0	16.	32.	64.	128.
SWT	2.1	3.7	6.5	11.	19.	34.	60.
sBWT	1.4	2.0	2.8	4.0	5.6	8.0	11.3

TABLE IV
$\kappa(T)$ FOR WAVELET TRANSFORMS: SWT AND SCALED BWT WITH l LEVELS

level $l =$	1	2	3	4
m=1, SWT	2.12	2.46	2.56	2.58
sBWT	1.41	2.00	2.35	2.60
m=2, SWT	3.75	4.75	5.03	5.10
sBWT	2.00	4.00	5.27	6.28
m=3, SWT	6.55	8.99	9.71	9.90
sBWT	2.82	8.00	11.6	15.4
m=4, SWT	11.4	16.9	18.7	19.1
sBWT	4.00	16.0	25.6	40.3
m=5, SWT	19.9	32.0	36.0	37.0
sBWT	5.65	32.0	56.6	113.
m=6, SWT	34.7	60.5	69.3	71.7
sBWT	8.00	64.0	125.	351.
m=7, SWT	60.7	114.	133.	138
sBWT	11.3	128.	280.	1209.

TABLE V
$\kappa(T)$ FOR FULL TREE WAVE PACKET TRANSFORMS: SWP AND SCALED BWP WITH l LEVELS

level $l =$	1	2	3	4
m=1, SWP	2.12	2.77	4.81	7.03
BWP	1.41	2.00	2.56	3.40
m=2, SWP	3.75	5.57	13.7	25.4
BWP	2.00	4.00	6.68	11.8
m=3, SWP	6.55	11.0	37.1	83.8
BWP	2.82	8.00	16.9	40.5
m=4, SWP	11.4	21.9	99.7	278.
BWP	4.00	16.0	43.1	138.
m=5, SWP	19.9	43.7	267.	921.
BWP	5.65	32.0	114.	486.
m=6, SWP	34.7	86.9	718.	3048.
BWP	8.00	64.0	300.	1704.
m=7, SWP	60.7	172.	1931.	10081.
BWP	11.3	128.	776.	5975.

prisingly, the BWP condition numbers grow with l somewhat slower than those for the SWP. Note that the opposite is true for the scaled BWT's and SWT's (see Table IV). Due to quite large condition numbers of wavelet packet transforms T for larger values of m, the condition numbers of impedance matrices Z' satisfying the inequality $\kappa(Z') \le \kappa(Z)\kappa^2(T)$ are generally too large for the effective use of any iterative solvers.

V. Matrix Sparsification with Wavelets and Wavelet Packets

In this section, we compare the sparsities of impedance matrices obtained with the help of various wavelet transforms from a discretization of an electromagnetic integral equation. To enable comparisons with some earlier papers [7], [9] the scattering of plane waves from perfectly conducting cylinders with constant cross sections bounded by surface contour C is considered here. Two contours: a circle and an L-shape, were selected for the experiments [9]. The far fields and the radar cross sections are determined by the surface current J computed from the combined field integral equation

$$\left(1+\frac{\partial}{\partial n_x}\right) E_z^{\text{inc}}(\boldsymbol{x}) = -\frac{i\omega\mu_0}{4}\left(1+\frac{\partial}{\partial n_x}\right)\int_C H_0^{(1)}(2\pi\lambda \boldsymbol{r}) J(\boldsymbol{x}')\, dl(\boldsymbol{x}') \quad (13)$$

where $H_0^{(1)}$ is the zero order Hankel function of the first kind, $\boldsymbol{r} = |\boldsymbol{x} - \boldsymbol{x}'|$, $\boldsymbol{x}$, $\boldsymbol{x}'$ denote points on C, n_x is the outer unit normal at point $\boldsymbol{x}$, and λ is the excitation wavelength.

Point matching and pulse-test functions were used to discretize integral (13) with a constant number of test functions per wavelength λ. Thus, N, the size of the impedance matrix Z was directly proportional to the electrical size of the contour. The support of pulse functions was $\lambda/10$ throughout all experiments.

To provide a comparison between different wavelet transforms, Table VI lists condition numbers of Z' for a circular cylinder with $N = 256$. The condition number of the original impedance matrix Z is $\kappa(Z) = 6.8$. The results show that condition numbers $\kappa(Z')$ grow both with the number of levels l (for a fixed m) and with number of vanishing moments m (for a fixed level l). A comparison of results from Tables IV and VI show that the bound $\kappa(Z)\kappa^2(T)$ is quite accurate. The growth of $\kappa(Z')$ with l is moderate for small values of m for both SWT's and scaled BWT's. For larger m and l the condition number for the scaled BWT's is quite large.

The choice of thresholding criteria for the elements of the transformed matrix Z' poses certain problems due to matrix conditioning. A popular choice in the literature [7], [9], [15] is to select the relative residual error

$$\|E' - Z'J'_{sp}\| / \|E'\|$$

or the relative matrix error

$$\|Z' - Z'_{sp}\| / \|Z'\|$$

TABLE VI
Condition Numbers $\kappa(Z')$ with l Levels, $\kappa(Z) = 6.8$

level l =	1	2	3	4
m=1, SWT	8.4E+0	1.1E+1	3.1E+1	3.1E+1
sBWT	7.3E+0	9.4E+0	1.3E+1	1.7E+1
m=2, SWT	2.6E+1	3.4E+1	1.1E+2	1.1E+2
sBWT	7.8E+0	2.8E+1	4.1E+1	6.3E+1
m=3, SWT	8.0E+1	1.2E+2	3.6E+2	3.5E+2
sBWT	1.5E+1	1.1E+2	2.0E+2	3.2E+2
m=4, SWT	2.4E+2	4.3E+2	1.2E+3	1.1E+3
sBWT	3.0E+1	4.5E+2	9.6E+2	2.1E+3
m=5, SWT	7.3E+2	1.5E+3	3.6E+3	3.4E+3
sBWT	6.0E+1	1.8E+3	4.7E+3	1.8E+4
m=6, SWT	2.2E+3	5.4E+3	1.1E+4	1.0E+4
sBWT	1.2E+2	7.0E+3	2.3E+4	1.8E+5
m=7, SWT	.7E+3	1.9E+4	3.4E+4	3.1E+4
sBWT	2.3E+2	2.8E+4	1.2E+5	1.4E+6

as a measure of accuracy, where J'_{sp} is the solution of matrix equation

$$Z'_{sp} J'_{sp} = E'$$

and Z'_{sp} is the thresholded version of the transformed matrix Z'. The use of such measures is justified when orthogonal wavelet transformation are used and the condition number for Z does not increase considerably with N. For large values of $\kappa(T)$ in the case of nonorthogonal transformations, the small size of any of the above measures can not assure that the relative error of the computed current

$$\|J - T^t J'_{sp}\| / \|J\|$$

is small. This can be seen from the easily derived inequality

$$\frac{\|J - T^t J'_{sp}\|}{\|J\|} \le \kappa(T)\kappa(Z)\,\frac{\|E' - Z'J'_{sp}\|}{\|E'\|}. \quad (14)$$

Thus, it can be expected that in some cases the relative error of computations exceeds the relative residual error by a factor of $\kappa(T)\kappa(Z)$.

This phenomenon is illustrated in Fig. 1, which shows the modulus of the computed current along the circular contour with $N = 256$ obtained with the orthogonal (OWT) and semi-orthogonal (SWT) wavelet transforms with $m = 7$ vanishing moments. These results were generated with the threshold level selected as to satisfy the relative residual error criterion

$$\|E' - Z'J'_{sp}\| / \|E'\| = (1 \pm 0.1)\%$$

in both cases. The matrix sparsities were 10.8 and 6.9%, respectively. The solution obtained with the OWT is visually identical with the solution of the original equation $ZJ = e$. It can be checked that the relative error $\|J - T^t J'_{sp}\|/\|J\|$ for the OWT is less than 1.6%. On the other hand, the SWT solution is much less accurate. The SWT relative error is about 20.6%, which is more than 20 times larger than the relative residual error in this case. Since, for the SWT with $m = 7$ the condition numbers are $\kappa(Z) \approx 6.7$ and $\kappa(T) \approx 140$, inequality (14) overestimates the discrepancy between the relative error and the relative residual.

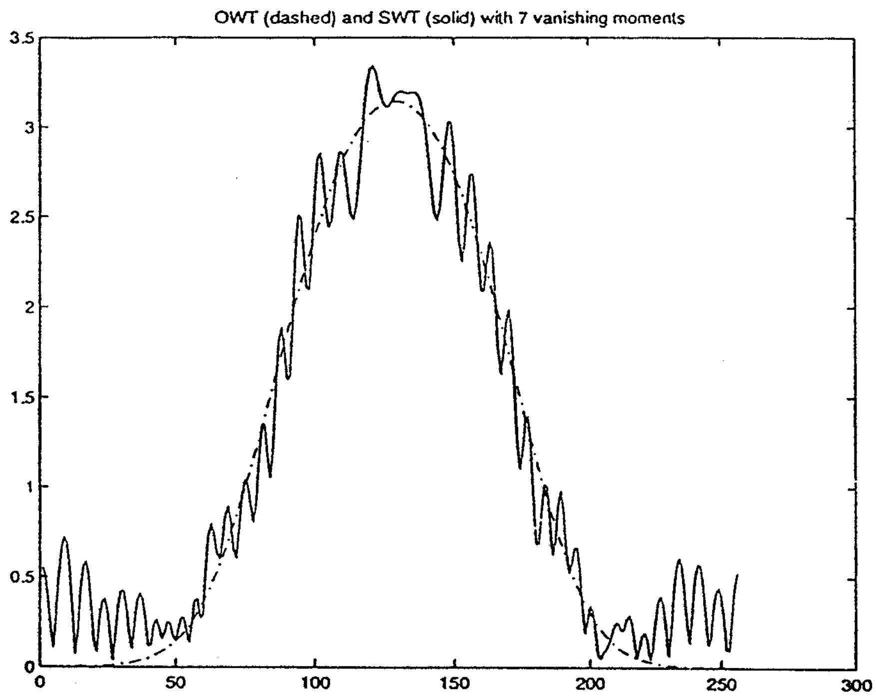

Fig. 1. Modulus of the current along the circular contour obtained with the OWT (dashed line) and the SWT (solid line) with seven vanishing moments. Relative residual error $\|e' - Z'J'_{\mathrm{comp}}\|/\|e'\| = (1 \pm 0.1)\%$. Matrix size $N = 256$.

Additional tests indicate that the SWT solution becomes as accurate as the OWT setting the relative residual error to 0.07% which yields the matrix sparsity of 11%. This is about the same sparsity as that obtained from the OWT.

Since, for nonorthogonal wavelets, the comparisons of matrix sparsity using relative residual errors may be misleading, a study of matrix sparsity was conducted by controlling the relative error

$$\|J - T^t J'_{sp}\|/\,\|J\|.$$

The system sizes studied ranged from $N = 128$ (contour length of $12.8\ \lambda$) to $N = 2048$ (contour length of $204.8\ \lambda$). The relative error was maintained within $(5 \pm 0.1)\%$. Both classical wavelet transforms (WT's), as well as wavelet packet transformations (WP's) with various number of vanishing moments m were used; the latter ones used an adaptive algorithm described in [9]. Tables VII–IX show results for orthogonal Daubechies wavelets (OWT's and OWP's), semiorthogonal B-spline wavelets (SWT's and SWP's), and bi-orthogonal B-spline wavelets (SBWT's).

The matrix sparsity results for the circular cylinder are presented in Table VII. The sparsest matrices were obtained with the orthogonal wavelet packet transform (OWP, $m = 7$). Among the semiorthogonal transforms (SWT), those with $m \geq 3$ produced better matrix sparsity for large N than the sparsity results obtained with the OWT, $m = 7$. The scaled bi-orthogonal transforms produced very poor sparsity for all $0 < m < 8$ and for this reason only two examples are given here.

VI. Conditioning and Iterative Solvers

Table VIII contains the condition numbers $\kappa(Z')$ for the system matrices for the circular cylinder. Several observations should be made. First, the condition numbers for the OWP and OWT are the same as those for the original matrices Z (because of the orthogonality of the transforms) and increase with N. This increase partially explains why the number of nonzero matrix elements in Table VII for the OWP does not decrease monotonically when the relative error is controlled (instead of relative residual as in [9]). Second, the size of condition numbers for the SWT's increases both with N (for fixed m) and with m for fixed N. The increase of $\kappa(Z')$ with N is a combination of the growth of $\kappa(Z)$ with N and the growth of $\kappa(T)$ with the number of levels l (see Table IV). The growth of $\kappa(Z')$ with m for fixed N (i.e., for the fixed number of wavelet levels) is also predicted by Table IV. Its harmful effect on matrix sparsity is partially offset by better approximation properties of wavelets with higher value of m. This offset may be responsible for slow improvement in matrix sparsity for larger values of m and N for the SWT's in the case of the circular cylinder. Last, the condition numbers for the SBWT are so large that better approximation for higher m do not produce any gains—the matrix sparsity in this case is very disappointing (see Table VII).

Finally, we discuss the performance of iterative solvers on the sparse systems generated with the help of wavelet transforms. It is well known that the rate of convergence of iterative solvers, e.g., the ratio of the norms of the current and initial residuals, is related to the condition number [27]. We will show that the relatively high-condition numbers of SWT and SBWT transformed

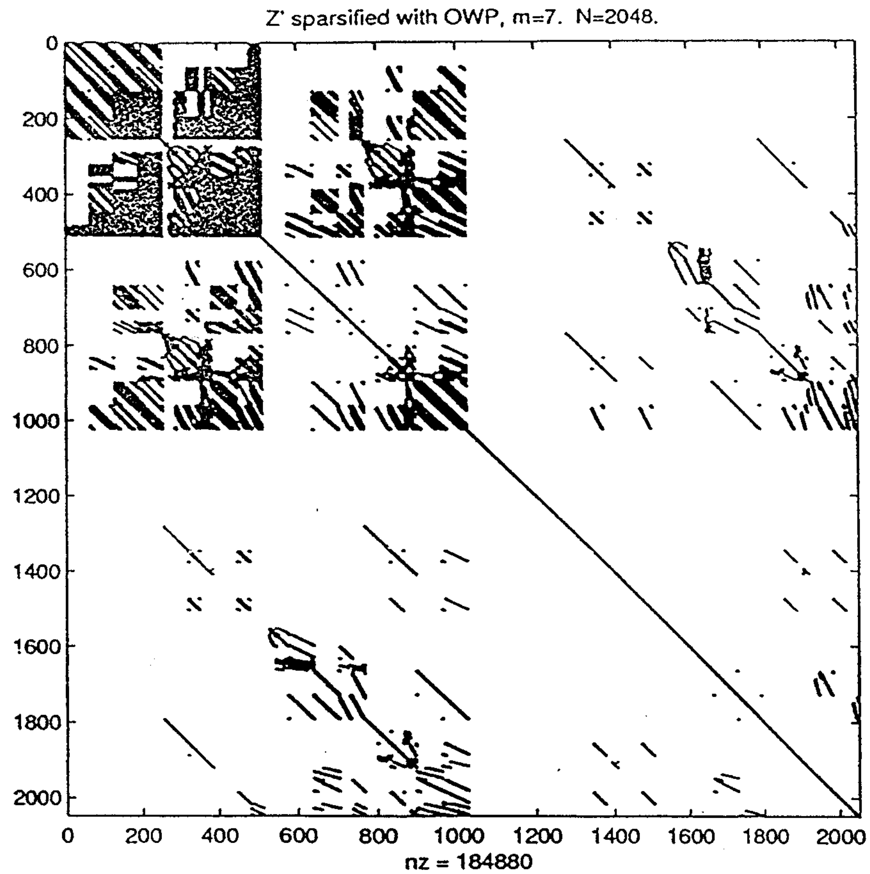

Fig. 2. Impedance matrix sparsity pattern for the circular cylinder, $N = 2048$ obtained with the OWP $m = 7$.

TABLE VII
NUMBER OF NONZERO ELEMENTS (IN %) FOR THE CIRCULAR CYLINDER WITH THE RELATIVE ERROR = $(5 \pm 0.1)\%$

N	128	256	512	1024	2048
OWP m=7	8.0	3.6	9.2	3.9	4.0
OWT m=7	8.3	6.2	10.3	12.5	10.6
SWT m=1	17.1	12.3	24.0	19.2	22.9
m=2	14.9	9.0	11.7	17.2	10.8
m=3	13.2	7.3	8.3	8.4	6.6
m=4	10.6	6.7	8.1	7.6	5.7
m=5	13.2	7.1	8.4	6.4	4.8
m=6	13.6	7.8	8.2	6.1	4.4
m=7	15.7	8.8	9.8	6.1	4.3
SWP m=4	4.4	4.8	4.2	4.1	4.9
m=7	9.3	5.4	6.2	5.1	4.1
sBWT m=1	44.2	41.5	71.9	66.3	55.9
m=7	46.9	62.9	60.60	67.4	72.4

TABLE VIII
CONDITION NUMBERS $\kappa(Z')$ FOR THE CIRCULAR CYLINDER

N	128	256	512	1024	2048
OWP m=7	1.7E1	6.7E0	4.7E2	1.8E2	2.7E2
OWT m=7	1.7E1	6.7E0	4.7E2	1.8E2	2.7E2
SWT m=1	7.7E1	2.7E1	3.1E2	5.2E2	6.8E2
m=2	3.0E2	9.1E1	4.2E2	1.2E3	1.4E3
m=3	1.2E3	4.8E2	2.3E3	2.4E3	4.7E3
m=4	4.2E3	1.7E3	8.0E4	5.7E3	1.0E4
m=5	1.2E3	3.4E3	3.6E4	1.5E4	2.7E4
m=6	4.1E3	1.1E4	1.3E5	4.3E4	6.3E4
m=7	1.4E4	3.1E4	4.8E5	1.2E5	1.3E5
SWP m=4	9.3E4	1.2E6	1.4E7	1.6E8	1.5E9
m=7	5.1E6	1.5E8	9.8E9	7.6E10	2.3E12
sBWT m=1	7.1E1	2.8E1	6.6E2	3.6E2	5.5E2
m=7	6.9E4	1.4E6	2.2E7	4.7E8	8.8E9

matrices lead to increased numbers of iterations, negating any savings obtained from mild sparsity improvements over those obtained with the OWT's.

To illustrate the complicated relation between the condition numbers and the rate of convergence of iterative solvers, the conjugate gradient for normal equations (CGNE) solver was selected due to its popularity. Similar tests conducted with the QMR and GMRES solvers generally confirm our findings. Theoretically, in the absence of roundoff errors, the CGNE method is guaranteed to yield the exact solution after N iterations. The presence of roundoff errors leads to loss of accuracy and finite termination is not guaranteed. However, the CGNE is normally used as a genuine iterative method and often a few iterations are required to achieve sufficient accuracy of the iterated solution. This usually happens when the system matrix is well conditioned or, in case of high-condition number, if the spectral values are highly clustered.

There was no preconditioning in the experiments and the iterations started with the zero vector. The lack of preconditioning

TABLE IX
NUMBER OF THE CGNE ITERATIONS WITH THE RELATIVE RESIDUAL $<5e-3$ FOR THE CIRCULAR CYLINDER (N INDICATES THAT ITERATIONS EXCEEDED THE SYSTEM SIZE)

N	128	256	512	1024	2048
OWP m=7	13	15	34	32	36
OWT m=7	14	16	34	34	36
SWT m=1	33	33	67	79	97
m=2	103	112	221	199	168
m=3	94	166	288	362	455
m=4	103	N	N	911	1353
m=5	N	N	N	N	N
m=6	N	N	N	N	N
m=7	N	N	N	N	N
SWP m=4	97	N	N	N	N
m=7	71	160	277	672	1245
sBWT m=1	47	38	93	59	59
m=7	41	101	190	316	400

TABLE X
SPARSITY (IN %) FOR THE L-SHAPE CYLINDER OBTAINED WITH THE RELATIVE ERROR = (5 ± 0.1)%. THE ITERATION NUMBERS ARE GIVEN IN PARENTHESES

N	128	256	512	1024
OWP m=7	18(29)	15(24)	14(33)	8(35)
OWT m=7	18(29)	14(27)	14(31)	10(35)
SWT m=1	24(63)	19(37)	19(32)	16(48)
m=4	25(128)	16(221)	13(226)	8(242)
m=7	35(128)	19(248)	12(376)	7(527)

has been dictated by a desire to illustrate the impact of various wavelet transforms on the conditioning of the system matrix. If a given wavelet transform increased the matrix condition number greatly, a much better preconditioner would have to be found just to undo this undesirable phenomenon. This paper indicates which classes of wavelets do not cause the excessive growth of condition numbers. As a measure of iterative solution accuracy, an easily computable quantity, the relative residual error

$$||E' - Z'_{sp}J'_{sp}|| / ||E'||$$

was used. From (3) it follows that this error measure is monotonically decreasing with number of iterations.

The iterations were stopped when the relative residual dropped below 0.005 or if the number of iterations exceeded the size of the system. The second stopping criterion was added because of inevitable round-off errors. Note that stopping iterations at the same level of the relative residual error yields more accurate results for solutions of systems with lower condition numbers.

The results Table IX generally support the notion that the higher condition number corresponds to more iterations. The moderate growth of $\kappa(Z')$ with N for the OWT and OWP transformed matrices translates into a moderate growth of the necessary iteration numbers. Since $\kappa(Z')$ obtained from nonorthogonal transformed matrices are generally larger than for the orthogonally transformed matrices, the iteration numbers are also much higher.

Since the condition number alone provides only an upper bound for the necessary number of iterations, convergence may be faster if the spectrum of the matrix is clustered. This could be the reason why, in spite of the huge matrix condition numbers for the sBWT with $m = 7$, the corresponding iteration numbers, although much larger than for the orthogonal transformations, are smaller than those for the SWT's.

To confirm the observations made above we performed numerical computations on an L-shape and the NACA0012 airfoil. The system sizes studied ranged from $N = 128$ (contour length of $12.8\ \lambda$) to $N = 2048$ (contour length of $204.8\ \lambda$). All the observations made from numerical experiments on circular cylinders held true for the other shapes. Table X lists the sparsity and the number of iterations obtained for an infinite cylinder with the L-shaped cross section for some representative wavelet transforms. The number of iterations with the SWT's is larger than with the orthogonal transformations especially for larger m. The results for the sBWT's are not listed—they produced very poor matrix sparsity.

VII. CONCLUSIONS

Recent articles in computational electromagnetics literature have shown that some nonorthogonal wavelet transforms produce sparser impedance matrices than the orthogonal wavelet transforms. Since the nonorthogonality increases the condition number of transformed matrices the improvement in matrix sparsity could be offset by the increased number of iterations needed for the solution to converge.

This paper studies the impact of the B-spline semiorthogonal and scaled bi-orthogonal wavelet and wavelet packet transforms on the matrix conditioning, matrix sparsity, and the performance of the CGNE iterative solver. Several conclusion have been reached. It is well known that the matrix condition number for impedance matrices (obtained from discretization with simple basis functions) grows rather slowly with matrix size N, when N is proportional to the electric size of the scattering object. This leads to a small increase of CGNE iterations with N. Orthogonal transformations do not change the condition numbers and the convergence behavior of the iterative solver.

We found that the condition numbers of matrices transformed with nonorthogonal transforms increased faster with N than those treated with the orthogonal transforms (Table VIII). This growth was due to the higher condition numbers of the transforms themselves (see Tables IV and V). The growth was only little faster for the SWT and the SBWT with small numbers of vanishing moments m. The condition numbers of large matrices transformed with the SWT and SBWT with $m > 4$ are several orders of magnitude larger than those from orthogonal transforms. The largest condition numbers, as predicted by theory, were produced by nonorthogonal wavelet packet transforms (the SWP and BWP).

A comparison of matrix sparsities obtained with different wavelet transforms for the problem of scattering from infinite cylinders indicates that generally an SWT with $m > 2$ yields better sparsity than any OWT. On the other hand, the SBWT and the BWP are consistently worse than both SWT and OWT. The best sparsity results are obtained by the orthogonal wavelet packet transforms (OWP).

A comparison of number of iterations needed to reduce the initial residual a fixed factor showed that matrices transformed with the SWT, SBWT, OWT, and BWP required many more iterations than those transformed with the OWT. The increase of iterations is large enough to completely offset any gains from the slightly sparser matrices obtained in most of the SWT cases.

The numerical results generally support the thesis that the much higher condition numbers of impedance matrices obtained with the nonorthogonal wavelets lead to more iterations and thus cancel any benefits from higher sparsity.

In practical computations effective preconditioners are necessary to speed up the convergence of iterations even without any sparsifying transformations. It appears that in order to use semiorthogonal wavelet transforms effectively inexpensive preconditioners must be found to drastically lower the number of iterations. Unless such preconditioners are found the orthogonal wavelets and orthogonal wavelet packets will remain more efficient overall.

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