On a Choice of Wavelet Bases in the Wavelet Transform Approach

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***Abstract*—The Daubechies orthogonal wavelet (DOW) is compared with the nonorthogonal cardinal spline wavelet (NCSW) in the wavelet transform approach and it is shown that the DOW is better than the NCSW in view of the computation cost. First, the computation cost required for the wavelet transform based on the DOW is less than that based on the NCSW because the DOW has smaller support provided the same number of vanishing moments of wavelets is used. Second, in contrast with the fact that the wavelet transform based on the DOW does not affect the condition number of the impedance matrix, that, based on the NCSW, has an effect to make it very large. As a result, even though the NCSW results in a sparser impedance matrix, it requires more computation cost for solving the resultant matrix equation in comparison with the DOW because the cost depends not only on the sparsity, but also on the condition number of the matrix.**

***Index Terms*—Method of moments (MoM), wavelet, wavelet transform approach.**

I. Introduction

THE wavelet transform approach, in combination with the conventional method of moments (MoM), becomes a familiar method for solving the electromagnetic integral equations. In this method, the impedance matrices obtained by the MoM are reduced to sparse ones by a wavelet matrix transform and, hence, the computation cost is dramatically lowered [1]–[4]. Xiang and Lu [1] compared the nonorthogonal cardinal spline wavelet (NCSW) with the Daubechies orthogonal wavelet (DOW), where advantages of the NCSW are emphasized mainly from a standpoint of matrix compression rate. A similar comparison has been done with a similar conclusion in the wavelet basis expansion method [5]. However, computation cost was not discussed in detail. The computation cost is mainly consumed in performing wavelet transform and solving the resultant impedance matrix equation: the former depends on the sparsity of the wavelet matrix, the latter depends not only on the sparsity but also on the condition number of the impedance matrix.

In this paper, the choice of wavelet bases for the wavelet transform approach will be discussed from a standpoint of computation cost. Two wavelets with compact support, the DOW and the NCSW, will be compared with each other. As is already known [1], [5], the NCSW actually results in a sparser impedance matrix than the DOW does. On the contrary, wavelet matrices have less nonzero elements in the DOW than in the NCSW, because the former has more compact support than the latter provided the number of vanishing moments of wavelets is the same. As the result, the computation cost required for the wavelet transform based on the DOW is less than that based on the NCSW. Furthermore, the wavelet transform based on the DOW does not affect the condition number of the impedance matrix while the NCSW transform has an effect to make it very large. Consequently, this property is also helpful for the DOW approach to reduce the computation cost. In conclusion, the DOW can be said to provide a better approach compared to the NCSW. In this paper, an effective construction method for the wavelet matrix will be proposed where the circulant elements of the wavelet matrix is given explicitly. In addition, both the computation cost and the allocation storage necessary for the wavelet matrix will be shown to be extremely reduced and the computation cost required for the wavelet transform will be estimated.

Manuscript received August 27, 1999; revised May 1, 2000.

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Publisher Item Identifier S 0018-926X(00)06925-8.

II. Formulation

A. Wavelet Transform Approach

By using the MoM, we obtain the matrix equation

$$ZI = V \tag{1}$$

where Z is a dense impedance matrix. Introducing a wavelet matrix W, the matrix equation is then transformed as

$$Z'I' = V' \tag{2}$$

where

$$Z' = WZW^T, \quad I' = \left(W^T\right)^{-1} I, \quad V' = WV. \tag{3}$$

Here T stands for the transpose of a matrix. Once I' is solved, the desired solution is obtained as

$$I = W^T I'. \tag{4}$$

The wavelet matrix W must be chosen so that the resultant impedance matrix Z' has a great number of very small elements that are deleted due to a preselected threshold without seriously affecting the solution. Another very important requirement is that W must be sparse enough to reduce the computation cost of the wavelet transform. Finally, the condition number of Z' must not be much larger than that of Z.

It is known that many types of wavelets can be used to sparsify an original matrix. However, wavelets with infinite support, such as the Battle–Lemarié or the Meyer wavelets, will not be taken into account here because they produce dense wavelet matrices. Both the DOW and the NCSW have compact support and it is worthwhile to compare with each other.

B. Wavelet Matrix

Both the DOW and the NCSW can be generated by the two-scale relations in terms of finite sets of the coefficients $\{p_i\}$ and $\{q_i\}$ [6]:

$$\phi(x) = \sqrt{2}\sum_{n=0}^{N_p} p_n \phi(2x-n) \tag{5}$$

$$\psi(x) = \sqrt{2}\sum_{n=0}^{N_q} q_n \phi(2x-n) \tag{6}$$

where ϕ and ψ are the scaling and the wavelet functions, respectively, N_p and N_q are related with the number of vanishing moments of the corresponding wavelet m

$$N_p = N_q = 2m-1 \tag{7}$$

for the DOW

$$N_p = m, \quad N_q = 3m-2 \tag{8}$$

for the NCSW.

Wavelet matrix W of order $2^J \times 2^J$ is constructed by using periodic wavelets [1], [3]

$$W = U_1 \cdots U_{J-1} U_J \tag{9}$$

where U_j is defined as

$$U_j = \begin{bmatrix} \begin{bmatrix} P^j \\ Q^j \end{bmatrix} & 0 \\ 0 & I_{2^J-2^j} \end{bmatrix}. \tag{10}$$

Here I_j denotes a unit matrix of rank j, P^j and Q^j are matrices of order $2^{j-1} \times 2^j$ and their elements are defined as

$$P^j_{nk} = p^j_{k-2n} \tag{11}$$

$$Q^j_{nk} = q^j_{k-2n} \tag{12}$$

$$p^j_i = \sum_l p_{i+2^j l} \tag{13}$$

$$q^j_i = \sum_l q_{i+2^j l}. \tag{14}$$

Obviously, $\{p^j_i\}$ and $\{q^j_i\}$ are periodic with the period of 2^j and only $\{p^j_i\colon 0 \le i \le N_p\}$ and $\{q^j_i\colon 0 \le i \le N_q\}$ are nonzero during one period. It should be noted that W is orthogonal for the DOW, but this is not for the NCSW.

C. Effective Construction of the Wavelet Matrix

In this section, we will propose an effective construction method for the wavelet matrix where the circulant elements of the matrix are given explicitly. The computation cost necessary for the construction as well as the allocation storage for the matrix will be estimated. Only the operation of multiplication is counted as the computation cost without loss of generality.

For the sake of convenience, we consider a new matrix

$$V_j = U_j \cdots U_J. \tag{15}$$

For example

$$\begin{aligned} V_{J-1} &= U_{J-1}U_J \\ &= \begin{bmatrix} \begin{bmatrix} P^{J-1} \\ Q^{J-1} \end{bmatrix} & 0 \\ 0 & I_{2^J-2^{J-1}} \end{bmatrix} \begin{bmatrix} P^J \\ Q^J \end{bmatrix} \\ &= \begin{bmatrix} P^{J-1}P^J \\ Q^{J-1}P^J \\ Q^J \end{bmatrix} \triangleq \begin{bmatrix} S^{J-1} \\ T^{J-1} \\ T^J \end{bmatrix}. \end{aligned} \tag{16}$$

Repeating this operation yields

$$V_j = \begin{bmatrix} S^j \\ T^j \\ T^{j+1} \\ \vdots \\ T^J \end{bmatrix}, \quad W = V_1 \tag{17}$$

where S^j and T^j are matrices of order $2^{j-1} \times 2^J$ given by

$$S^j_{nk} = s^j_{k-2^{J-j+1}n} \tag{18}$$

$$T^j_{nk} = t^j_{k-2^{J-j+1}n} \tag{19}$$

$$s^j_i = \sum_{l=0}^{2^j-1} p^j_l s^{j+1}_{i-2^{J-j}l}, \quad s^J_i = p^J_i \tag{20}$$

$$t^j_i = \sum_{l=0}^{2^j-1} q^j_l s^{j+1}_{i-2^{J-j}l}, \quad t^J_i = q^J_i. \tag{21}$$

$\{s^j_i\}$ and $\{t^j_i\}$ are periodic with the period of 2^J and only $\{s^j_i\colon 0 \le i \le N^s_j = N^s_{j+1} + 2^{J-j}N_p\}$ and $\{t^j_i\colon 0 \le i \le N^t_j = N^s_{j+1} + 2^{J-j}N_q\}$ are nonzero during one period, where $N^s_J = \min(2^J, N_p)$. It is easily obtained that

$$N^s_j = \min\left[2^J, \left(2^{J-j+1}-1\right)N_p\right] \tag{22}$$

$$N^t_j = \min\left[2^J, \left(2^{J-j+1}-1\right)N_p + 2^{J-j}N_q\right]. \tag{23}$$

The computation cost for construction of W is less than

$$\begin{aligned} &\sum_{j=1}^{J-1}\left[\left(N^s_j+1\right)\left(N_p+1\right) + \left(N^t_j+1\right)\left(N_q+1\right)\right] \\ &\quad < \left[J-1+\left(2^{J+1}-J-3\right)N_p\right]\left(N_p+1\right) \\ &\qquad + \left[J-1+\left(2^J-J-1\right)N_p + \left(2^J-2\right)N_q\right]\left(N_q+1\right) \end{aligned} \tag{24}$$

or $O(N)$ with $N = 2^J$. Only $\{s^1_i\}$ and $\{t^j_i\}^J_{j=1}$ need to be stored into memory, then the allocation storage is

$$\begin{aligned} &\sum_{j=1}^{J}\left(N^t_j+1\right) + \left(N^s_1+1\right) \\ &\quad < \left(2^{J+1}-J-2\right)N_p + \left(2^J-1\right)N_q + J + 1 \end{aligned} \tag{25}$$

or less than $O(N)$. Finally, the number of nonzero elements of W is

$$N_W = \sum_{j=1}^{J} 2^{j-1}\left(N_j^t + 1\right) + \left(N_1^s + 1\right) < J2^{J-1}N_p + \left(2^J - 1\right)N_q + 2^J - 1 \quad (26)$$

or less than $O(N \log N)$.

III. Computation Cost

A. Wavelet Transform

The computation cost required for the wavelet transform depends on the number of nonzero elements of the wavelet matrix N_W. Let the number of unknowns be N and the cost is $O(NN_W)$ or less than $O(N^2 \log N)$.

As will be demonstrated in the next section, this cost dominates the total computation cost and should be reduced in the top priority. The cost depends only on the sparsity of the wavelet matrix and is determined by the length of the coefficients of the corresponding wavelet. Wavelets with infinite support are not suitable for the wavelet transform approach since they lead to dense wavelet matrices.

Among compact supported wavelets, the more compact the support of wavelet becomes, the sparser the wavelet matrix becomes. Consequently, a relative small number of vanishing moments of wavelets should be chosen for both the DOW and the NCSW. The DOW always yields a sparser wavelet matrix than the NCSW provided the same number of vanishing moments is used. Table I lists the sparsity of the wavelet matrix for both of the wavelets, where the sparsity is defined as the ratio of the number of nonzero elements to the total number of elements of the matrix.

B. Sparse Solver

After the wavelet transforming and the thresholding, a sparse matrix equation can be obtained. The sparse matrix equation must be treated by an effective sparse solver. An iterative method is, in general, more suitable for the matrix equation, which has a random sparsity structure rather than a direct method [7]. When the iterative method such as the conjugate gradient (CG) method is applied, the computation cost for each iteration depends on the sparsity of the matrix, but the convergence of solution, that is, the number of the iteration necessary for a converged solution, depends on the sparsity structure or the condition number of the matrix which is defined as

$$\kappa(A) = \frac{\beta_{\max}}{\beta_{\min}} \quad (27)$$

where βs are the nonzero singular values of A.

Actually, it is known that the NCSW yields a sparser impedance matrix than the DOW [1]. On the other hand, the transformed impedance matrix Z' based on the DOW has exactly the same condition number as the original matrix Z, but Z' based on the NCSW has much larger condition number than Z due to the nonorthogonality of the NCSW

$$\kappa(Z') = \kappa\left(WZW^T\right) \le \kappa^2(W)\kappa(Z). \quad (28)$$

TABLE I
Sparsity of Wavelet Matrix W

Base	m	Sparsity (%) N=512	1024	2048	4096
DOW	2	4.495	2.540	1.416	0.781
	3	6.653	3.812	2.149	1.196
	4	8.612	4.985	2.833	1.587
	5	10.40	6.067	3.470	1.954
	6	11.98	7.047	4.057	2.296
	7	13.55	8.026	4.643	2.638
	8	15.12	9.006	5.230	2.980
	9	16.60	9.912	5.774	3.299
	10	17.80	10.70	6.264	3.592
NCSW	2	4.689	2.637	1.465	0.806
	3	7.037	4.005	2.246	1.245
	4	9.189	5.276	2.979	1.660
	5	11.16	6.451	3.664	2.051
	6	12.92	7.527	4.299	2.417
	7	14.69	8.603	4.934	2.784
	8	16.45	9.679	5.569	3.150
	9	18.07	10.67	6.158	3.493
	10	19.45	11.55	6.696	3.810

The CG method is used here and the corresponding computation cost will be discussed in the next section.

IV. Numerical Results

In this section, the scattering from an array of 2-D metal strips shown in Fig. 1 is numerically calculated, where the number of the strips is five, all the strips have the width of λ, the wavelength of the incident wave, and are separated by also λ. A TM plane wave is assumed to be incident with an angle of 45°. The number of vanishing moments of both the DOW and the NCSW is set to four because this relative small number gives a faster solution [8]. Decomposition is carried out as far as possible for construction of the wavelet matrix. For example, the decomposition level is nine for $N = 512$, ten for $N = 1024$, etc.

The elements of Z' whose magnitudes are smaller than τM are set to zero, where τ is a preselected threshold and M is the largest elements of Z'.

The sparsity of truncated matrix and the associated relative error of the scattering cross section σ which is defined as

$$\sigma(\phi) = 2\pi r \left| \frac{E_z^{(s)}(\phi)}{E_z^{(i)}} \right|^2 \quad (29)$$

are shown in Tables II and III, respectively. Here the relative error ϵ caused by the truncation is defined by

$$\epsilon = \frac{\|\sigma_o - \sigma_\tau\|_2}{\|\sigma_o\|_2} \quad (30)$$

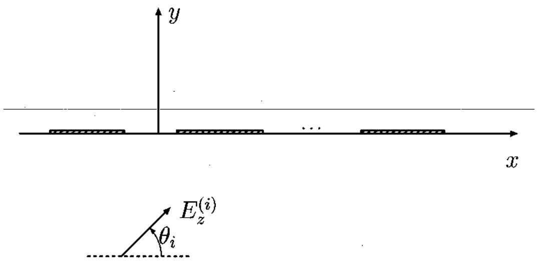

Fig. 1. Configuration for analysis.

TABLE II
SPARSITY OF TRUNCATED Z'

Base	τ	Sparsity (%) $N = 1024$	2048	4096
DOW	10^{-4}	2.969	1.104	0.4120
	10^{-5}	5.076	2.088	0.8475
	10^{-6}	7.791	3.279	1.361
	10^{-7}	10.77	4.657	1.994
	10^{-8}	14.86	6.432	2.757
	10^{-9}	21.19	8.977	3.832
	10^{-10}	28.74	12.06	5.158
NCSW	10^{-4}	0.06104	0.01526	0.003791
	10^{-5}	0.2907	0.05803	0.01425
	10^{-6}	0.8460	0.2503	0.04423
	10^{-7}	1.634	0.5539	0.1670
	10^{-8}	3.381	1.068	0.3137
	10^{-9}	4.721	1.800	0.6819
	10^{-10}	6.292	2.422	0.9903

TABLE III
RELATIVE ERROR OF SCATTERING CROSS SECTION

Base	τ	Relative error $N = 1024$	2048	4096
DOW	10^{-4}	2.9×10^{-3}	5.2×10^{-3}	1.0×10^{-2}
	10^{-5}	2.2×10^{-4}	9.0×10^{-5}	1.4×10^{-4}
	10^{-6}	2.9×10^{-6}	4.9×10^{-6}	1.0×10^{-5}
	10^{-7}	7.3×10^{-9}	9.4×10^{-8}	2.3×10^{-7}
	10^{-8}	4.4×10^{-9}	3.3×10^{-9}	2.8×10^{-9}
	10^{-9}	9.8×10^{-10}	8.8×10^{-10}	4.3×10^{-10}
	10^{-10}	7.7×10^{-10}	9.1×10^{-10}	5.5×10^{-10}
NCSW	10^{-4}	9.3×10^{0}	1.2×10^{1}	1.5×10^{1}
	10^{-5}	2.2×10^{0}	4.9×10^{0}	8.1×10^{0}
	10^{-6}	2.5×10^{0}	4.9×10^{-1}	1.0×10^{0}
	10^{-7}	4.8×10^{-3}	1.2×10^{-2}	1.4×10^{0}
	10^{-8}	4.6×10^{-4}	4.6×10^{-4}	2.3×10^{-3}
	10^{-9}	2.2×10^{-5}	4.4×10^{-5}	2.7×10^{-4}
	10^{-10}	1.6×10^{-6}	2.6×10^{-6}	8.8×10^{-6}

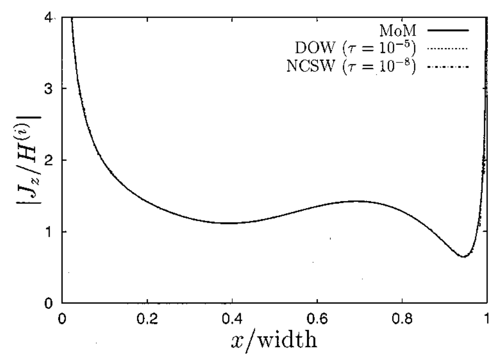

Fig. 2. Current distribution on the first strip from left side ($N = 4096$).

where σ_o is the solution obtained by the MoM and σ_τ is that obtained from the truncated matrix and $\|\cdot\|_2$ denotes the L^2 norm. It is actually shown that the sparsity by the NCSW transform is much smaller than that by the DOW one, owing to the smoothness of the NCSW functions. However, Table III shows that smaller threshold must be used for the NCSW than for the DOW if the same order of accuracy is required for both bases. The fact that the error is quite large and even not monotonic with τ for NCSW with large value of τ is caused by the rank deficiency of resulting matrix.

Fig. 2 illustrates the current distribution on the first strip from the left side and shows that the solutions obtained by both the DOW and the NCSW transforms coincide with that obtained by the MoM except very close proximity to the strip edges.

Fig. 3 shows the condition number of W constructed by the NCSW versus the matrix size. The condition number becomes larger as the matrix size increases and, at the same time, the larger the number of vanishing moments, the larger the condition number. It is noteworthy that the condition number of W constructed by the DOW with any number of vanishing moments is unity.

Fig. 4 shows the condition number of corresponding transformed impedance matrix Z' where the upper limit $\kappa^2(W)\kappa(Z)$ is also plotted as a reference. Z' has much larger condition number than the original matrix Z. This implies that more computation cost is required for reasonable solutions.

Fig. 5 shows the number of iteration required for a converged solution in a CG method based sparse solver from the SLATEC Common Mathematical Library [9]. The number based on the NCSW is larger than that based on the DOW. It should be pointed out that no solution or only solutions with very poor accuracy can be obtained for matrices with very large condition numbers.

Fig. 6 shows the computation time required for the sparse solver. The computation time required for performing the

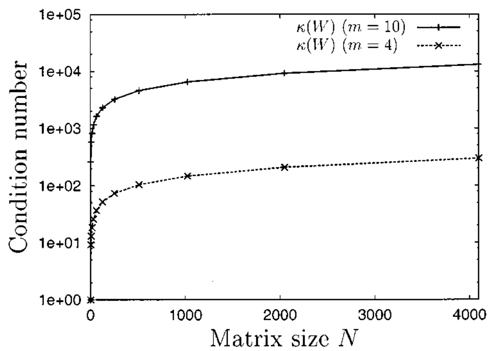

Fig. 3. Condition number of W constructed by the NCSW.

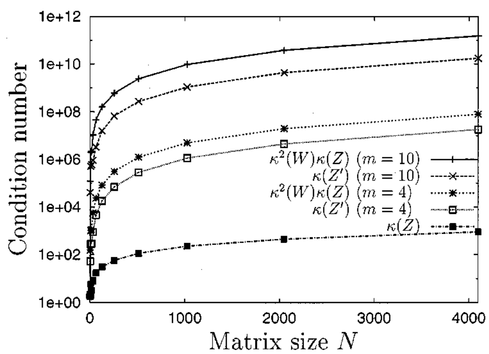

Fig. 4. Condition number of Z' transformed by the NCSW.

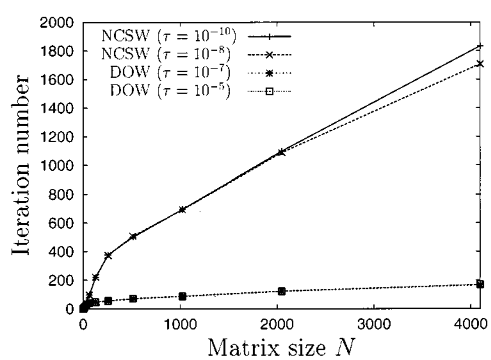

Fig. 5. Iteration number required by the CG method.

wavelet transform is also plotted. The computation time required for recovery of I from I' is not plotted together because it is too small, e.g., 0.18 s for the DOW and 0.19 s for the NCSW when $N = 4096$. The computations were performed on a PC with a 333 MHz Pentium II central processing unit (CPU). It is shown that the NCSW needs more computation cost than the DOW, not only in the wavelet transform, but also in the sparse solver. In conclusion, we can say that the DOW is superior to the NCSW as far as the computation cost is concerned. Considering that the DOW is much better than the NCSW with regard to the cost required by the sparse solver, the advantage is expected to be more significant for ill-posed problems since the cost consumed by the sparse solver will take a large percentage of the total cost in that case. It is also obvious that the computation cost required for the wavelet transform is larger than that required for the sparse solver and then dominates the total computation cost as the number of unknowns increases.

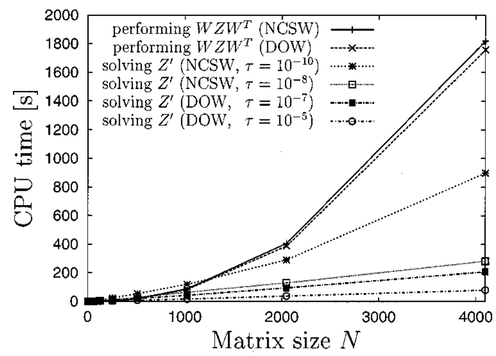

Fig. 6. CPU time for the sparse solver and the wavelet transform.

V. CONCLUSION

The DOW is compared with the NCSW in the wavelet transform approach. Even though the NCSW yields a sparser impedance matrix than the DOW does, the impedance matrix obtained by the NCSW has much larger condition number than that obtained by the DOW. As the result, the NCSW based impedance matrix requires more computation cost for its solution than the DOW. Especially for the computation cost required for performing the wavelet transform, which is a dominant part of the total computation cost, the DOW is better than the NCSW.

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