Letters

Scattering from an Ionized Column in the Earth–Ionosphere Space

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***Abstract*—An extension of an earlier paper [1] on this subject is hereby developed to allow for the generalization of the Born approximation to account for eddy currents in the column. Also, the relevance to VLF sprites is pointed out; these are ionized columns extending from the cloud tops to the lower ionosphere.**

***Index Terms*—Earth–ionosphere waveguide, electromagnetic (EM) scattering.**

A crucial idealization earlier [1] was the invoking of the first-order Born approximation to estimate the induced currents in the column. The key step here is to note that the induced current $I(z')$ within the column extending from $z' = z_a$ to z_b can be represented more generally by

$$I(z') = E_r(d_1, z')/[Z_c(z') + Z_{\rm ex}]. \tag{1}$$

The notation and symbols are as defined in the earlier paper [1]. In particular, see [1, Fig. 1]. To remind the reader, E_r is the radial electric field of the distant very low frequency (VLF) source and d_1 is the great-circle distance from the source to the sprite column. $Z_c(z')$ is the axial (z' dependent) internal impedance of the column, while $Z_{\rm ex}$ is the external impedance of the column [2] that is assumed here not to depend on the mode order of the dominant waveguide modes, which excite the column currents. As in the earlier paper [1], the time factor is $\exp(jwt)$. Appropriate expressions for the impedances [2] are

$$Z_c = (j\mu w/2\pi) I_0(\gamma_c \hat{a})/[\gamma_c \hat{a} I_1(\gamma_c \hat{a})] \tag{2}$$

where I_0 and I_1 are modified Bessel functions, $\gamma_c = [j\mu w(\sigma + j\varepsilon w)]^{1/2}$ where σ, ε, and μ are the electromagnetic (EM) properties of the column, and where $\hat{a}$ is the column radius. Also

$$Z_{\rm ex} = (j\mu_0 w/2\pi) K_0(jk\hat{a}), \qquad (k = 2\pi/\text{wavelength}) \tag{3}$$

where K_0 is the MacDonald function of order zero.

Now because $k\hat{a} \ll 1$ for all purposes here

$$Z_{\rm ex} \cong (\mu_0 w/2) + j\mu w/2\pi)[\ln(2/k\hat{a}) - 0.5773].$$

At sufficiently low frequencies and/or low conductivities, where $|\gamma_c \hat{a}| \ll 1$, it follows from (2) that $Z_c \cong (\pi \hat{a}^2 \sigma)^{-1}$ assuming $\varepsilon w \ll \sigma$. The z' dependence of Z_c and σ is understood. Then, if $Z_{\rm ex} \ll Z_c$, (1) is reduced to

$$I(z') = E_r(d_1, z')/Z_c(z') \cong \pi \hat{a}^2 \sigma(z') E_r(d_1, z') \tag{4}$$

which corresponds to the Born approximation as employed before [1]. In the present context, it is valid if $|\gamma_c \hat{a}|^2 \ln(2/k\hat{a}) \ll 1$. As indicated earlier [2], such would apply to column radii of 1 m or less and column conductivities not greater than 10^{-2} S/m at frequencies less than 30 kHz.

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A simple modification of the working equations in [1, given by (23)–(27)], is to replace the actual column conductivity $\sigma(z')$ by an effective (complex) conductivity $\sigma_\theta(z')$. The implementation of this step is $\sigma(z') n\hat{a}^2 \Rightarrow n\hat{a}^2 \sigma_\theta(z') = [Z_c(z') + Z_{\rm ex}]^{-1}$. To further simplify the discussion, we follow the assumption in [1] and regard the actual conductivity constant over the vertical extent of the column or to regard it as an average denoted by $\overline{\sigma}$ and the corresponding effective expression by $\overline{\sigma}_\theta$.

In the specific application to a sprite column, the mode conversion coefficient in [1] is now given by

$$C_{mn} \cong (1/h) \int_{z_a}^{h} G_m(y') G_n(y')\, dz' \tag{5}$$

where a (not $\hat{a}$) is the earth's radius and $y' = (2/ka)^{1/3} kz'$. Here the integration extends from the bottom of the column at height z_a above the earth surface to the height of the lower edge of the ionosphere. We now change the variable to y' and denote $y_a = (2/ka)^{1/3} kz_a$ and $y_0 = (2/ka)^{1/3} kh$. Also we exploit the orthogonality of the height gain functions $G_m(y')$ and $G_n(y')$ to arrive at the useful version of (5)

$$C_{mn} \cong \delta_{mn} (2\Lambda_m)^{-1} - (1/y_0) \int_0^{y_a} G_m(y')\, dy' \tag{8}$$

where $\delta_{mn} = 0$ for $m \neq n$ and =1 for $m = n$ and where

$$\Lambda_m \cong (y_0/2) \left[\int_0^{y_0} [G_m(y')]^2\, dy' \right]^{-1} \tag{9}$$

is the excitation function as conventionally defined [3] for earth–ionosphere waveguide modes at VLF.

To deal with integral in (8), we make use of the boundary condition also used before [1], which reads

$$[dG_m(y)/dy - qG_m(y)]_{y=0} = 0 \tag{10}$$

where

$$q = -j(ka/2)^{1/3} Z_s/(120\pi)$$

in terms of the surface impedance of the ground where it is safe here to ignore the dependence of Z_s on the mode index m. The height gain functions are normalized such that $G_m(0) = G_n(0) = 1$. Also, we utilize the fact for the region $0 < y < y_0$, $G_m(y)$ satisfies Stokes differential equation [3] given by

$$d^2 G_m(y)/dy^2 - (t_m - y) G_m(y) = 0 \tag{11}$$

where t_m is the eigenvalue for mode of order m [1], [3]. Employing (10) and (11), it is not difficult to show that

$$G_m(y) = 1 - qy + (t_m/2) y^2 - (1 + t_m q)(y^3/6) + \cdots \tag{12}$$

and thus

$$\begin{aligned} G_m(y) G_n(y) = {} & 1 - 2qy + (t_m + t_n + 2q^2)(y^2/2) \\ & - (1 + 2t_m q + 2t_n q)(y^3/3) + \cdots . \end{aligned} \tag{13}$$

On replacing y by y' in (13), the integration in (8) can be performed to yield the desired expression for the mode conversion

$$C_{mn} = \delta_{mn}/(2\Lambda_m) - (y_a/y_0)[1 - 2qy_a + (t_m + t_n + 2q^2) \cdot (y_a^2/6) - (1 + 2t_m q + 2t_n q)(y_a^3/12) + \cdots]. \quad (14)$$

In the case where $m \neq n$, the mode conversion is characterized by

$$C_{mn} = -(z_a/h)[1 + jkz_a(Z_s/120\pi) + \text{terms in } z_a^2,\ z_z^3 \ldots].$$

What we have done here is to show that within the confines of the Born approximation, the conclusions arrived at earlier [1] remain intact; but also, we have indicated how the earlier formulation can be generalized to permit application to “denser” ionized columns, which may also have a larger cross-sectional area. The specific relevance to scattering from sprites has also been pointed out. Further generalizations now are needed to cope with multiple scattering of clusters of columns that have been observed optically [4]. An analytical and numerical model to treat this problem has already been carried out [5], but such was confined to a two-dimensional geometry and earth curvature effects were not considered.

References

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