Ground Wave of an Idealized Lightning Return Stroke

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***Abstract*—We model a lightning return stroke by a vertical traveling wave of current with a complex propagation constant. The Sommerfeld-integral analysis is similar to that of a vertical electric dipole over a lossy earth except that the source is distributed in height. When the integration over the source current is performed analytically, an extra term appears in addition to the classical Sommerfeld attenuation function. This term is a result of the height-gain function of the distributed source due to an effective elevated height of the source dipole moment. In most cases of interest, the extra term is small and the height-gain function is not much larger than one. The results have application to remote sensing of lightning from a ground-based observer.**

***Index Terms*—Ground wave, height-gain function, lightning return stroke, Sommerfeld integral, vertical electric dipole.**

I. Introduction

USUALLY, in analyzing the electromagnetic fields produced by lightning [1], it is assumed that the ground plane is perfectly conducting [2], [3]. For many applications such an idealization is permissible. But one aspect of the problem is then overlooked. Specifically the excited ground wave at zero-elevation angle is profoundly influenced by the finite earth conductivity [4].

It is interesting to note that a fairly simple analytical solution for the fields of a traveling wave current is available based on what we know about Sommerfeld-type integrals [5]. Here we derive both an integral form for the solution and approximate forms that are closely related to the Sommerfeld attenuation function. The results have application to remote sensing of lightning from ground-based observations.

II. Theoretical Model

The simple model is indicated in Fig. 1. With respect to a cylindrical coordinate system (ρ, ϕ, z), the ground plane at $z = 0$ is characterized by a surface impedance Z, which, for simplicity, is assumed known. The source is a traveling wave of current $I(h)$ at $\rho = 0$, where h is the height above $z = 0$. Elementary field theory tells us that the resultant fields in the free-space region $z > 0$ can be determined by a vector potential $\boldsymbol{A}$, which has only a z component, here denoted simply by A. We consider only the frequency-domain case, but the corresponding time-domain solution could be obtained by an inverse Fourier transform. For a time-harmonic dependence $\exp(j\omega t)$, A is given by [5]

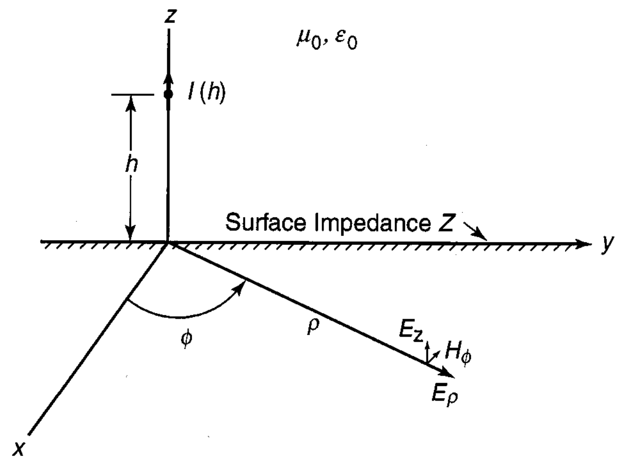

Fig. 1. Vertical traveling wave current located over a flat earth of surface impedance Z. The observer is located at the surface at a distance ρ from the base of the current channel.

$$A = \frac{1}{4\pi} \int_0^\infty I(h) \cdot \left\{ \int_0^\infty \frac{g}{u} \left[\exp(-u|z-h|) + \frac{u - jk\Delta}{u + jk\Delta} \exp[-u(z+h)] \right] \cdot J_0(g\rho)\, dg \right\} dh \tag{1}$$

where $u = \sqrt{g^2 - k^2}$, $\Delta = Z/\eta_0$, $k^2 = (\omega/c)^2$, $\eta_0 = \sqrt{\mu_0/\varepsilon_0}$, $c = 1/\sqrt{\mu_0 \varepsilon_0}$, $J_0(g\rho)$ is the Bessel function of order zero and argument $g\rho$, g is the radial wavenumber, ε_0 is the permittivity of free-space, and μ_0 is the magnetic permeability of free-space. The fields at the observer (ρ, z) can be obtained from

$$E_p = \frac{1}{j\omega\varepsilon_0} \frac{\partial^2 A}{\partial \rho \partial z} \tag{2}$$

$$E_z = \frac{1}{j\omega\varepsilon_0} \left(k^2 + \frac{\partial^2}{\partial z^2} \right) A \tag{3}$$

$$H_\phi = -\frac{\partial A}{\partial \rho}. \tag{4}$$

Equation (1) is the exact solution of the problem and is basically the known result for a vertical electric dipole $I(h)\, dh$ that has been integrated over the line source extending from $h = 0$ to $h = \infty$. With due care, the order of the integrations can be

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interchanged. But before we proceed, it is useful to note that the normalized surface impedance Δ for a homogeneous half-space model of the earth is given by [5]

$$\Delta = N^{-1}\sqrt{1-N^{-2}S^2} \tag{5}$$

where $S = g/k$, $N = \sqrt{(\sigma + j\omega\varepsilon)/(j\omega\varepsilon_0)}$ and g is the radial wavenumber as in (1) above. Here, the whole region is assumed to be nonmagnetic, so the permeability is μ_0 everywhere. The permittivity and conductivity of the lower region are ε and σ respectively. For layered-earth models, other forms for Δ are available [5].

In the present case, we are focusing our attention on the ground-wave field, so it is permissible to let $z \to 0$ in (1) so that

$$A = \frac{1}{2\pi}\int_0^\infty I(h)\int_0^\infty \frac{\exp(-uh)}{u+jk\Delta} J_0(g\rho)g\,dg\,dh. \tag{6}$$

Now we adopt the traveling wave model [2], [6], [7]

$$I(h) = I_0 \exp(-\Gamma h) \quad (\text{for } 0 < h < \infty) \tag{7}$$

where I_0 is the current at the air-ground interface and $\Gamma = j(\omega/v) + \alpha$ in terms of the phase velocity v and attenuation rate α. Because $\mathrm{Re}(\Gamma) > 0$, the h integration in (6) is allowable. (If one wishes to be more rigorous, the integration over h could be done before setting $z = 0$. The results would be the same.) Then

$$A = \frac{I_0}{2\pi}\int_0^\infty (u+jk\Delta)^{-1}(u+\Gamma)^{-1}J_0(g\rho)g\,dg \tag{8}$$

which is equivalent to

$$A = \frac{I_0}{2\pi(\Gamma - jk\Delta)}(F-G) \tag{9}$$

where

$$F = \int_0^\infty (u+jk\Delta)^{-1}J_0(g\rho)g\,dg \tag{10}$$

$$G = \int_0^\infty (u+\Gamma)^{-1}J_0(g\rho)g\,dg \tag{11}$$

and we evaluate Δ at $S = g/k = 1$.

An explicit expression for the magnetic field at (ρ, 0) without further restrictions is obtained from (4) operating on (9) to give

$$H_\phi = \frac{I_0}{2\pi(\Gamma - jk\Delta)}(F' - G') \tag{12}$$

where

$$F' = \int_0^\infty (u+jk\Delta)^{-1}J_1(g\rho)g^2\,dg = -\frac{\partial F}{\partial \rho} \tag{13}$$

and

$$G' = \int_0^\infty (u+\Gamma)^{-1}J_1(g\rho)g^2\,dg = -\frac{\partial G}{\partial \rho}. \tag{14}$$

Formally, the results given above are exact for the model adopted, and they are exact for all distances ρ along the ground plane. But the reader will note that F, G, F', and G' are divergent integrals as written. The results can be made more rigorous by noting the definition

$$\int_0^\infty f(g)J_m(g\rho)g^n\,dg = \lim_{z\to 0}\left[\int_0^\infty f(g)\exp(-uz)J_m(g\rho)g^n\,dg\right] \tag{15}$$

where m and n are positive integers.

Exact expressions for the electric field components can also be obtained from (1) when (2) and (3) are used *prior* to setting $z = 0$. These are not given here.

III. Approximations

Here, we will be content with approximate analytical expressions for the integrals F and G. Not too surprisingly, F as given by (10) is identical to the form for ground-wave propagation for a ground-based vertical electric dipole when the observer is also at $z = 0$. In the case when Δ is assumed not to depend on g, where $k\rho \gg 1$ and indeed where $|\Delta|^2 \ll 1$, the Sommerfeld–Norton form is equivalent to [5]

$$F \approx [1 - j\sqrt{\pi p}\,\exp(-p)\mathrm{erfc}(j\sqrt{p})]\,\frac{\exp(-jk\rho)}{\rho} \tag{16}$$

where the numerical distance p is defined by $p = -jk\rho\Delta^2/2$ and erfc is the error-function complement of the indicated complex argument. The term in square brackets is usually known as the ground-wave attenuation function (GAF), and it has been well studied numerically [5]. A convergent (but not always useful) series expansion is given by [5]

$$\mathrm{gaf} = 1 - j(\pi p)^{1/2} + 2p + j\pi^{1/2}p^{3/2} + \cdots \tag{17}$$

while an asymptotic (sometimes useful) expansion is

$$\mathrm{gaf} \approx -(2p)^{-1} - 3(2p)^{-2} - 15(2p)^{-3} - \cdots \tag{18}$$

valid for $|p| \to \infty$. The leading term in (18) is adequate for high frequencies (e.g., above 10 MHz) and normal soil (e.g., $\sigma = 5 \times 10^{-3}$ S/m and $\varepsilon/\varepsilon_0 = 10$). Then

$$\mathrm{gaf} \approx (jk\rho\Delta^2)^{-1} \tag{19}$$

where

$$\Delta^2 = N^{-2}(1 - N^{-2}) \tag{20}$$

and we assume $S \approx 1$.

But we also need to examine the G integral as defined by (11). It appears to have the same form as the F integral which is true. But if we set $\Gamma = jk\Delta_e$, the effective or analogous function to Δ would be given by $\Delta_e = \Gamma/(jk) \approx c/v$. Thus, $|\Delta_e|^2$ is not small compared to one. But now a simple asymptotic estimate will suffice, rather than trying to modify the GAF. To this end, we note that

$$\begin{aligned}(u+\Gamma)^{-1} &= \Gamma^{-1}[1 + (u/\Gamma)]^{-1}\\ &\approx \Gamma^{-1}[1 - (u/\Gamma) + (u/\Gamma)^2 - \cdots]\end{aligned} \tag{21}$$

is highly convergent since $u = \sqrt{g^2 - k^2} \approx jk$ because the principal value of g is jk for propagation along the interface where $k\rho \gg 1$. The next step is then to note that

$$G = \lim_{z \to 0} \left[\int_0^\infty (u + \Gamma)^{-1} \exp(-uz) J_0(g\rho) g\, dg \right] \tag{22a}$$

$$= \lim_{z \to 0} \left[\Gamma^{-1} \int_0^\infty \exp(-uz) J_0(g\rho) g\, dg - \Gamma^{-2} \int_0^\infty \exp(-uz) u J_0(g\rho) g\, dg + \Gamma^{-3} \int_0^\infty \exp(-uz) u^2 J_0(g\rho) g\, dg - \cdots \right]. \tag{22b}$$

The first integral is identically zero in the limit $z \to 0$.

The second and third integrals are

$$G = \lim_{z \to 0} \left\{ \left[-\Gamma^{-2} \frac{\partial^2}{\partial z^2} - \Gamma^{-3} \frac{\partial^3}{\partial z^3} - \cdots \right] \cdot \int_0^\infty \frac{g}{u} \exp(-uz) J_0(g\rho)\, dg \right\}. \tag{23}$$

The integral here is just $r^{-1} \exp(-jkr)$, where $r = \sqrt{\rho^2 + z^2}$. When these derivative operations are carried out and assuming $k\rho \gg 1$, we find that

$$G \approx [-(2p_e)^{-1} - 3(2p_e)^{-2} - \cdots] \rho^{-1} \exp(-jk\rho) \tag{24}$$

where $p_e = -jk\rho\Delta_e^2/2 = jk\rho(\Gamma/k)^2/2 \approx -jk\rho(c^2/v^2)/2$. (The attenuation rate α is assumed to be small.) Thus, the first-order estimate for G is

$$G \approx \frac{1}{jk\rho} \frac{v^2}{c^2} \frac{\exp(-jk\rho)}{\rho}. \tag{25}$$

Thus, for most cases G can be neglected compared with F in (9) for the ground wave when $k\rho \gg 1$. Thus, for $k\rho \gg 1$, the tangential magnetic field can be well approximated by

$$H_\phi \approx \frac{I_0}{2\pi(\Gamma - jk\Delta)} F' \tag{26}$$

where $F' = -\partial F/\partial p \approx jkF$ and F is given by (16). Also for $kr \gg 1$, it is noted that E_z at $z = 0$ is given by $E_z \approx -\eta_0 H_\phi$. Furthermore the horizontal electric field at $z = 0$ is given by $E_p \approx -\Delta\eta_0 H_\phi$.

IV. Equivalent Dipole Interpretation

The analytical approximations for the distributed source in the previous section have a somewhat different appearance from the well-known ground-wave expressions for a vertical electric dipole source located over imperfect ground [5]. However, in this section we will show that the fields for sufficiently large $k\rho$ can be interpreted as the usual ground wave for a surface-based dipole times a standard height-gain function [8] to account for the effective height of the distributed line source.

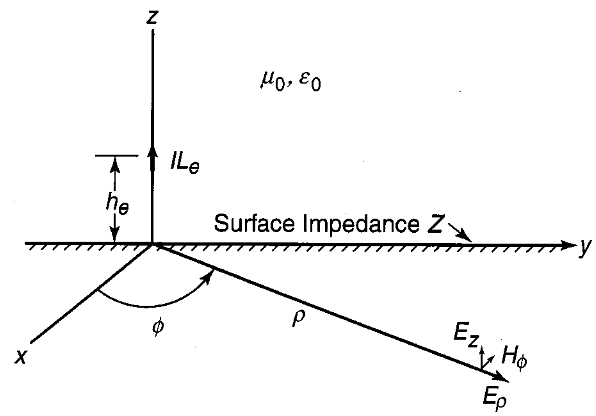

Fig. 2. Effective source dipole moment IL_e located at a complex height h_e.

The starting point is to determine the leading terms in inverse ρ for both F' and G' to be substituted into H_ϕ as given by (12). From (13), (16), and (19), we can approximate F' as

$$F' \approx \frac{\exp(-jk\rho)}{\rho^2 \Delta^2}. \tag{27}$$

Similarly, from (14) and (24), we can approximate G' as

$$G' \approx -\frac{\exp(-jk\rho)}{\rho^2 (\Gamma/k)^2}. \tag{28}$$

If we substitute (27) and (28) into (12), we can approximate H_ϕ as

$$H_\phi \approx \frac{I_0}{2\pi\Gamma(1 - jk\Delta/\Gamma)} \frac{\exp(-jk\rho)}{\rho^2\Delta^2} \left(1 + \frac{k^2\Delta^2}{\Gamma^2}\right). \tag{29}$$

We can factor the quantity in the right-hand parentheses to obtain

$$H_\phi \approx \frac{I_0 \exp(-jk\rho)}{2\pi\Gamma\rho^2\Delta^2} \left(1 + \frac{jk\Delta}{\Gamma}\right). \tag{30}$$

This result can be interpreted as the classic ground wave multiplied by the height-gain function as shown in the next section.

A. Ground Wave and Height-Gain Function

For ground-wave excitation, the effective dipole moment IL_e of the vertical line current is

$$IL_e = \int_0^\infty I(h)\, dh = I_0 \int_0^\infty \exp(-\Gamma h)\, dh = I_0/\Gamma \tag{31}$$

where we assume $\alpha > 0$ so that the integral converges. The effective height h_e can be determined by averaging over h

$$h_e = \frac{1}{IL_e} \int_0^\infty h I(h)\, dh = \Gamma \int_0^\infty h \exp(-\Gamma h)\, dh = 1/\Gamma. \tag{32}$$

Since Γ is complex, both IL_e and h_e are complex. Even so, they can be used to rewrite (30) as

$$H_\phi \approx H_{g\phi}(\rho) G_h(h_e) \tag{33}$$

TABLE I
NORMALIZED SURFACE IMPEDANCE AND HEIGHT-GAIN FUNCTION ($v/c = 0.5$).

Frequency (MHz)	$\varepsilon/\varepsilon_0$	σ (S/m)	Δ	G_h
1	6	10^{-3}	0.228 ∠34.3°	1.096 ∠3.4°
1	10	10^{-2}	0.075 ∠43.4°	1.027 ∠1.4°
10	6	10^{-3}	0.368 ∠6.8°	1.182 ∠1.1°
10	10	10^{-2}	0.217 ∠29.2°	1.096 ∠2.8°

where

$$H_{g\phi}(\rho) = \frac{IL_e \exp(-jk\rho)}{2\pi\rho^2\Delta^2} \tag{34}$$

and

$$G_h(h_e) = 1 + jk\Delta h_e. \tag{35}$$

Now we can make the physical interpretation of (33) that the magnetic field is the product of the ground-wave field $H_{g\phi}$ for a surface-based dipole with an effective moment IL_e and the height-gain function G_h for an effective height h_e. This is pictured in Fig. 2, but it should be kept in mind that h_e is complex because Γ is complex. The same height-gain function [8] can be used to account for an elevated observer $(z > 0)$:

$$H_\phi \approx H_{g\phi}(\rho) G_h(h_e) G_h(z) \tag{36}$$

where $G_h(z) = 1 + jk\Delta z$. The height-gain approximation in (35) and (36) neglects terms of order $(k\Delta h_e)^2$.

B. Space Wave at Low Angles

The height-gain function provides a convenient method for determining the field of an elevated source or observer [9], [10] when the ground wave is already known. However, equivalent results can also be obtained by adding the space wave to the ground wave. To illustrate the procedure, consider the case of the vertical line source with the observation point at the surface ($z = 0$). Then the inverse-distance space wave $H_{s\phi}$ can be written as the sum of an incident and reflected wave [8] integrated over the vertical line source. For both $k\rho$ and ρ/z large, $H_{s\phi}$ can be approximated by

$$H_{s\phi} \approx \frac{jkI_0 \exp(-jk\rho)}{4\pi\rho} \int_0^\infty (1 + R_v) \exp(-\Gamma h)\, dh \tag{37}$$

where

$$R_v = \frac{C - \Delta}{C + \Delta} \text{ and } C \approx \frac{h}{\rho}. \tag{38}$$

By substituting (38) into (37), we can approximate $H_{s\phi}$ as

$$\begin{aligned} H_{s\phi} &\approx \frac{jkI_0 \exp(-jk\rho)}{2\pi\rho^2\Delta} \int_0^\infty h \exp(-\Gamma h)\, dh \\ &\approx jk\Delta h_e \frac{IL_e \exp(-jk\rho)}{2\pi\rho^2\Delta^2}. \end{aligned} \tag{39}$$

Now we can relate the space wave to the ground wave by using (34), (35), and (39)

$$H_{s\phi} \approx [G_h(h_e) - 1] H_{g\phi}. \tag{40}$$

Hence, the sum of the ground and space waves is

$$H_{g\phi} + H_{s\phi} \approx H_{g\phi} G_h(h_e) \tag{41}$$

which is consistent with (36) for the observer located at the surface ($z = 0$). For large heights z, the space wave dominates the ground wave and that case has been covered separately [11].

V. DISCUSSION

Typically the attenuation rate α of the traveling wave current is fairly small compared to the imaginary part of the propagation constant Γ [2]. So Γ can be approximated as

$$\Gamma \approx j(\omega/v) = jk(c/v). \tag{42}$$

If we substitute (32) and (42) into (35), we obtain the following approximation for the height-gain function:

$$G_h(h_e) \approx 1 + \Delta\frac{v}{c}. \tag{43}$$

Even though (43) has a simple form, G_h is frequency dependent and complex because Δ is frequency dependent and complex. For all ground-wave calculations, Δ is approximated by (5) at grazing ($S = 1$)

$$\Delta \approx N^{-1}\sqrt{1 - N^{-2}}. \tag{44}$$

In Table I, we show numerical results for the normalized surface impedance and height-gain function at 1 and 10 MHz for typical ground and lightning parameters. For the current phase velocity [1], we assume $v/c = 1/2$. For dry ground we take $\varepsilon/\varepsilon_0 = 6$ and $\sigma = 10^{-3}$ S/m and for moist ground we take $\varepsilon/\varepsilon_0 = 10$ and $\sigma = 10^{-2}$ S/m. The important point is that G_h is typically close to one. The largest departure of G_h from one occurs at 10 MHz for dry ground because $|\Delta|$ has its largest value for that case. However, the ground-wave field strength $H_{g\phi}$, which multiplies the height-gain function would be smaller for that case because $H_{g\phi}$ is proportional to Δ^{-2}, as shown in (34).

VI. CONCLUSION

We have modeled a lightning return stroke by a vertical, traveling wave current with a complex propagation constant. A Sommerfeld-integral analysis yields an exact expression for the field at arbitrary height in (6). For a ground-based observer, we obtain an exact expression for the Hertz potential and the magnetic field in terms of two Sommerfeld integrals in (9)–(14). These integrals could be evaluated numerically, but we have obtained analytical approximations in Section III. The F integral is proportional to the standard ground-wave attenuation function for a surface-based vertical electric dipole as shown in (16). The G integral is a result of the distributed current, and it has a simple asymptotic expansion as shown in (25). For most cases of interest, the dominant field contribution comes from the F integral.

At sufficiently large horizontal distances, the field can be written as the product of the ground wave of a surface-based electric dipole and the height-gain function due to the effective height of the distributed current source, as shown in (33)–(36). Because the propagation constant of the traveling wave current is complex, the effective height h_e is also complex as shown in (32). Even so, the form of the height-gain function in (35) is the same as the usual form for an elevated source located at a real height [8], [9]. For the usual case where the real part of the propagation constant of the current is small, the height-gain function takes a simple form involving the normalized current-phase velocity and surface impedance, as shown in (43). Some typical numerical values for the height-gain function and the normalized surface impedance are shown in Table I. The largest departure of the height-gain function from one is approximately 18%, and it occurs at 10 MHz for dry ground.

Since both the surface impedance and the height-gain function are frequency dependent, the ground-wave pulse will suffer dispersion. A useful extension would be to compute some typical time-domain waveforms to see how ground-based remote sensing and coupling to cables near the ground [12] would be affected. The effect of earth curvature [13] has been neglected here, but could be included to make the analysis more general.

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