Matched Asymptotic Expansion for the Low-Frequency Scattering by a Semi-Circular Trough in a Ground Plane

Robert W. Scharstein and Anthony M. J. Davis

Abstract—**Plane wave scattering by an electrically small circular trough cut in an infinite ground plane is solved analytically for both the TM and TE polarizations. A quasi-static solution for the inner field based upon a transformation to bipolar coordinates exploits the failure of the narrow trough to react to the detailed wave nature of the incident field and forms the starting point for the method of matched asymptotic expansions. The distant behavior of the inner field must agree with the near behavior of the outer field, which is a radiative solution of the Helmholtz equation. In addition to yielding several analytic terms of the solution in low-order powers and the logarithm of the trough wave size ka the matching process provides an account of the interplay between all of the physical parameters.**

Index Terms—**Asymptotic analysis, electromagnetic scattering, electromagnetic scattering from a gap, ground plane, quasi-static.**

I. Introduction

THE SCATTERING of a plane electromagnetic wave by the concave trough and several variants is treated by [1]–[4]. This geometry is pervasive throughout applied electromagnetics and is simple enough to be considered a canonical scatterer. For example, it is a basic version of the cavity-backed aperture that is a central topic in aircraft radar signature studies.

Integral equation techniques based upon the application of the equivalence principle to a curved boundary between the partial cylindrical cavity and a half-space are developed in [1], [2], and [4] and allow for the presence of different dielectrics in the interior and exterior regions. In such cases, the picture is that of a "partially embedded dielectric cylinder" [2]. Some simplification in the mathematical details occurs when the cylinder is half buried, producing the special case of the semicircular trough [3], [4]. Although the initial approach in [2] is a moment-method expansion of the aperture field, the computations there quickly become equivalent to the "dual series" approaches in [1], [3], and [4] and whereby Fourier series of cylindrical harmonics are truncated and forced to agree in the mean-square sense over the semi-circular boundary surface. As is common in such mixed boundary value problems, any advantages of using ordinary eigenfunctions of the wave operator are shadowed by the forfeit of orthogonality over the split boundary. Such methods are practically restricted to the low-frequency regime, because of the nonuniform convergence of the modal expansions. However, the persistence and patience of these authors [2]–[4] with computers has resulted in some impressive numerical results within the accuracy limits implied by "numerical" or "self" convergence tests, presumably for troughs as large as $ka \approx 12$. The wavenumber is $k = \omega/c = 2\pi/\lambda$ and $2a$ is the trough diameter.

The physical source of the convergence difficulty in the Fourier series is the well-known singular behavior of the fields in the vicinity of the two corners where the curved channel joins the flat ground plane. The tenable objective of the present paper is the derivation of several dominant terms in ascending powers of ka that comprise a perturbation expansion for the trough-scattered field based on the intuitive idea that the far field sees the trough as a point singularity while the near field sees very long waves. Both TM (soft) and TE (hard) polarizations of the incident electromagnetic wave are considered. Each case begins with a static solution which is successively modified and linked to the proper radiation field under the framework of the method of matched asymptotic expansions [5]. The starting point for the method is the exact solution to the static Laplace's equation, which is forced by several terms in the low-frequency expansion of the boundary behavior of the incident geometrical optics field. This exact solution is derived as a Fourier integral in bipolar coordinates, in terms of which the physical domain is an infinite strip. The far behavior of this inner expansion is matched to the near behavior of the appropriate radiative outer field, in a careful succession of steps that group terms of common, low-order powers of ka. In addition to supplying the usual Hankel functions for the outer expansion, the Helmholtz equation provides the mechanism to generate higher order terms in the inner expansion via a perturbative sequence of Poisson equations. In the development, contributions of order $(ka)^4 \ln ka$ appear and the expansions are adjusted to include such intermediate terms of magnitude between $(ka)^3$ and $(ka)^4$. The resultant low-frequency expressions are in agreement with the applicable results of [3], for example. An entirely different, and more difficult, high-frequency approximation is required to characterize the interaction of the scatterer of Fig. 1 with short radio waves.

The TM excitation of the semicircular trough is formulated as a scalar boundary value problem in Section II, the applicable static solution is obtained in Section II-A, and the perturbative correction to account for dynamic effects via the Helmholtz equation is obtained in Section II-B. The quasi-static field reacts

Manuscript received May 22, 1998; revised April 2, 1999.
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Publisher Item Identifier S 0018-926X(00)04376-3.

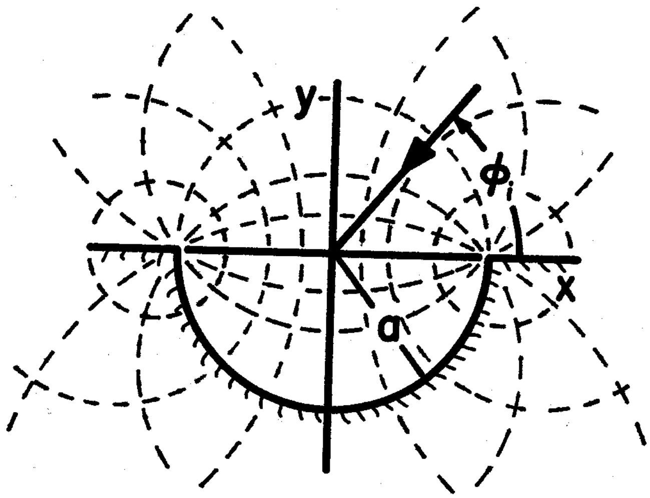

Fig. 1. Semicircular trough in ground plane.

directly with the corners and curvature of the trough and, therefore, the lengthy inner field calculations are performed first. This permits a clean account of the motivating ideas and detailed evolution of the matching procedure in Section II-C, resulting in the desired far-field expansion for the trough-scattered TM field. Section III is a concise summary of the changes required by the TE-polarized wave. Modifications to accommodate dielectric loading of the trough region are the subject of continuing research by the authors. Likewise, circular indentations other than the semi-circle are amenable to the present method, but the mathematics is more clearly presented as a sequel. If the indented boundary lies at the planar interface between penetrable media, i.e., two dielectric half-spaces, then the mathematical procedure of this paper can be suitably adapted, in principle, to extract the dominant low-frequency terms in that scattered field. In such a geometry, the excitation could also be a surface wave. The vector problem of a plane electromagnetic wave incident upon a partial spherical depression is a candidate for the general technique of this paper. Toroidal coordinates are appropriate for such a three-dimensional geometry, which is further simplified through the orthogonality of the azimuthal Fourier modes of a "body of revolution."

Analytic methods such as matched asymptotic expansions derive their power from the exploitation of specific coordinate systems and symmetries. Therefore, compared with general numerical approaches, the realizable scope of the present method is restricted. Furthermore, attempts to obtain the next term of the asymptotic expansion beyond the three terms identified herein are beset with complicated algebra whereupon the reasonably clear physical interactions between the small scatterer and the radiation field become obscured and the method loses its fundamental appeal. Not only are the resultant *closed-form* answers invaluable characterizations of the scattering problem, but the back-and-forth interplay between the near (inner) and far (outer) fields of the mathematical procedure give the reader a new view of the important physics. Unlike numerical approaches that involve a matrix inversion, for example, each important physical parameter can be traced through to the final result.

II. TM Excitation of the Semicircular Trough

The TM electromagnetic field in this two-dimensional geometry is completely specified in terms of the z-directed component of the complex phasor electric field, denoted here by the scalar $\psi(x, y)$. With $\exp(-i\omega t)$ time-harmonic behavior, the sum of the unit-amplitude plane wave incident from the direction of ϕ_i (Fig. 1) and the geometrical-optics reflection from the ground plane is

$$\begin{aligned}\psi_i &= e^{-ikr\,\cos(\phi-\phi_i)} - e^{-ikr\,\cos(\phi+\phi_i)} \\ &= -2i\,\sin(ky\,\sin\,\phi_i)\,e^{-ikx\,\cos\,\phi_i} \end{aligned} \tag{1}$$

in terms of both Cartesian and polar coordinates. The total field is defined as the sum of this "incident" (geometrical optics) field ψ_i that would suffice in the absence of the trough, plus the scattered field, denoted simply as ψ, which is the direct contribution of the trough. The sought-after scattered field satisfies the Dirichlet boundary value problem

$$(\nabla^2 + k^2)\psi(x, y) = 0, \qquad \psi(x, y) = -\psi_i(x, y) \text{ on } C \tag{2}$$

where the boundary surface C includes both the flat sections $|x| \geq a$ and the semicircular arc $r = a, \pi \leq \phi \leq 2\pi$. Naturally, ψ consists of outgoing waves at infinity, in compliance with the usual Sommerfeld radiation condition.

A. Quasi-Static Solution

The dominant interaction between the electrically small trough and the incident wave (1) is adequately captured by the action of Laplace's equation

$$\nabla^2\Psi^{(0)}(x, y) = 0 \tag{3}$$

as the static limit of the Helmholtz operator of (2). To leading order, the narrow trough does not "see" the detailed wave nature of the incident field. However, even this zeroth-order static solution contains critical information about the full dynamic field, as it is forced by the boundary data of the true incident wave-field of (1). A Taylor series expansion of the incident field (1) for small kr gives

$$\begin{aligned}\psi_i(X, Y) = &\,\epsilon[-2i\,\sin\,\phi_i Y] + \epsilon^2[-\sin\,2\phi_i XY] \\ &+ \epsilon^3\left\{\frac{i}{4}\left[\sin\,\phi_i(X^2Y + Y^3)\right.\right. \\ &\left.\left. + \sin\,3\phi_i\left(X^2Y - \tfrac{1}{3}Y^3\right)\right]\right\} + O(\epsilon^4)\end{aligned} \tag{4}$$

in ascending powers of the Helmholtz parameter $\epsilon = ka$, and with the inner coordinates

$$X = \frac{x}{a}, \qquad Y = \frac{y}{a}. \tag{5}$$

At $R = 1$, $X = \cos\,\phi$, $Y = \sin\,\phi$ and the incident wave is

$$\begin{aligned}(\psi_i)_{R=1} = &\,\epsilon[-2i\,\sin\,\phi_i\,\sin\,\phi] + \epsilon^2\left[-\tfrac{1}{2}\,\sin\,2\phi_i\,\sin\,2\phi\right] \\ &+ \epsilon^3\left\{\frac{i}{4}\left[\sin\,\phi_i\,\sin\,\phi + \frac{1}{3}\,\sin\,3\phi_i\,\sin\,3\phi\right]\right\} \\ &+ O(\epsilon^4).\end{aligned} \tag{6}$$

The tangential derivative on the curved surface

$$\left(\frac{1}{R}\frac{\partial\psi_i}{\partial\phi}\right)_{R=1} = \epsilon[-2i\sin\phi_i X] + \epsilon^2[-\sin 2\phi_i(X^2-Y^2)] + \epsilon^3\left\{\frac{i}{4}[\sin\phi_i X + \sin 3\phi_i(X^3-3XY^2)]\right\} + O(\epsilon^4) \tag{7}$$

is useful to connect this TM solution via conjugate functions to the TE field in Section III.

Laplace's equation is invariant under a mapping to bipolar coordinates (ξ, η) [6]

$$X = \frac{\sinh\xi}{\cosh\xi - \cos\eta}, \qquad Y = \frac{\sin\eta}{\cosh\xi - \cos\eta} \tag{8}$$

with metric coefficients

$$h_\xi = h_\eta = \frac{1}{\cosh\xi - \cos\eta} \tag{9}$$

from the Jacobian of the transformation. Several surfaces of constant ξ and η are drawn as dashed curves on the x–y plane of Fig. 1. Note that

$$X^2 + Y^2 = \frac{\cosh\xi + \cos\eta}{\cosh\xi - \cos\eta} = 1 \quad \text{when} \quad \eta = \pi/2,\, 3\pi/2 \tag{10}$$

and the semicircular trough boundary in the original coordinate system is mapped to the line $\eta = 3\pi/2$, while the flat sections $|x| > a$ are projected onto the line $\eta = 0$. Therefore, the entire domain of the original trough problem of Fig. 1, consisting of the trough channel plus the complete upper half of the x–y plane is mapped via (8) to the strip $-\infty < \xi < \infty$, $0 \le \eta \le 3\pi/2$ in the ξ–η plane. Hence, the static boundary value problem is

$$\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)\Psi^{(0)}(\xi, \eta) = 0 \tag{11}$$

subject to

$$\Psi^{(0)}(\xi, 0) = 0 \tag{12}$$

on the flat portion of the boundary C and from (6)

$$\Psi^{(0)}(\xi, 3\pi/2) \approx \epsilon\left[\frac{-2i\sin\phi_i}{\cosh\xi}\right] + \epsilon^2\left[-\sin 2\phi_i\frac{\sinh\xi}{\cosh^2\xi}\right] + \epsilon^3\frac{i}{4} \cdot\left[\frac{\sin\phi_i}{\cosh\xi} + \sin 3\phi_i\left(\operatorname{sech}\xi\tanh^2\xi - \frac{1}{3}\operatorname{sech}^3\xi\right)\right] \tag{13}$$

on the curved surface. Even and odd symmetry components of the solution are written as Fourier integrals

$$\Psi^{(0)}_{e,o}(\xi, \eta) = \int_0^\infty \left\{\begin{matrix} P(\kappa)\cos\kappa\xi \\ Q(\kappa)\sin\kappa\xi \end{matrix}\right\}\sinh\kappa\eta\, d\kappa. \tag{14}$$

Successive differentiation of the spectral representation [7]

$$\frac{1}{\cosh\xi} = \int_0^\infty \frac{\cos(\kappa\xi)}{\cosh(\pi\kappa/2)}\, d\kappa \tag{15}$$

supplies the additional forms

$$\left.\begin{matrix}\operatorname{sech}^3\xi \\ \operatorname{sech}\xi\tanh^2\xi\end{matrix}\right\} = \frac{1}{2}\int_0^\infty \frac{(1\pm\kappa^2)\cos(\kappa\xi)}{\cosh(\pi\kappa/2)}\, d\kappa. \tag{16}$$

Insertion of (15) and (16) into the boundary function (13) permits identification of the Fourier transforms $P(\kappa)$ and $Q(\kappa)$ in (14) when evaluated at $\eta = 3\pi/2$. The static solution can now be written as

$$\Psi^{(0)}(\xi, \eta) = \epsilon(-2i\sin\phi_i)I(\xi, \eta) + \epsilon^2(\sin 2\phi_i)\frac{\partial}{\partial\xi}I(\xi, \eta) + \epsilon^3\frac{i}{4}\left[\sin\phi_i I(\xi, \eta) + \frac{1}{3}\sin 3\phi_i \cdot\left(2\frac{\partial^2}{\partial\xi^2}I(\xi, \eta) + I(\xi, \eta)\right)\right] \tag{17}$$

in terms of the integral function

$$I(\xi, \eta) = \int_0^\infty \frac{\sinh(\kappa\eta)\cos(\kappa\xi)}{\sinh(3\pi\kappa/2)\cosh(\pi\kappa/2)}\, d\kappa = \frac{1}{2}\int_{-\infty}^\infty \frac{\sinh(\kappa\eta)e^{i\kappa\xi}}{\sinh(3\pi\kappa/2)\cosh(\pi\kappa/2)}\, d\kappa. \tag{18}$$

The integrand has two classes of poles on the imaginary axis of the complex $\kappa = u + iv$ plane given by $v_n = 2n/3$ and $v_m = 2m+1$ and is analytic at the origin of the κ plane. When ξ is positive, the symmetric integral (18) is computed as the sum of residues in the upper-half plane

$$\begin{aligned} I(\xi, \eta) &= -\frac{2}{3}\sum_{n=1}^\infty (-1)^n\frac{\sin(2n\eta/3)}{\cos(n\pi/3)}e^{-2n\xi/3} - 2\sum_{m=0}^\infty \sin[(2m+1)\eta]e^{-(2m+1)\xi} \\ &= -2\sum_{n=1}^\infty \sin(n\eta)e^{-n\xi} + \tfrac{4}{3}\sum_{n=1}^\infty \sin(2n\eta/3)e^{-2n\xi/3} \\ &= -\Im\left[\coth\tfrac{1}{2}\zeta\right] + \tfrac{2}{3}\Im\left[\coth\tfrac{1}{3}\zeta\right] \\ &= -\frac{\sin\eta}{\cosh\xi - \cos\eta} + \frac{2}{3}\frac{\sin(2\eta/3)}{\cosh(2\xi/3) - \cos(2\eta/3)} \end{aligned} \tag{19}$$

with $\zeta = \xi - i\eta$. This is an appropriate expansion for the far behavior of the inner field because ξ and η both approach zero as $r \to \infty$. The first several terms of I are

$$I = \Im[-A_1\zeta - A_3\zeta^3 - A_5\zeta^5 - \cdots] \tag{20}$$

even in ξ as required by (18) and with constants

$$A_1 = \tfrac{5}{54}, \qquad A_3 = -\tfrac{13}{5832}, \qquad A_5 = \tfrac{19}{314\,928}. \tag{21}$$

In terms of the polar representation $\zeta = \chi e^{-i\beta}$, I, and its required ξ-derivatives are

$$\begin{aligned} I &= A_1\chi\sin\beta + A_3\chi^3\sin 3\beta + A_5\chi^5\sin 5\beta \\ \frac{\partial I}{\partial\xi} &= 3A_3\chi^2\sin 2\beta + 5A_5\chi^4\sin 4\beta \\ \frac{\partial^2 I}{\partial\xi^2} &= 6A_3\chi\sin\beta + 20A_5\chi^3\sin 3\beta. \end{aligned} \tag{22}$$

In terms of $Z = \mathrm{R}e^{i\phi}$, where

$$\frac{\zeta}{2} = \tanh^{-1}\frac{1}{Z} = \frac{1}{Z} + \frac{1}{3Z^3} + \frac{1}{5Z^5} + \cdots \tag{23}$$

the three forms above are

$$
\begin{aligned}
I =& \Im\left[-2A_1\left(\frac{1}{Z}+\frac{1}{3Z^3}+\frac{1}{5Z^5}\right)\right.\\
&\left.-8A_3\left(\frac{1}{Z}+\frac{1}{3Z^3}\right)^3-\frac{32A_5}{Z^5}+\cdots\right]\\
&\sim 2A_1\frac{\sin\phi}{R}+\left(\frac{2}{3}A_1+8A_3\right)\frac{\sin 3\phi}{R^3}\\
&+\left(\frac{2}{5}A_1+8A_3+32A_5\right)\frac{\sin 5\phi}{R^5}
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
\frac{\partial I}{\partial\xi} &\sim \Im\left[-3A_3\cdot 4\left(\frac{1}{Z}+\frac{1}{3Z^3}\right)^2-\frac{5A_5\cdot 16}{Z^4}\right]\\
&\sim 12A_3\frac{\sin 2\phi}{R^2}+(8A_3+80A_5)\frac{\sin 4\phi}{R^4}
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
\frac{\partial^2 I}{\partial\xi^2} &\sim \Im\left[-6A_3\cdot 2\left(\frac{1}{Z}+\frac{1}{3Z^3}\right)-\frac{20A_5\cdot 8}{Z^3}\right]\\
&\sim 12A_3\frac{\sin\phi}{R}+(2A_3+160A_5)\frac{\sin 3\phi}{R^3}.
\end{aligned}
\tag{26}
$$

The "static" solution (17) in the inner field is therefore

$$
\begin{aligned}
&\Psi^{(0)}(R,\phi)\\
&\sim -\epsilon\, 2i\sin\phi_i\left[2A_1\frac{\sin\phi}{R}+\left(\frac{2}{3}A_1+8A_3\right)\right.\\
&\quad\left.\cdot\frac{\sin 3\phi}{R^3}+\left(\frac{2}{5}A_1+8A_3+32A_5\right)\frac{\sin 5\phi}{R^5}\right]\\
&\quad+\epsilon^2\sin 2\phi_i\left[12A_3\frac{\sin 2\phi}{R^2}+(8A_3+80A_5)\frac{\sin 4\phi}{R^4}\right]\\
&\quad+\epsilon^3\frac{i}{4}\left\{\sin\phi_i\left[2A_1\frac{\sin\phi}{R}+\left(\frac{2}{3}A_1+8A_3\right)\frac{\sin 3\phi}{R^3}\right]\right.\\
&\quad+\frac{1}{3}\sin 3\phi_i\left[(2A_1+24A_3)\frac{\sin\phi}{R}\right.\\
&\quad\left.\left.+\left(\frac{2}{3}A_1+12A_3+320A_5\right)\frac{\sin 3\phi}{R^3}\right]\right\}\\
&\quad+\text{terms of higher order in }\epsilon.
\end{aligned}
\tag{27}
$$

B. Perturbative Correction

A perturbative correction to this inner field proceeds by expressing the scaled Helmholtz operator in inner coordinates

$$(\nabla_R^2+\epsilon^2)\Psi(R,\phi)=0 \tag{28}$$

and writing the (partial) solution as the series

$$\Psi(R,\phi)=\epsilon\Psi_1(R,\phi)+\epsilon^2\Psi_2(R,\phi)+\epsilon^3\Psi_3(R,\phi)+\cdots \tag{29}$$

in which there exists the possibility of inserting terms with scale factors involving $\ln\epsilon$. Insertion of the series (29) into the Helmholtz equation (28) yields upon equating like powers of ϵ, an iterative sequence of Poisson equations

$$\nabla_R^2\Psi_{n+2}=-\Psi_n. \tag{30}$$

The first two terms of $O(\epsilon)$ and $O(\epsilon^2)$ are the first two terms in the static solution $\Psi^{(0)}(R,\phi)$ of (27). That is, both Ψ_1 and Ψ_2 satisfy Laplace's equation while

$$\nabla_R^2\Psi_3=-\Psi_1. \tag{31}$$

A particular solution that vanishes at $\eta=0,\ 3\pi/2$ is required of

$$\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)\Psi_3^{(p)}(\xi,\eta)=2i\sin\phi_i\frac{I(\xi,\eta)}{(\cosh\xi-\cos\eta)^2}. \tag{32}$$

Direct use of the Green's function is precluded by the singular behavior at $(\xi,\eta)=(0,0)$, which, therefore, must be removed by means of a function that mimics the singularity but fails to satisfy (32) exactly. With the far behavior of $I(\xi,\eta)$ given by (22) and with

$$
\begin{aligned}
\cosh\xi-\cos\eta &= \sum_{n=0}^{\infty}\frac{\xi^{2n}-(i\eta)^{2n}}{(2n)!}\\
&=(\xi^2+\eta^2)\left[\frac{1}{2!}+\frac{\xi^2-\eta^2}{4!}+\cdots\right]
\end{aligned}
\tag{33}
$$

the forcing term of (32) is

$$
\begin{aligned}
&\frac{I(\xi,\eta)}{(\cosh\xi-\cos\eta)^2}\\
&\quad\sim 4A_1\frac{\sin\beta}{\chi^3}+\left(4A_3-\frac{1}{3}A_1\right)\frac{\sin 3\beta}{\chi}+\frac{1}{3}A_1\frac{\sin\beta}{\chi}
\end{aligned}
\tag{34}
$$

in terms of the cylindrical bipolar coordinates. Particular solutions corresponding to these three forcing terms are defined by

$$
\begin{aligned}
\nabla_\zeta^2[\chi^{-1}\ln\chi\sin\beta] &= -2\chi^{-3}\sin\beta\\
\nabla_\zeta^2[\chi\ln\chi\sin\beta] &= 2\chi^{-1}\sin\beta\\
\nabla_\zeta^2[\chi\sin 3\beta] &= -8\chi^{-1}\sin 3\beta.
\end{aligned}
\tag{35}
$$

The desired asymptotic solution of (32) is now assembled and written as

$$
\begin{aligned}
\Psi_3^{(p)} \sim 2i\sin\phi_i&\left[-2A_1\chi^{-1}\ln\chi\sin\beta+\tfrac{1}{6}A_1\chi\ln\chi\sin\beta\right.\\
&+\tfrac{1}{2}(A_1/12-A_3)\chi\sin 3\beta\\
&\left.+\left(\frac{B_1}{\chi}+B_2\chi\right)\sin\beta\right].
\end{aligned}
\tag{36}
$$

Inclusion of the two homogeneous terms, with constants B_1 and B_2 derived in Appendix A, sustains the order consistency of this expansion. These constants are needed to justify the matching in Section II-C, but their determination requires more information than an asymptotic solution can provide since there is a Dirichlet condition at $\eta=3\pi/2$ to be satisfied.

The pertinent combinations of the far-field bipolar coordinates are now expressed in terms of the inner polar coordinates via

$$
\begin{aligned}
\ln\chi &= \frac{1}{2}\ln(\xi^2+\eta^2)\sim\ln\left(\frac{2}{R}\right)+\frac{\cos 2\phi}{3R^2}\\
\chi e^{-i\beta} &\sim 2\left(\frac{e^{-i\phi}}{R}+\frac{e^{-3i\phi}}{3R^3}\right)\\
\frac{e^{i\beta}}{\chi} &\sim \frac{1}{2}\left(\mathrm{R}e^{i\phi}-\frac{e^{-i\phi}}{3R}\right)\\
\chi e^{-3i\beta} &= \frac{\zeta^2}{\bar{\zeta}}\sim 2\frac{\overline{Z}}{Z^2}=\frac{2}{R}e^{-3i\phi}.
\end{aligned}
\tag{37}
$$

These expressions enable the particular solution (36) to be written

$$\frac{\Psi_3^{(p)}}{2i\,\sin\,\phi_i} \sim A_1 R\,\ln(R/2)\,\sin\,\phi + \frac{1}{R}[(A_1/6 + 2B_2)\,\sin\,\phi - (A_1/12 + A_3)\,\sin\,3\phi] \qquad (38)$$

as $R \to \infty$.

C. Outer Field Expansion and Matching Results

The outer field consists of outwardly propagating solutions to the Helmholtz equation that vanish on the ground plane where $\phi = 0,\, \pi$. Evidently, an appropriate expansion for the outer field in terms of the outer coordinate $\rho = kr = \epsilon R$ is

$$\psi(\rho,\,\phi) = c_1(\epsilon)H_1^{(1)}(\rho)\,\sin\,\phi + c_2(\epsilon)H_2^{(1)}(\rho)\,\sin\,2\phi + \cdots. \qquad (39)$$

The region of overlap between the inner and outer expansions is characterized by $\rho = \epsilon^\nu$ with $0 < \nu < 1$ such that small ρ corresponds to large $R = \epsilon^{\nu-1}$. The following development is facilitated by the explicit near-in behavior of the first three Hankel functions

$$H_1^{(1)}(\rho) \sim \frac{1}{\pi i}\left[\frac{2}{\rho} - \rho\,\ln(\rho/2) + \rho(1/2 - \gamma + \pi i/2)\right]$$
$$H_2^{(1)}(\rho) \sim \frac{1}{4\pi i}\left[\frac{16}{\rho^2} + 4 - \rho^2\,\ln(\rho/2) + \rho^2(3/4 - \gamma + \pi i/2)\right]$$
$$H_3^{(1)}(\rho) \sim \frac{1}{24\pi i}\left[\frac{384}{\rho^3} + \frac{48}{\rho} + 6\rho - \rho^3\,\ln(\rho/2) + \rho^3(11/12 - \gamma + \pi i/2)\right]. \qquad (40)$$

A careful matching of the outer behavior $(R \to \infty)$ of the inner field $\Psi(R,\,\phi)$ with the inner behavior $(\rho \to 0)$ of the outer field $\psi(\rho,\,\phi)$ is accomplished by grouping terms of common ϵ-dependence and permits the asymptotic construction of the trough-scattered field. The back-and-forth interplay between the inner and outer fields can now proceed directly, unencumbered by the above detailed derivations of the various inner fields. The static inner field is forced by the boundary data of the incident wave (6), whereupon the largest component of Ψ, in powers of ϵ, is

$$\Psi = \epsilon\Psi_1 + \cdots \qquad (41)$$

where (27) supplies

$$\Psi_1 = -2i\,\sin\,\phi_i\left[2A_1\frac{\sin\,\phi}{R} + \cdots\right]. \qquad (42)$$

In terms of outer coordinates, this first term of the inner field is

$$\epsilon\Psi_1 = -2i\,\sin\,\phi_i\left[\epsilon^2 2A_1\frac{\sin\,\phi}{\rho} + \cdots\right]. \qquad (43)$$

The corresponding first term in the outer field is, therefore, $\psi \sim \epsilon^2\psi_2$ with near behavior

$$\psi_2 \sim -4iA_1\,\sin\,\phi_i\frac{\sin\,\phi}{\rho} \qquad \text{as } \rho \to 0. \qquad (44)$$

This must be the close-in limit of the cylindrical wave function

$$H_1^{(1)}(\rho)\sin\,\phi \sim \frac{2}{\pi i\rho}\,\sin\,\phi \qquad (45)$$

and thus

$$\psi_2 = 2\pi A_1\,\sin\,\phi_i H_1^{(1)}(\rho)\,\sin\,\phi + \cdots. \qquad (46)$$

Expressed in inner coordinates, this first term of the outer field is

$$\epsilon^2\psi_2 = -2iA_1\,\sin\,\phi_i\left[\frac{2\epsilon}{R} - \epsilon^3 R\,\ln\,\epsilon + \epsilon^3 R\{-\ln(R/2) + 1/2 - \gamma + \pi i/2\}\right]\sin\,\phi \qquad (47)$$

which introduces terms of order ϵ^3 and $\epsilon^3\,\ln\,\epsilon$ into the inner field. Note that ψ_2 does not affect the $O(\epsilon^2)$ term in Ψ. The inner field expansion, modified from (29), is now of the form

$$\Psi = \epsilon\Psi_1 + \epsilon^2\Psi_2 + \epsilon^3\,\ln\,\epsilon\,\Psi_{31} + \epsilon^3\Psi_3 + \cdots. \qquad (48)$$

The dominant far-field behavior of Ψ_2 is known from the previous quasi-static result of (27),

$$\epsilon^2\Psi_2 = \epsilon^2\,\sin\,2\phi_i\left[12A_3\frac{\sin\,2\phi}{R^2} + \cdots\right] = \sin\,2\phi_i\left[12A_3\epsilon^4\frac{\sin\,2\phi}{\rho^2} + \cdots\right]. \qquad (49)$$

This inner field triggers a corresponding term of $O(\epsilon^4)$ in the outer field such that

$$\psi \sim \epsilon^2\psi_2 + \epsilon^4\psi_4 \qquad (50)$$

where

$$\psi_4 \sim \sin\,2\phi_i 12A_3\frac{\sin\,2\phi}{\rho^2} \qquad \text{as } \rho \to 0 \qquad (51)$$

must be the near behavior of

$$\psi_4 = 3\pi i\,\sin\,2\phi_i A_3 H_2^{(1)}(\rho)\sin\,2\phi. \qquad (52)$$

The two terms of the cumulative outer field (50) are of $O(\epsilon)$ and $O(\epsilon^2)$ when expressed in inner coordinates. According to (47), the distant behavior $(R \to \infty)$ of the next two inner field components is

$$\Psi_{31} \sim 2iA_1\,\sin\,\phi_i R\,\sin\,\phi$$
$$\Psi_3 \sim 2iA_1\,\sin\,\phi_i[\ln\,R/2 - 1/2 + \gamma - \pi i/2]R\,\sin\,\phi. \qquad (53)$$

The harmonic Ψ_{31} satisfies homogeneous boundary conditions at $\eta = 0,\, 3\pi/2$. Since the asymptotic form of $\Psi_{31} \sim R\,\sin\,\phi = Y$ is both harmonic and proportional to the boundary behavior of Ψ_1 on $\eta = 3\pi/2$, the desired function is the combination

$$\Psi_{31} = A_1[2i\,\sin\,\phi_i Y - \Psi_1]. \qquad (54)$$

The perturbation series (29) reveals that Ψ_3 is a solution to a Poisson equation (32), which must cancel the nonzero incident field of (4) on the trough $\eta = 3\pi/2$. The static field $\Psi_3^{(0)}$ of (17) furnishes the needed nonhomogeneous boundary behavior, and $\Psi_3^{(p)}$ of (38) displays the correct $R\,\ln(R/2)\,\sin\phi$ far-field variation in (53). Hence, in addition to the sum $\Psi_3^{(0)} + \Psi_3^{(p)}$, a harmonic function is required that vanishes on $\eta = 0,\, 3\pi/2$ and has the far-field form $\sim R\,\sin\phi = Y$ of the remaining terms in (53). As in (54), this additional function is $\propto [2i\,\sin\phi_i Y - \Psi_1]$, where (53) supplies the scale factor, resulting in

$$\Psi_3 = \Psi_3^{(0)} + \Psi_3^{(p)} + A_1(-1/2 + \gamma - \pi i/2) \cdot [2i\,\sin\phi_i\, Y - \Psi_1]. \quad (55)$$

The four components of the inner field (48) now permit its expression in outer coordinates up to $O(\epsilon^4)$. Proper arrangement of these terms and matching to the near behavior of the three Hankel functions yields the final form for the scattered field

$$\begin{aligned}\psi(\rho,\phi) =& \pi[2\,\sin\phi_i\{\epsilon^2 A_1 - \epsilon^4[5A_1/24 + B_2 \\ &+ A_1^2(\ln\epsilon - 1/2 + \gamma - \pi i/2)]\} \\ &+ \sin 3\phi_i \epsilon^4(A_1/12 + A_3)]H_1^{(1)}(\rho)\,\sin\phi \\ &+ 3\pi i\,\sin 2\phi_i\epsilon^4 A_3 H_2^{(1)}(\rho)\,\sin 2\phi + \pi\,\sin\phi_i\epsilon^4 \\ &\cdot (A_1/12 + A_3)H_3^{(1)}(\rho)\,\sin 3\phi + O(\epsilon^6\,\ln\epsilon).\end{aligned} \quad (56)$$

III. TE Polarization

The scalar field of interest is now the single component of the magnetic field, which is polarized in the axial (z) direction. Some of the ensuing analysis mimics that for the TM polarization and is consequently abbreviated. The Neumann boundary condition, vanishing of the normal derivative on the boundary C, applies and the incident or geometrical optics field is

$$\begin{aligned}\psi_i &= e^{-ikr\,\cos(\phi-\phi_i)} + e^{-ikr\,\cos(\phi+\phi_i)} \\ &= 2\,\cos(ky\,\sin\phi_i)e^{-ikx\,\cos\phi_i} \\ &= 2\,\cos(\epsilon Y\,\sin\phi_i)e^{-i\epsilon X\,\cos\phi_i} \\ &\sim 2 - 2i\epsilon X\,\cos\phi_i - \epsilon^2(Y^2\,\sin^2\phi_i + X^2\,\cos^2\phi_i) \\ &\quad + i\epsilon^3\,\cos\phi_i\left(XY^2\,\sin^2\phi_i + \tfrac{1}{3}X^3\,\cos^2\phi_i\right).\end{aligned} \quad (57)$$

On the curved boundary of the electrically small trough, the normal derivative is

$$\begin{aligned}&\left(\frac{\partial\psi_i}{\partial R}\right)_{R=1,\,Y<0} \\ &= \epsilon[-2i\,\cos\phi_i X] + \epsilon^2[-1 - \cos 2\phi_i(X^2 - Y^2)] \\ &\quad + \epsilon^3\left\{\frac{i}{4}[3\,\cos\phi_i X + \cos 3\phi_i(X^3 - 3XY^2)]\right\} \\ &\quad + O(\epsilon^4).\end{aligned} \quad (58)$$

The scattered field arises to cancel this nonzero normal derivative. The static component of the scattered inner field $\Psi^{(0)}$ is a solution of Laplace's equation subject to these Neumann conditions for which the compatibility condition will necessitate a flux at infinity.

Parallel to the static field development from the TM boundary data of (4), the $O(\epsilon)$ component of the TE field is now written

$$\Psi_1^{(0)} = (-2i\,\cos\phi_i)J(\xi,\eta) \quad (59)$$

where, since

$$\frac{1}{R}\frac{\partial I}{\partial\phi} = \frac{\partial J}{\partial R} \quad (60)$$

$J + iI$ is an analytic function of $X + iY = \mathrm{R}e^{i\phi} = Z$. In view of (8), it is also analytic in the bipolar coordinates $\zeta = \xi - i\eta = \chi e^{-i\beta}$, where the pertinent Cauchy–Riemann equations

$$\frac{\partial J}{\partial\chi} = -\frac{1}{\chi}\frac{\partial I}{\partial\beta}, \qquad \frac{1}{\chi}\frac{\partial J}{\partial\beta} = \frac{\partial I}{\partial\chi} \quad (61)$$

determine the harmonic conjugate

$$J = -A_1\chi\,\cos\beta - A_3\chi^3\,\cos 3\beta - A_5\chi^5\,\cos 5\beta \quad (62)$$

anticipated in (19)

$$J + iI = \tfrac{2}{3}\,\coth\,\tfrac{1}{3}\zeta - \coth\,\tfrac{1}{2}\zeta. \quad (63)$$

Similarly, the $O(\epsilon^2)$ portion of the static TE field is

$$\Psi_2^{(0)} = -\cos 2\phi_i\frac{\partial J}{\partial\xi} + \Psi_2^* \quad (64)$$

where the additional ϕ_i-independent term arises from

$$\left(\frac{\partial\Psi_2^*}{\partial R}\right)_{R=1,\,Y<0} = 1 \quad (65)$$

in (58). This flux out of the trough generates a monopole field and must be balanced by an equal flux from infinity. The two requirements

$$\left(\frac{\partial\Psi_2^*}{\partial\eta}\right)_{\eta=0} = 0 \text{ for } \xi \neq 0$$

and

$$\int_{-\infty}^{\infty}\left(\frac{\partial\Psi_2^*}{\partial\eta}\right)_{\eta=0} d\xi = \pi \quad (66)$$

on the image of the flat portion of S indicate the presence of a Dirac–delta function

$$\left(\frac{\partial\Psi_2^*}{\partial\eta}\right)_{\eta=0} = \pi\delta(\xi) = \int_0^\infty \cos\kappa\xi\,d\kappa. \quad (67)$$

This singularity exists at the origin of the transform coordinates (ξ,η) corresponding to infinity in the original x–y plane. A suitable Fourier transform representation for the monopole term is thus evaluated by residue calculus

$$\begin{aligned}\Psi_2^* =& \int_0^\infty \frac{\cosh\kappa\eta - \cosh(\pi\kappa/2)\cosh\kappa(3\pi/2 - \eta)}{\sinh(3\pi\kappa/2)\cosh(\pi\kappa/2)} \\ &\cdot \frac{\cos\kappa\xi}{\kappa}\,d\kappa \\ =& \tfrac{3}{2}\,\ln\left[2\,\cosh\,\tfrac{2}{3}\xi - 2\,\cos\,\tfrac{2}{3}\eta\right] - \ln\left[2\,\cosh\,\xi - 2\,\cos\,\eta\right] \\ \sim& \ln(2/R) + \frac{2}{9}\frac{\cos 2\phi}{R^2} + 3\,\ln(2/3).\end{aligned} \quad (68)$$

According to (58), the $O(\epsilon^3)$ static field is

$$\Psi_3^{(0)} = \frac{i}{4}\left[\cos\phi_i 3J + \frac{1}{3}\cos 3\phi_i\left(2\frac{\partial^2 J}{\partial\xi^2} + J\right)\right]. \quad (69)$$

By exploiting the conjugate functions J and I, algebra similar to that in Section II can be avoided, and the static TE inner field deduced in the form

$$\begin{aligned}\Psi^{(0)}(R,\phi) &\sim \epsilon 2i\cos\phi_i\left[2A_1\frac{\cos\phi}{R} + \left(\frac{2}{3}A_1 + 8A_3\right)\frac{\cos 3\phi}{R^3}\right.\\ &\left.\quad + \left(\frac{2}{5}A_1 + 8A_3 + 32A_5\right)\frac{\cos 5\phi}{R^5}\right]\\ &+ \epsilon^2\left\{\ln(2/R) + \frac{2}{9}\frac{\cos 2\phi}{R^2} + \cos 2\phi_i\left[12A_3\frac{\cos 2\phi}{R^2}\right.\right.\\ &\left.\left.\quad + (8A_3 + 160A_5)\frac{\cos 4\phi}{R^4}\right]\right\}\\ &- \epsilon^3\frac{i}{4}\left\{\cos\phi_i\left[6A_1\frac{\cos\phi}{R} + (2A_1 + 24A_3)\frac{\cos 3\phi}{R^3}\right]\right.\\ &\quad + \frac{1}{3}\cos 3\phi_i\left[(2A_1 + 24A_3)\frac{\cos\phi}{R}\right.\\ &\left.\left.\quad + \left(\frac{2}{3}A_1 + 12A_3 + 320A_5\right)\frac{\cos 3\phi}{R^3}\right]\right\}\\ &+ \text{terms of higher order in } \epsilon \end{aligned} \quad (70)$$

analogous to (27). Next, a particular solution is required of

$$\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)\Psi_3^{(p)}(\xi,\eta) = 2i\cos\phi_i\frac{J(\xi,\eta)}{(\cosh\xi - \cos\eta)^2} \quad (71)$$

subject to $\partial\Psi_3^{(0)}/\partial\eta = 0$ at $\eta = 0, 3\pi/2$, but now the advantages of conjugate functions are reduced by the presence of the metric $(\cosh\xi - \cos\eta)^{-2}$ which is not analytic in ζ. Thus, for example, there is a sign change in the formula analogous to (34), namely

$$\frac{J(\xi,\eta)}{(\cosh\xi - \cos\eta)^2} \sim -4A_1\frac{\cos\beta}{\chi^3} - \left(4A_3 - \frac{1}{3}A_1\right)\frac{\cos 3\beta}{\chi} + \frac{1}{3}A_1\frac{\cos\beta}{\chi}. \quad (72)$$

Note that since J is an odd function of ξ, the compatibility condition is satisfied. Hence, the use of a formula conjugate to (35) gives the particular solution

$$\begin{aligned}\Psi_3^{(p)} &\sim 2i\cos\phi_i\left[2A_1\chi^{-1}\ln\chi\cos\beta + \tfrac{1}{6}A_1\chi\ln\chi\cos\beta\right.\\ &\quad - \frac{1}{2}(A_1/12 - A_3)\chi\cos 3\beta\\ &\left.\quad + \left(\frac{D_1}{\chi} + D_2\chi\right)\cos\beta\right]\end{aligned} \quad (73)$$

from which (37) reveals to exhibit the distant R behavior

$$\begin{aligned}\frac{\Psi_3^{(p)}}{2i\cos\phi_i} &\sim -A_1 R\ln(R/2)\cos\phi + \frac{1}{R}\\ &\quad\cdot[(A_1/6 + 2D_2)\cos\phi + (A_3 + A_1/12)\cos 3\phi]\end{aligned} \quad (74)$$

analogous to (38). The constant D_2 is evaluated in Appendix B.

Matching the near-field behavior of the Hankel function of order zero

$$\begin{aligned}H_0^{(1)}(\rho) &\sim \frac{2i}{\pi}[\ln(\rho/2) + \gamma]\\ &\quad + 1 - \frac{1}{4}\rho^2\left\{1 + \frac{2i}{\pi}[\ln(\rho/2) + \gamma - 1]\right\}\end{aligned} \quad (75)$$

with the ϕ-independent component of Ψ_2 in the inner field, demonstrates that the outer scattered field has the term

$$\psi \sim \pi i\epsilon^2 H_0^{(1)}(\rho)/2 \quad (76)$$

and so the inner field must have terms

$$\Psi \sim -\frac{1}{4}\epsilon^4 R^2\left\{1 + \frac{2i}{\pi}[\ln(\epsilon R/2) + \gamma - 1]\right\}\frac{\pi i}{2} \quad (77)$$

which includes $(1/4)\epsilon^4\ln\epsilon R^2$. Comparison with the monopole term $(-1/2)\epsilon^2 R^2$ in the incident field ψ_i of (57) shows that the monopole term in the outer scattered field is modified to

$$\psi \sim \frac{\pi i\epsilon^2}{2}\left[1 - \frac{1}{2}\epsilon^2\ln\epsilon\right]H_0^{(1)}(\rho). \quad (78)$$

This is in agreement with [3], since Hinders and Yaghjian work with conjugate expressions.

By using the exponential form of ψ_i in (57) it can be readily shown that

$$\int_{-\pi}^{0}\left(\frac{\partial\psi_i}{\partial R}\right)_{R=1} d\phi = -2\pi\epsilon J_1(\epsilon) \sim -\pi\epsilon^2\left(1 - \frac{1}{8}\epsilon^2\right). \quad (79)$$

The monopole must cancel this flux, i.e., its strength up to $O(\epsilon^4)$ is described by modifying (78) to the form

$$\psi \sim \frac{\pi i\epsilon^2}{2}\left[1 - \frac{1}{2}\epsilon^2\ln\epsilon - \frac{1}{8}\epsilon^2\right]H_0^{(1)}(\rho). \quad (80)$$

The higher order multipole fields in the inner field expression (70) indicate that the outer field $\psi(\rho,\phi)$ also contains the terms

$$\begin{aligned}&\pi\epsilon^2\left[-2A_1\cos\phi_i + \epsilon^2\left(\frac{3A_1}{4}\cos\phi_i\right.\right.\\ &\left.\left.\quad + (A_1/12 + A_3)\cos 3\phi_i\right)\right]H_1^{(1)}(\rho)\cos\phi\\ &- \pi i\epsilon^4 3A_3\cos 2\phi_i H_2^{(1)}(\rho)\cos 2\phi\\ &- \pi\epsilon^4(A_1/12 + A_3)\cos\phi_i H_3^{(1)}(\rho)\cos 3\phi.\end{aligned} \quad (81)$$

As in the TM case, the inner field is henceforth of the form (48), with

$$\Psi_{31} = A_1[-2i\cos\phi_i R\cos\phi + \Psi_1] \quad (82)$$

and

$$\begin{aligned}\Psi_3 &= \Psi_3^{(0)} + \Psi_3^{(p)} + A_1(\gamma - 1/2 - i\pi/2)\\ &\quad\cdot[-2i\cos\phi_i R\cos\phi + \Psi_1]\end{aligned} \quad (83)$$

such that the scattered field expansion follows:

$$\begin{aligned}\psi(\rho,\,\phi) \sim &\frac{\pi i\epsilon^2}{2}\left[1-\frac{1}{2}\epsilon^2\,\ln\,\epsilon-\frac{1}{8}\epsilon^2\right]H_0^{(1)}(\rho)\\ &+\pi\epsilon^2\left[-2A_1\,\cos\,\phi_i+\epsilon^2\left(\frac{3A_1}{4}\,\cos\,\phi_i\right.\right.\\ &\left.\left.+(A_1/12+A_3)\,\cos\,3\phi_i\right)\right]\\ &\cdot H_1^{(1)}(\rho)\cos\phi-2\pi\epsilon^4\,\cos\,\phi_i\\ &\cdot\left[A_1^2(\gamma-1/2-i\pi/2+\ln\,\epsilon)+A_1/12+D_2\right]\\ &\cdot H_1^{(1)}(\rho)\,\cos\,\phi+\pi i\epsilon^4(1/18-3A_3\,\cos\,2\phi_i)\\ &\cdot H_2^{(1)}(\rho)\cos\,2\phi-\pi\epsilon^4(A_1/12+A_3)\,\cos\,\phi_i\\ &\cdot H_3^{(1)}(\rho)\,\cos\,3\phi+O(\epsilon^5\,\ln\,\epsilon).\end{aligned}\tag{84}$$

IV. Conclusion

The method of matched asymptotic expansions yields several terms in a perturbation series for the scattered field due to the TM and TE excitation of the semicircular trough in a ground plane. These low-frequency analytic solutions are expressed in ascending powers of the electrical size of the trough ka and also involve the logarithm of ka. The simple form of the results explicitly shows the dependence of the trough-scattered field upon the source and observation angles as well as the frequency via ka. The unmistakably dominant feature of the TM-polarized far field is the dipole term of $O(ka)^2$, with amplitude in agreement with the dual-series results of Hinders and Yaghjian [3], where the constant $2A_1 = 5/27$ is accurately computed as 0.185. Clearly, the next terms of $O(ka)^4$ in the multipole expansion have insignificant effect upon the radiation field. In the case of the TE-polarized wave, the scattered field also contains a monopole component of $O(ka)^2$ that is independent of the incidence angle. No conclusions applicable to electrically larger troughs follow from this low-frequency solution, which quickly becomes invalid as ka increases beyond unity.

Appendix A
Evaluation of the Constant B_2

Use of the exact form of $I(\xi,\,\eta)$ from (19) shows that the error term in (34) is $O(\chi)$. The Green's function for the problem is

$$\begin{aligned}&G(\xi,\,\eta;\xi_0,\,\eta_0)\\ &\quad=\frac{1}{4\pi}\,\ln\left[\frac{\cosh\,\frac{2}{3}(\xi-\xi_0)-\cos\,\frac{2}{3}(\eta-\eta_0)}{\cosh\,\frac{2}{3}(\xi-\xi_0)-\cos\,\frac{2}{3}(\eta+\eta_0)}\right]\end{aligned}\tag{A.1}$$

but the right-hand side of (32) is too singular at $\chi = 0$ for direct use of $G(\xi-\xi_0,\,\eta;\;\eta_0)$. Note that $4\pi G$ can be written as an integral of the second term in $I(\xi-\xi_0,\,\eta')$, i.e.,

$$\begin{aligned}4\pi G &= -\frac{2}{3}\int_{\eta-\eta_0}^{\eta+\eta_0}\frac{\sin\,\frac{2}{3}\eta'}{\cosh\,\frac{2}{3}(\xi-\xi_0)-\cos\,\frac{2}{3}\eta'}\,d\eta'\\ &= -4\sum_{n=1}^{\infty}\frac{1}{n}\,\exp\left(-\tfrac{2}{3}n|\xi-\xi_0|\right)\sin\,\tfrac{2}{3}n\eta\,\sin\,\tfrac{2}{3}n\eta_0.\end{aligned}\tag{A.2}$$

Hence, the pertinent far-field form

$$\begin{aligned}\frac{\pi}{\eta}\,G \xrightarrow[(\xi,\,\eta)\to(0,\,0)]{} &-\frac{1}{3}\,\frac{\sin\,\frac{2}{3}\eta_0}{\cosh\,\frac{2}{3}\xi_0-\cos\,\frac{2}{3}\eta_0}\\ &\sim\frac{-\eta_0}{\xi_0^2+\eta_0^2}\quad\text{as}\quad(\xi_0,\,\eta_0)\to(0,\,0).\end{aligned}\tag{A.3}$$

Thus, if the terms in (34) are removed from the right-hand side of (32), the remaining particular solution behaves as a multiple of $\eta = \chi\,\sin\,\beta$ as $\chi \to 0$. This removal is achieved by suitably combining three independent functions which vanish at $\eta = 0,\,3\pi/2$ and whose Laplacians have leading terms of the type $\sin\,\beta/\chi^3$, as in (34).

Consider the product $F^{(1)} = f_1 f_2$ of the two harmonic functions

$$f_1=\frac{\sin\,\frac{2}{3}\eta}{\cosh\,\frac{2}{3}\xi-\cos\,\frac{2}{3}\eta},\qquad f_2=\ln(2\,\cosh\,\xi-2\,\cos\,\eta).\tag{A.4}$$

The Laplacian

$$\begin{aligned}\nabla^2F^{(1)} &= 2\nabla f_1\cdot\nabla f_2\\ &= 2\frac{-\frac{2}{3}\,\sin\,\frac{2}{3}\eta\,\sinh\,\frac{2}{3}\xi}{\left(\cosh\,\frac{2}{3}\xi-\cos\,\frac{2}{3}\eta\right)^2}\cdot\frac{\sinh\,\xi}{\cosh\,\xi-\cos\eta}\\ &\quad+2\left[\frac{\frac{2}{3}\,\cos\,\frac{2}{3}\eta}{\cosh\,\frac{2}{3}\xi-\cos\,\frac{2}{3}\eta}-\frac{\frac{2}{3}\,\sin^2\,\frac{2}{3}\eta}{\left(\cosh\,\frac{2}{3}\xi-\cos\,\frac{2}{3}\eta\right)^2}\right]\\ &\quad\cdot\frac{\sin\,\eta}{\cosh\,\xi-\cos\,\eta}\end{aligned}\tag{A.5}$$

exhibits the required singularity. Its details are more easily obtained by first expanding $f_1,\;f_2$ and then evaluating $\nabla f_1\cdot\nabla f_2$. The asymptotic form of f_2 is

$$\begin{aligned}f_2 &\sim \ln\left[\xi^2+\tfrac{1}{12}\xi^4+\eta^2-\tfrac{1}{12}\eta^4\right]\\ &= \ln(\xi^2+\eta^2)+\ln\left[1+\tfrac{1}{12}(\xi^2-\eta^2)\right]\\ &\sim 2\,\ln\,\chi+\tfrac{1}{12}\chi^2\,\cos\,2\beta.\end{aligned}\tag{A.6}$$

Define

$$f_2^*=\ln\left[\tfrac{9}{4}(2\,\cosh\,\tfrac{2}{3}\xi-2\,\cos\,\tfrac{2}{3}\eta)\right]\tag{A.7}$$

with asymptotic form

$$f_2^*\sim 2\,\ln\,\chi+\tfrac{1}{27}\chi^2\,\cos\,2\beta=\ln(\xi^2+\eta^2)+\tfrac{1}{27}(\xi^2-\eta^2).\tag{A.8}$$

Hence

$$f_1=\frac{3}{2}\,\frac{\partial f_2^*}{\partial\eta}\sim\frac{3\eta}{\xi^2+\eta^2}-\frac{3\eta}{27}=3\,\sin\,\beta\left(\frac{1}{\chi}-\frac{\chi}{27}\right)\tag{A.9}$$

and the required Laplacian is

$$\begin{aligned}\nabla^2F^{(1)} &= 2\nabla f_1\cdot\nabla f_2\\ &= 2\frac{\partial f_1}{\partial\chi}\,\frac{\partial f_2}{\partial\chi}+\frac{2}{\chi^2}\,\frac{\partial f_1}{\partial\beta}\,\frac{\partial f_2}{\partial\beta}\\ &\sim -\frac{12\,\sin\,\beta}{\chi^3}-\frac{4\,\sin\,\beta}{9\chi}-\frac{\sin\,3\beta}{\chi}.\end{aligned}\tag{A.10}$$

The limiting form of the Laplacian of a second function $F^{(2)} = f_1 f_2^*$ is similarly

$$\nabla^2 F^{(2)} \sim -\frac{12 \sin \beta}{\chi^3} - \frac{4 \sin \beta}{9\chi} - \frac{4 \sin 3\beta}{9\chi}. \qquad \text{(A.11)}$$

A third function

$$F^{(3)} = -\sum_{n=-\infty}^{\infty} \frac{\eta + 3n\pi}{\xi^2 + (\eta + 3n\pi)^2} \ln[\xi^2 + (\eta + 3n\pi)^2] \qquad \text{(A.12)}$$

has the singular behavior of the first term on the right-hand side of (36), but also vanishes at $\eta = 0,\, 3\pi/2$ because if f is an odd function, then the image system is such that

$$\sum_{n=-\infty}^{\infty} f(\eta + 2nL) = 0 \quad \text{at} \quad \eta = 0,\, \pm L,\, \pm 2L,\, \cdots. \qquad \text{(A.13)}$$

Its Laplacian follows from (35)

$$\nabla^2 F^{(3)} = 4 \sum_{n=-\infty}^{\infty} \frac{\eta + 3n\pi}{[\xi^2 + (\eta + 3n\pi)^2]^2} \sim \frac{4 \sin \beta}{\chi^3}. \qquad \text{(A.14)}$$

The desired linear combination of the above three functions follows from writing

$$\Psi_3^{(p)} = \frac{2i \sin \phi_i}{81} \Big[C_1 F^{(1)} + C_2 F^{(2)} + C_3 F^{(3)} + \text{bounded terms} \Big] \qquad \text{(A.15)}$$

whereupon

$$\nabla^2 \Psi_3^{(p)} \sim \frac{2i \sin \phi_i}{81} \cdot \left\{ [C_3 - 3(C_1 + C_2)] \frac{4 \sin \beta}{\chi^3} - (C_1 + C_2) \frac{4 \sin \beta}{9\chi} - \left(C_1 + \frac{4}{9} C_2 \right) \frac{\sin 3\beta}{\chi} \right\}. \qquad \text{(A.16)}$$

The scale factor 81 trims the ensuing arithmetic. By comparison with (34), set

$$\begin{aligned} C_3 - 3(C_1 + C_2) &= 81 \cdot A_1 \\ -\tfrac{4}{9}(C_1 + C_2) &= 81 \cdot A_1/3 \\ C_1 + \tfrac{4}{9} C_2 &= 81 \cdot (A_1/3 - 4A_3). \end{aligned} \qquad \text{(A.17)}$$

Now check the behavior of $\Psi_3^{(p)}$

$$\begin{aligned} F^{(1)} &\sim 6 \frac{\ln \chi}{\chi} \sin \beta - \frac{2}{9} \chi \ln \chi \sin \beta + \frac{1}{4} \chi \sin \beta \cos 2\beta \\ F^{(2)} &\sim 6 \frac{\ln \chi}{\chi} \sin \beta - \frac{2}{9} \chi \ln \chi \sin \beta + \frac{1}{9} \chi \sin \beta \cos 2\beta \\ F^{(3)} &\sim -2 \frac{\ln \chi}{\chi} \sin \beta + \left(\frac{2}{3\pi} \right)^2 \chi \sin \beta \sum_{n=1}^{\infty} \frac{\ln(3n\pi) - 1}{n^2}. \end{aligned} \qquad \text{(A.18)}$$

Thus

$$\begin{aligned} \tfrac{1}{81} \Big[C_1 F^{(1)} + C_2 F^{(2)} + C_3 F^{(3)} \Big] &\sim -2A_1 \frac{\ln \chi}{\chi} \sin \beta + \frac{1}{6} A_1 \chi \ln \chi \sin \beta \\ &\quad + \tfrac{1}{2}(A_1/12 - A_3) \chi (\sin 3\beta - \sin \beta) \\ &\quad - \frac{5A_1}{9\pi^2} \chi \sin \beta \sum_{n=1}^{\infty} \frac{\ln(3n\pi) - 1}{n^2} \end{aligned} \qquad \text{(A.19)}$$

agrees with (36). Evidently $B_1 = 0$; in fact, it was precluded by the $9/4$ factor in f^*. Adding $\ln(9/4)$ to f^* corresponds to adding a homogeneous solution to $F^{(2)}$. The required solution is therefore

$$\begin{aligned} \Psi_3^{(p)}(\xi,\, \eta) = {} & \frac{2i \sin \phi_i}{81} \Big[C_1 F^{(1)}(\xi,\, \eta) + C_2 F^{(2)}(\xi,\, \eta) + C_3 F^{(3)}(\xi,\, \eta) \Big] \\ & + \frac{2i \sin \phi_i}{4\pi} \int_{-\infty}^{\infty} \int_0^{3\pi/2} \cdot \left\{ \frac{I(\xi_0, \eta_0)}{(\cosh \xi_0 - \cos \eta_0)^2} - \frac{\nabla_0^2}{81} \cdot \Big[C_1 F^{(1)}(\xi_0,\, \eta_0) + C_2 F^{(2)}(\xi_0,\, \eta_0) + C_3 F^{(3)}(\xi_0,\, \eta_0) \Big] \right\} \\ & \cdot \ln \left[\frac{\cosh \frac{2}{3}(\xi - \xi_0) - \cos \frac{2}{3}(\eta - \eta_0)}{\cosh \frac{2}{3}(\xi - \xi_0) - \cos \frac{2}{3}(\eta + \eta_0)} \right] \cdot d\xi_0 \, d\eta_0 \end{aligned} \qquad \text{(A.20)}$$

whereupon its far-field limit $(\xi,\, \eta) \to (0,\, 0)$ yields, from (A.3) and (A.19), the remaining constant in (36)

$$\begin{aligned} B_2 = {} & \frac{1}{2}(A_3 - A_1/12) - \frac{5A_1}{9\pi^2} \sum_{n=1}^{\infty} \frac{\ln(3n\pi) - 1}{n^2} - \frac{1}{3\pi} \\ & \cdot \int_{-\infty}^{\infty} \int_0^{3\pi/2} \left\{ \frac{I(\xi_0,\, \eta_0)}{(\cosh \xi_0 - \cos \eta_0)^2} - \frac{\nabla_0^2}{81} \cdot \Big[C_1 F^{(1)}(\xi_0,\, \eta_0) + C_2 F^{(2)}(\xi_0,\, \eta_0) + C_3 F^{(3)}(\xi_0,\, \eta_0) \Big] \right\} \\ & \cdot \frac{\sin \frac{2}{3}\eta_0}{\cosh \frac{2}{3}\xi_0 - \cos \frac{2}{3}\eta_0} \, d\xi_0 \, d\eta_0. \end{aligned} \qquad \text{(A.21)}$$

APPENDIX B
EVALUATION OF THE CONSTANT D_2

The Green's function for the Poisson equation (71) with Neumann boundary conditions at $\eta = 0,\, 3\pi/2$ must be defined by

$$\nabla^2 G(\xi,\, \eta) = [\delta(\xi - \xi_0) - \delta(\xi + \xi_0)] \delta(\eta - \eta_0) \quad \begin{pmatrix} \xi_0 > 0 \\ 0 < \eta_0 < 3\pi/2 \end{pmatrix} \qquad \text{(B.1)}$$

whereupon

$$G = \frac{1}{4\pi} \ln \left[\frac{\cosh \frac{2}{3}(\xi - \xi_0) - \cos \frac{2}{3}(\eta - \eta_0)}{\cosh \frac{2}{3}(\xi + \xi_0) - \cos \frac{2}{3}(\eta - \eta_0)} \cdot \frac{\cosh \frac{2}{3}(\xi - \xi_0) - \cos \frac{2}{3}(\eta + \eta_0)}{\cosh \frac{2}{3}(\xi + \xi_0) - \cos \frac{2}{3}(\eta + \eta_0)} \right] \qquad \text{(B.2)}$$

with the forcing term restricted to $\xi_0 > 0$. This formulation of G is, of course, consistent with the right hand side of (71) being an odd function of ξ. For $0 \le |\xi| < \xi_0$, (B.2) can be rewritten as

$$
\begin{aligned}
4\pi G =& -\frac{2}{3}\int_{\xi_0-\xi}^{\xi_0+\xi}\left[\frac{\sinh \frac{2}{3}\xi'}{\cosh \frac{2}{3}\xi' - \cos\frac{2}{3}(\eta-\eta_0)}\right.\\
&\left.+\frac{\sinh \frac{2}{3}\xi'}{\cosh \frac{2}{3}\xi' - \cos\frac{2}{3}(\eta+\eta_0)}\right] d\xi'\\
=& -8\left\{\frac{1}{3}\xi + \sum_{n=1}^{\infty}\frac{1}{n}e^{-2n\xi_0/3}\right.\\
&\left.\cdot \sinh \tfrac{2}{3}n\xi \cos \tfrac{2}{3}n\eta \cos \tfrac{2}{3}n\eta_0\right\} \qquad \text{(B.3)}
\end{aligned}
$$

and so the far-field form is such that

$$
\begin{aligned}
\frac{\pi}{\xi}G \xrightarrow[(\xi,\,\eta)\to(0,\,0)]{} & -\frac{2}{3}\,\frac{\sinh \frac{2}{3}\xi_0}{\cosh\frac{2}{3}\xi_0 - \cos\frac{2}{3}\eta_0}\\
\sim & \frac{-2\xi_0}{\xi_0^2+\eta_0^2} \quad \text{as} \quad (\xi_0,\,\eta_0)\to(0,\,0). \qquad \text{(B.4)}
\end{aligned}
$$

In this case, the remaining particular solution behaves as a multiple of $\xi = \chi \cos\beta$ as $\chi \to 0$.

Introduction of the function

$$
f_1 = \frac{\sinh\frac{2}{3}\xi}{\cosh\frac{2}{3}\xi - \cos\frac{2}{3}\eta} = \frac{3}{2}\frac{\partial f_2^*}{\partial \xi} \sim 3\cos\beta\left(\frac{1}{\chi}+\frac{\chi}{27}\right) \qquad \text{(B.5)}
$$

together with f_2, f_2^* and the same functional forms of $F^{(1)}$ and $F^{(2)}$ of Appendix A, gives the asymptotic variations

$$
\nabla^2 F^{(1)} \sim -\frac{12\cos\beta}{\chi^3} + \frac{4\cos\beta}{9\chi} - \frac{\cos 3\beta}{\chi} \qquad \text{(B-6)}
$$

$$
\nabla^2 F^{(2)} \sim -\frac{12\cos\beta}{\chi^3} + \frac{4\cos\beta}{9\chi} - \frac{4\cos 3\beta}{9\chi}. \qquad \text{(B.7)}
$$

The image system

$$
F^{(3)} = -\sum_{n=-\infty}^{\infty}\frac{\xi}{\xi^2+(\eta+3n\pi)^2}\ \ln[\xi^2+(\eta+3n\pi)^2] \qquad \text{(B.8)}
$$

satisfies the specified Neumann conditions and exhibits the required behavior in its Laplacian

$$
\nabla^2 F^{(3)} = 4\sum_{n=-\infty}^{\infty}\frac{\xi}{[\xi^2+(\eta+3n\pi)^2]^2} \sim \frac{4\cos\beta}{\chi^3}. \qquad \text{(B.9)}
$$

Then

$$
\begin{aligned}
\Psi_3^{(p)} =& -\frac{2i\cos\phi_i}{81}\\
&\cdot\left[C_1F^{(1)} + C_2F^{(2)} + C_3F^{(3)} + \text{ bounded terms}\right] \qquad \text{(B.10)}
\end{aligned}
$$

with the same constants as in (A.15). The asymptotic behavior of the $F^{(j)}$ is

$$
\begin{aligned}
F^{(1)} &\sim 6\frac{\ln\chi}{\chi}\cos\beta + \frac{2}{9}\chi\ln\chi\cos\beta + \frac{1}{4}\chi\cos\beta\cos 2\beta\\
F^{(2)} &\sim 6\frac{\ln\chi}{\chi}\cos\beta + \frac{2}{9}\chi\ln\chi\cos\beta + \frac{1}{9}\chi\cos\beta\cos 2\beta\\
F^{(3)} &\sim -2\frac{\ln\chi}{\chi}\cos\beta - \left(\frac{2}{3\pi}\right)^2\chi\cos\beta\sum_{n=1}^{\infty}\frac{\ln(3n\pi)}{n^2}
\end{aligned} \qquad \text{(B.11)}
$$

i.e.,

$$
\begin{aligned}
-\tfrac{1}{81}&\left[C_1F^{(1)} + C_2F^{(2)} + C_3F^{(3)}\right]\\
&\sim 2A_1\frac{\ln\chi}{\chi}\cos\beta + \frac{1}{6}A_1\chi\ln\chi\cos\beta\\
&\quad - \tfrac{1}{2}(A_1/12 - A_3)\chi(\cos 3\beta + \cos\beta)\\
&\quad + \frac{5A_1}{9\pi^2}\chi\cos\beta\sum_{n=1}^{\infty}\frac{\ln(3n\pi)}{n^2}
\end{aligned} \qquad \text{(B.12)}
$$

and therefore $D_1 = 0$ in (73). The desired solution is thus

$$
\begin{aligned}
\Psi_3^{(p)}(\xi,\,\eta) =& -\frac{2i\cos\phi_i}{81}\left[C_1F^{(1)}(\xi,\,\eta) + C_2F^{(2)}(\xi,\,\eta)\right.\\
&\left.+C_3F^{(3)}(\xi,\,\eta)\right]\\
&+\frac{2i\cos\phi_i}{4\pi}\int_0^{\infty}\int_0^{3\pi/2}\\
&\cdot\left\{\frac{J(\xi_0,\,\eta_0)}{(\cosh\xi_0-\cos\eta_0)^2} + \frac{\nabla_0^2}{81}\left[C_1F^{(1)}(\xi_0,\,\eta_0)\right.\right.\\
&\left.\left.+C_2F^{(2)}(\xi_0,\,\eta_0) + C_3F^{(3)}(\xi_0,\,\eta_0)\right]\right\}\\
&\cdot\ln\left[\frac{\cosh\frac{2}{3}(\xi-\xi_0) - \cos\frac{2}{3}(\eta-\eta_0)}{\cosh\frac{2}{3}(\xi+\xi_0) - \cos\frac{2}{3}(\eta-\eta_0)}\right.\\
&\left.\cdot\frac{\cosh\frac{2}{3}(\xi-\xi_0) - \cos\frac{2}{3}(\eta+\eta_0)}{\cosh\frac{2}{3}(\xi+\xi_0) - \cos\frac{2}{3}(\eta+\eta_0)}\right]\\
&\cdot d\xi_0\,d\eta_0
\end{aligned} \qquad \text{(B.13)}
$$

which yields in the limit as $(\xi,\,\eta)\to(0,\,0)$

$$
\begin{aligned}
D_2 =& \frac{1}{2}(A_1/12 - A_3) - \frac{5A_1}{9\pi^2}\sum_{n=1}^{\infty}\frac{\ln(3n\pi)}{n^2}\\
&-\frac{2}{3\pi}\int_0^{\infty}\int_0^{3\pi/2}\left\{\frac{J(\xi_0,\,\eta_0)}{(\cosh\xi_0-\cos\eta_0)^2} - \frac{\nabla_0^2}{81}\right.\\
&\cdot\left[C_1F^{(1)}(\xi_0,\,\eta_0) + C_2F^{(2)}(\xi_0,\,\eta_0)\right.\\
&\left.\left.+C_3F^{(3)}(\xi_0,\,\eta_0)\right]\right\}\\
&\cdot\frac{\sinh\frac{2}{3}\xi_0}{\cosh\frac{2}{3}\xi_0 - \cos\frac{2}{3}\eta_0}\,d\xi_0\,d\eta_0. \qquad \text{(B.14)}
\end{aligned}
$$

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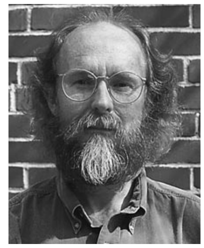

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