Antenna Pattern Synthesis Utilizing Spherical Bessel Functions

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***Abstract*—Pattern synthesis of linear antennas utilizing the spherical bessel functions is presented. This leads to antenna current distribution by the Legendre polynomials of the first kind, which are of finite support. Some examples are given to illustrate this procedure.**

***Index Terms*—Antenna pattern synthesis.**

I. Introduction

THE inverse Fourier transform of the current distribution on an antenna structure yields the far-field pattern. Since the current distribution is of spatially finite duration, its transform cannot strictly be band limited. Indeed, the far-field exists for the angular variable $u(=\cos\theta)$ from $-\infty$ to $+\infty$. However, the value of u between -1 to $+1$ or (θ between o to $+\pi$) is of interest to us as this is the visible region of the far field and the invisible region is given by $|u| \geq 1$.

We are often given the far-field pattern between $|u| \leq Q < 1$ in the visible region and the objective is to synthesize the current distribution that is going to match the given field in the visible region. A popular technique is to expand the far field in term of a set of orthogonal basis functions and then obtain the current distribution by the truncation of the inverse Fourier transform of each orthogonal functions between suitable spatial intervals. This procedure is only approximate [1]–[3]. Borgotti [4] used a generalization of the Woodward–Lawson [5] synthesis method for a wide class of apertures. By resorting to a rather general type of bi-dimensional Fourier series expansion, the aperture distribution is obtained as a superposition of orthogonal constant amplitude linearly phased components whose complex amplitude coefficients are the values of the radiation pattern in a regular lattice of "cardinal points" on which the desired function is exactly matched. Hence, there is no guarantee that the aperture distribution will be of finite support.

Another approach is usually to solve the antenna pattern synthesis problem in terms of an optimization (either by linear programming or quadratic error minimization) procedure. In this procedure, the error between the given field pattern and the pattern produced by a current distribution obtained from the minimization technique is reduced. One problem with such a technique is that it may lead to supergain conditions for the current distribution (i.e., the radiated power may be relatively small compared to the power fed to the antenna), which is updated iteratively [1]–[3]. A third approach is to expand the current distribution into a set of orthogonal polynomials and then their transform would match the pattern. Rhodes [6] utilized the prolate spheroidal functions to match the antenna pattern and, hence, its transform is guaranteed to be of finite support. In this paper, we present the spherical Bessel functions as they are easy to compute numerically instead of the prolate spheroidal functions. Some of the salient features of this alternate approach (use of the spherical Bessel functions) are also presented.

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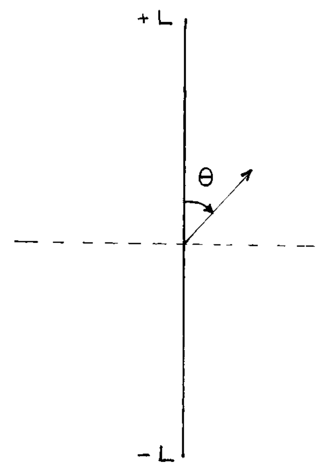

Fig. 1. Antenna geometry and the coordinate system.

II. Antenna Pattern Synthesis by Spherical Bessel Functions

Consider a linear antenna of total electrical length $2L$ extending from $-L$ to $+L$ as shown in Fig. 1. The electrical length is defined by the physical length over the wavelength. The current distribution on the antenna is given by $J(t)$. We assume that the current distribution on the antenna can be expanded in terms of the Legendre polynomials, i.e.,

$$J(t) = \sum_n a_n P_n(t), \qquad \text{for } |t = \frac{z}{L}| \leq 1 \tag{1}$$

where a_n are the unknown coefficients for the Legendre polynomials. The far-field pattern from the current distribution is given by the Fourier transform, i.e., (with $i = \sqrt{-1}$)

$$\begin{aligned} F(u) = F(\cos\theta) &= \int_{-L}^{L} e^{i2\pi zu} J(z)\, dz \\ &= L \int_{-1}^{+1} e^{it(2\pi Lu)} J(t)\, dt \qquad (\text{with } t = \frac{z}{L}). \end{aligned} \tag{2}$$

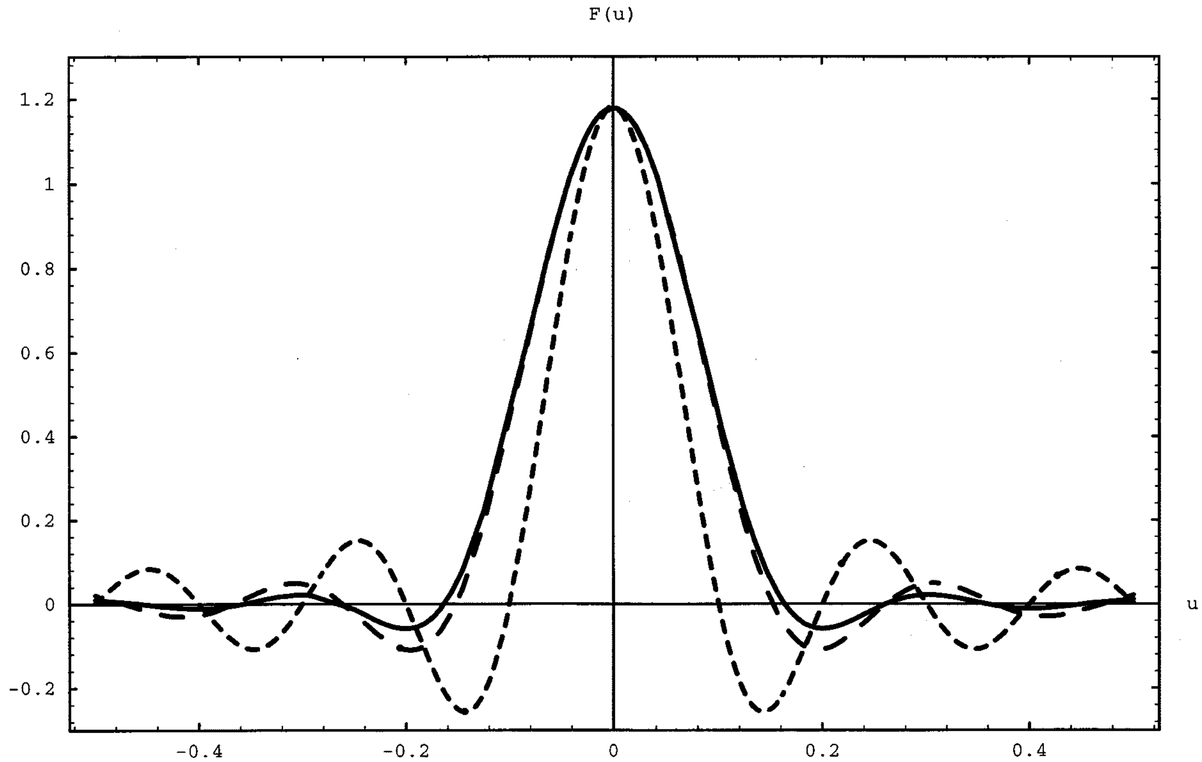

Fig. 2. Convergence of the far-field pattern for one (small dash); two (large dash); and three (solid) terms.

By substituting (1) into (2), one obtains

$$F(u) = \sum_n a_n L \int_{-1}^{+1} e^{it(2\pi \mathrm{Lu})} P_n(t)\, dt. \tag{3}$$

Next, we introduce the Fourier transform relationship between the Legendre polynomials P_n and the spherical bessel functions of the first kind j_n. The Fourier transform pair is given by

$$\int_{-\infty}^{\infty} e^{-i\omega p} j_n(p)\, dp = \begin{cases} \pi(-i)^n P_n(\omega), & \text{for } |\omega| < 1 \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

and

$$j_n(p) = \frac{(-i)^n}{2} \int_{-1}^{+1} P_n(\omega) e^{i\omega p}\, d\omega. \tag{5}$$

Substitution of (5) into (3) results in

$$F(u) = \sum_n a_n \frac{2L}{(-i)^n} j_n(2\pi \mathrm{Lu}). \tag{6}$$

To obtain the expansion coefficients a_n, we multiply both sides of (6) by $j_m(2\pi \mathrm{Lu})$ and integrate from $-\infty$ to $+\infty$. By utilizing orthogonality relationship for the spherical Bessel functions [7]

$$\int_{-\infty}^{\infty} j_n(\alpha t) j_n(\alpha t)\, dt = \begin{cases} 0, & \text{for } m \neq n \\ \dfrac{\pi}{\alpha(2n+1)}, & \text{for } m = n \end{cases} \tag{7}$$

one obtains

$$a_n = (2n+1)(-i)^n \int_{-1}^{1} F(u) j_n(2\pi \mathrm{Lu})\, du. \tag{8}$$

The current distribution is then computed from (1) and the far-field pattern is given by (6) in the entire region $-\infty < u < \infty$.

On the other hand, if we expand the far-field pattern into a series of spherical Bessel functions, then

$$F(u) = \sum_n b_n j_n(2\pi \mathrm{Lu}) \tag{9}$$

where

$$b_n = 2L(2n+1) \int_{-1}^{1} F(u) j_n(2\pi \mathrm{Lu})\, du \tag{10}$$

and the current distribution on the antenna is given by

$$J(y) = \sum_n \frac{(-i)^n}{2L} b_n P_n(y), \quad \text{with } |y = \frac{z}{L}| \leq 1. \tag{11}$$

By using the orthogonality property of the Legendre polynomials

$$\int_{-1}^{+1} P_n(t) P_m(t)\, dt = \begin{cases} 0, & \text{for } n \neq m \\ \dfrac{2}{2n+1}, & \text{for } n = m \end{cases} \tag{12}$$

it is seen that

$$a_n = \frac{(-i)^n b_n}{2L}. \tag{13}$$

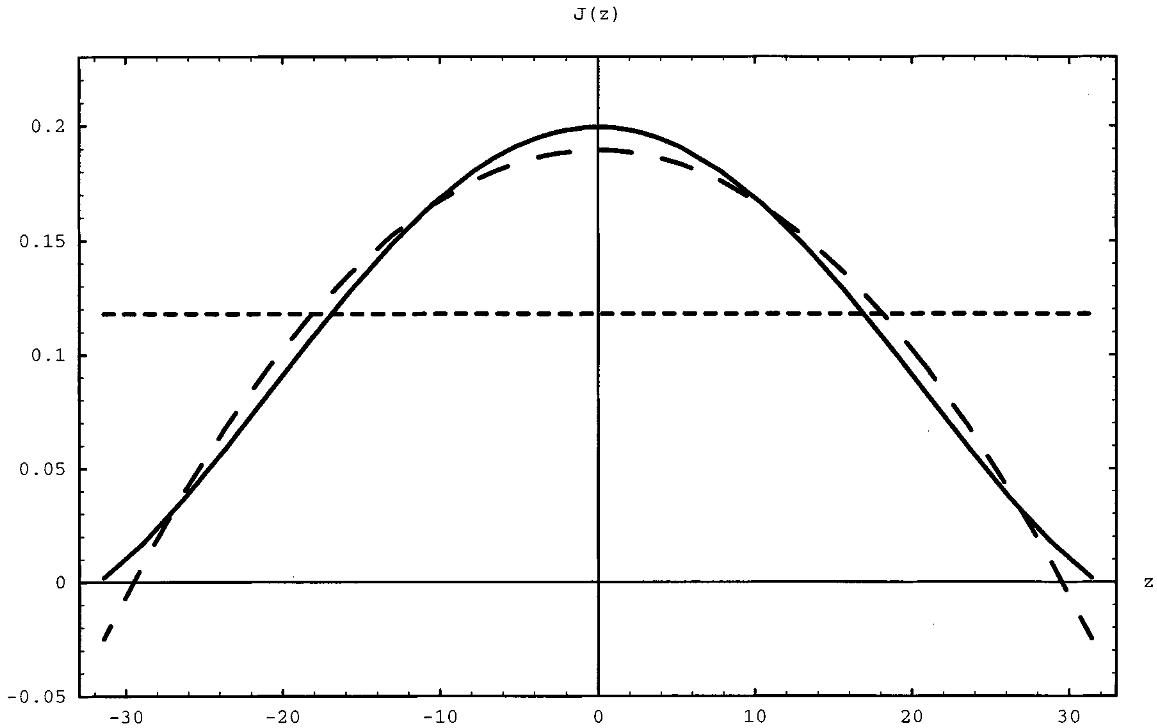

Fig. 3. Convergence of the synthesized current for one (small dash), two (large dash), and three (solid) terms.

III. Properties of the Proposed Methodology

Even though the current converges in the $\mathcal{L}^2$ norm, the pattern converges in the uniform norm. This can be shown through

$$F(u) - F_N(u) = \frac{1}{2\pi} \int_{-1}^{1} [J(y) - J_N(y)] \exp[iuy]\, dy. \quad (14)$$

By applying the Schwartz inequality, one obtains

$$|F(u) - F_N(U)|^2 \leq \frac{2}{(2\pi)^2} \int_{-1}^{+1} [J(Y) - J_N(y)]^2\, dy. \quad (15)$$

Note that the right-hand side goes to zero as $N - \infty$ and, therefore, the series expressing F_N converges uniformly. Hence, the spherical bessel functions are complete over the class of square integrable functions whose transforms are of finite support [8]. This is different from the case where we expand the current in terms of piecewise constant pulse functions so that they are orthogonal. Hence, the completeness of spherical bessel functions in approximating the pattern is different from the use of piecewise pulse functions which results in the set $j_o(2\pi y - n\pi)$ for $n = -\infty$ to $P{+}\infty$ being complete [8].

IV. Efficiency of the Synthesis Procedure

The efficiency is defined as the power radiated by the antenna divided by the input power to the antenna. A high efficiency will indicate that most of the power fed to the antenna will radiate. This is in contrast to the supergain condition, which often arises in array pattern synthesis. During the supergain condition, an arbitrary-shaped radiation pattern can be realized, however, a large portion of the power fed to the antenna does not get radiated, rather remains stored as reactive energy in the system. Under those conditions, the current distribution on the antenna is very large in magnitude requiring high-input power, but the radiated power is very small. This is the reason it is necessary to evaluate the radiation efficiency in any antenna synthesis procedure. We define the efficiency η by

$$\begin{aligned}\eta &= \frac{\displaystyle\int_{-1}^{+1} |F(u)|^2\, du}{\displaystyle\int_{-\infty}^{\infty} |F(u)|^2\, du} \\ &= \frac{\text{power in the visible region of the spectrum}}{\text{power fed to the array}} \\ &= \frac{\displaystyle\int_{-1}^{+1} \left|\sum_n b_n j_n(2\pi \mathrm{Lu})\right|^2 du}{\displaystyle\int_{-\infty}^{\infty} \left|\sum_n b_n j_n(2\pi \mathrm{Lu})\right|^2 du}. \end{aligned} \quad (16)$$

By utilizing the orthogonality relationships of the spherical Bessel functions we obtain

$$\eta = \frac{\displaystyle\int_{-1}^{+1} \left|\sum_n b_n j_n(2\pi \mathrm{Lu})\right|^2 du}{\displaystyle\frac{1}{2L}\sum_n \frac{|b_n|^2}{(2n+1)}}. \quad (17)$$

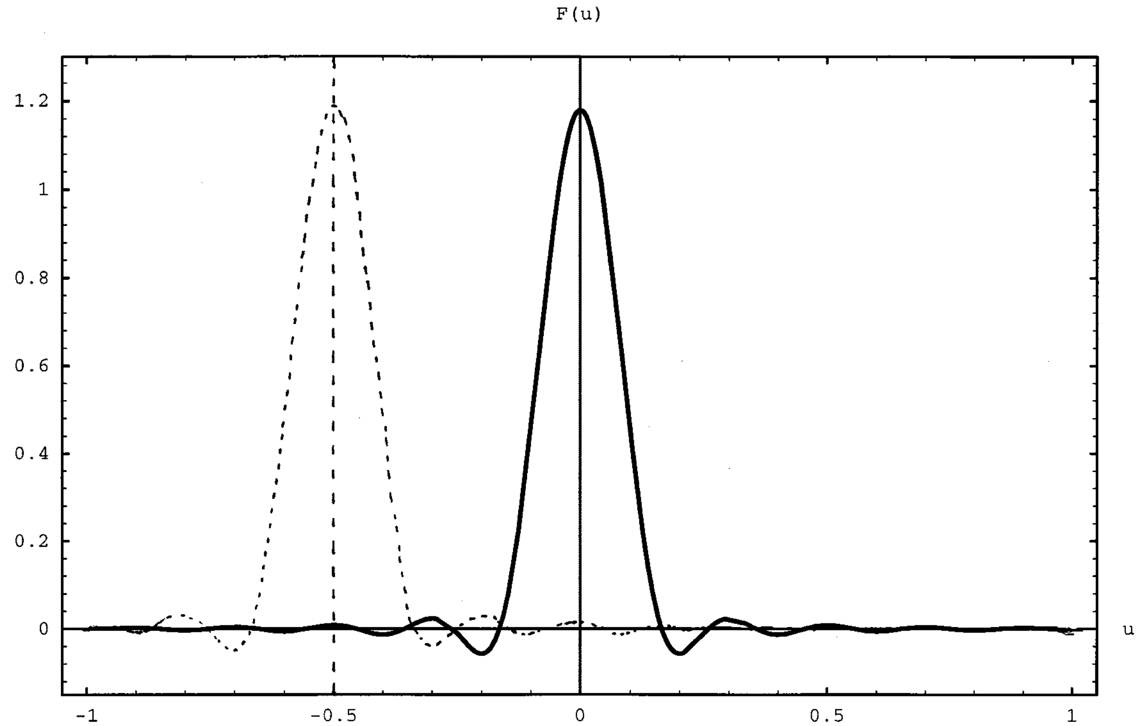

Fig. 4. Synthesized far-field pattern in the entire visible region for example 1 (solid) synthesized far-field pattern in the entire visible region for example 2 (dotted).

We would like to restrict further the domain of the numerator to $-Q$ to $+Q$ instead of $Q = 1$ to illustrate how efficient this particular design has been. Hence, we define

$$\eta_Q = \frac{\displaystyle\int_{-Q}^{Q} \left| \sum_n b_n j_n(2\pi \mathrm{Lu}) \right|^2 du}{\displaystyle\frac{1}{2L} \sum_n \frac{|b_n|^2}{(2n+1)}}. \quad (18)$$

TABLE I
COEFFICIENTS FOR THE LEGENDRE POLYNOMIALS

b_0	1.179
b_2	1.427
b_4	0.268
b_6	0.020
b_8	0.840×10^{-3}
b_{10}	0.217×10^{-4}
b_{12}	0.384×10^{-6}

V. EXAMPLES

As a first example, consider the synthesis of a far-field pattern, which is assumed to be of uniform amplitude unity for $|u| \leq 0.1$ and zero everywhere else, so that

$$F(u) = \begin{matrix} 1, & \text{for } |u| \leq Q = 0.1 \\ 0, & \text{otherwise.} \end{matrix}$$

Then, using (10), one computes b_n. Here, α half the electrical length in radians of the antenna is chosen as $\alpha(0.1) = \pi$. This assures that the dominant part of the pattern given by $j_0(u)$ has its first null at 0.1, thus yielding a value of α equal to 10π and a physical length of the antenna $L = 5\lambda$.

In evaluating the antenna current distribution, only a few terms of b_n in the summation (10) are of numerical significance. The odd terms of b_n are all zero, as the pattern is symmetrical about $u = 0$. The values of b_n obtained for the even terms are given in Table I.

From Table I, it is seen that convergence is very rapid and only a few terms of the summation in (10) are needed. The approximation to the far field is given by Fig. 2. It is seen that as the number of terms in (10) is increased, the convergence of the pattern is very rapid. The addition of terms greater than three does make no visual change of the graph. Similarly, the current distribution on the structure is shown in Fig. 3 as the number of terms are increased. The current distribution on the structure is real, for this example. The synthesized pattern in $|u| \leq 0.1$ is shown in Fig. 4 to illustrate its deviation from that of the uniform

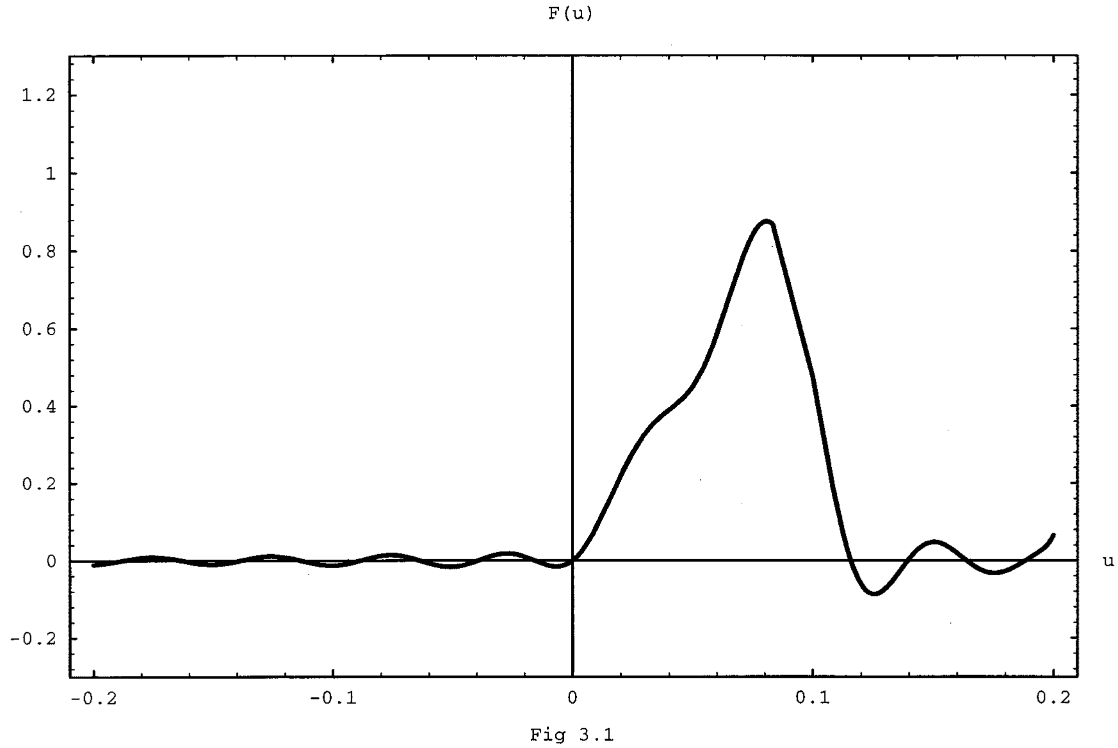

Fig. 5. Synthesized far field pattern in part of the visible region.

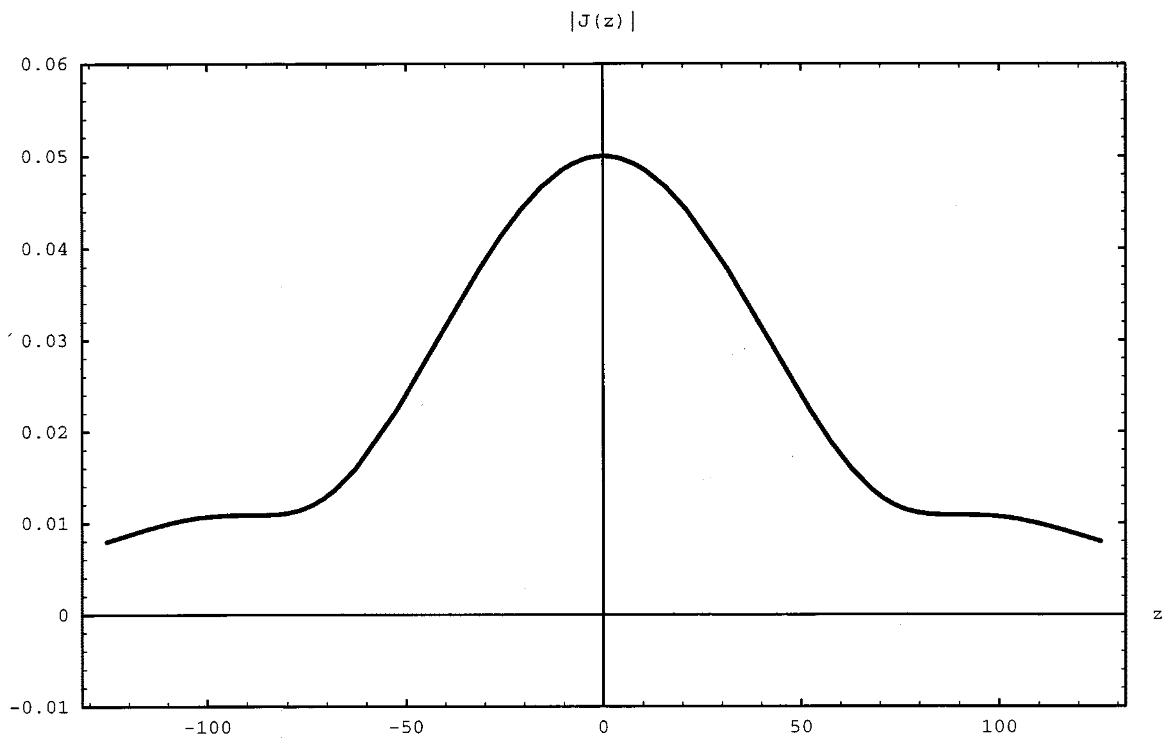

Fig. 6. Magnitude of the synthesized current distribution.

pattern. The antenna efficiency η for this example is given by (13) and is approximately 100%. However, η_Q has been computed to be 0.96. This implies that 96% of the input power is radiated and focussed in $|u| \leq 0.1$.

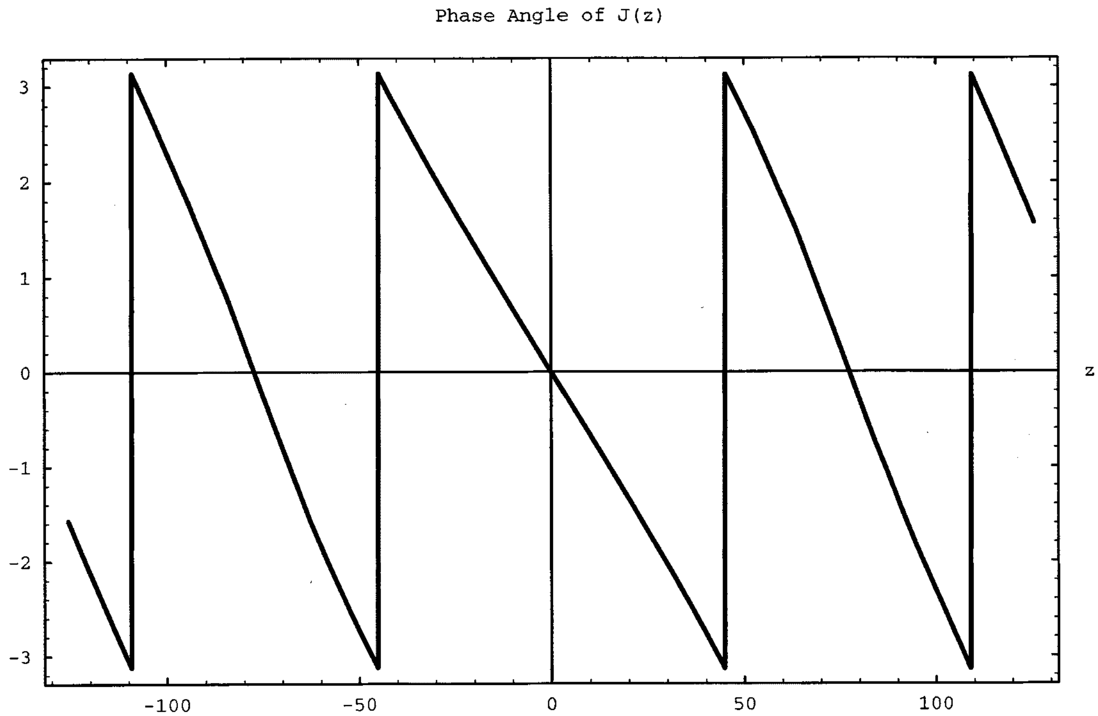

Fig. 7. Phase of the synthesized current distribution.

For the second example, we assume that the far-field pattern is of uniform amplitude of unity between $-0.6 \le u \le -0.4$ and zero everywhere else so that

$$F(u) = \begin{cases} 1, & -0.6 \le u \le -0.4 \\ 0, & \text{elsewhere.} \end{cases}$$

In this case, α is chosen as 10π or $L = 5\lambda$ as before. The synthesized far-field pattern is shown in Fig. 4 as the dotted pattern. In this case, the current distribution is complex and its magnitude is identical to that of Fig. 3, as expected. For this example, η has been computed to be approximately one.

For the last example, we consider a linearly tapered field pattern of the form

$$F(u) = \begin{cases} 10u, & \text{for } o \le u \le 0.1 \\ 0, & \text{otherwise.} \end{cases}$$

For this case, we choose $\alpha = 40\pi$ or equivalently $L = 20\lambda$. The synthesized far-field pattern is shown in Fig. 5 for the visible region. The complex current distribution is shown in Fig. 6 for its magnitude and Fig. 7 for its phase. For this example, the first 21 coefficients of b_n have any significant values (i.e., $|b_n| > 0.0003$). The antenna efficiency is computed $\eta = 0.99.9$, i.e., 99.9% of the power is radiated in the visible region.

VI. Conclusion

A method is presented for synthesizing a continuous current distribution to fit any given far-field antenna pattern through the use of the spherical Bessel and Legendre function pair.

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