Linear Inverse Problems in Wave Motion: Nonsymmetric First-Kind Integral Equations

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***Abstract*—We present a general framework to study the solution of first-kind integral equations. The integral operator is assumed to be compact and nonself-adjoint and the integral equation can possess a nonempty null space. An approach is presented for adding contributions from the null-space to the minimum-energy solution of the integral equation through the introduction of weighted Hilbert spaces. Stability, accuracy, and nonuniquenes of the solution are discussed through the use of model resolution, data fit, and model covariance operators. The application of this study is to inverse problems that exhibit nonuniqueness.**

***Index Terms*—Integral equation methods, inverse problems, weighted Hilbert space.**

I. Introduction

In the study of linear inverse problems, first-kind Fredholm integral equations play an important role. These problems have been widely studied and reported in the pages of the journal *Inverse Problems* [1], for example. Inverse problems present a particular difficulty in that they are often *ill posed* in the sense of Hadamard [2]. A problem is *well posed* if the following three statements are true:

1) a solution to the problem exists;
2) there is at most one solution to the problem;
3) the solution depends continously on the data.

If any of these three statements is untrue, the problem is ill posed. By far the major emphasis in the study of ill-posed problems has been on matters involving the dependence of the solution on the data (Statement 3). Such studies have led to regularization strategies [3]–[11] to improve solution stability in ill-posed problems. Indeed, there has recently been made available a Matlab package [12] concerned with the analysis of and solution to discrete ill-posed problems, with emphasis on regularization.

Our applications are in inverse problems in wave motion, where nonuniqueness is a principal issue [13]–[16]. It is, therefore, the violation of the condition in Statement 2 that commands our attention in this paper. This is not to minimize the importance of continuous dependence on the data. Indeed, many of our problems involve violation of both Statements 2 and 3.

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In many applications, the inversion of first-kind integral equations is widely used to estimate parameters and map geometries in areas as diverse as geophysical prospecting, medical imaging, and nondestructive evaluation. Principal among our interests in wave motion are applications in geophysical exploration [17]–[21]. A first-kind Fredholm integral equation arises, for example, in the employment of the Born approximation.

In this paper, we consider characteristics of first-kind integral equations and their solution(s) under the following two conditions:

1) the integral operator is compact, but not necessarily self adjoint;
2) the integral equation posseses a nonempty null space.

It is the second condition that sets our paper apart from most of the literature on the subject.

In Section II, we give some preliminaries consisting of the following:

1) a review the spectral theorem for compact self-adjoint operators;
2) a review of the singular-value decomposition theorem and its application to compact, nonself-adjoint operators.

In Section III, we consider compact, but not necessarily self-adjoint first-kind integral equations. We derive some basic results using singular-value decomposition. In Section IV, we consider a completeness relationship that involves eigenfunctions with nonzero eigenvalues and eigenfunctions in the null space. In Section V, we consider properties of the solution to the integral equation. In particular, we discuss the fact that classical methods produce a minimum norm solution in terms of the eigenfunctions with nonzero eigenvalues. However, the solution is nonunique because these methods do nothing to remove the arbitrary character of the portion of the solution in the null space. In Section VI, we discuss solution accuracy and sensitivity and develop measures to quantify the effects of the null space.

The principal purpose of this paper is to incorporate portions of the solution that are in the null space of the integral operator into the overall solution. We accomplish this goal by using weighting functions to include the null-space contributions. Mathematically, we construct a Hilbert space containing a weighting operator that generates the required weighting function. The resulting Hilbert space includes the original Hilbert space as a subspace. In Section VII, we introduce the weighting function and derive the minimum norm solution on the weighted Hilbert space. In Section VIII, we take a slightly different approach and weight the data with the weighting operator on the original Hilbert space. Finally, in Section IX, we produce our principal result. In the weighted Hilbert space, we produce a solution that consists of the minimum norm solution in the original

Hilbert space, plus a contribution from the null space of the integral operator. This null-space contribution can be adjusted by specific choice of the weighting operator, a choice that can incorporate our practical knowledge of the problem physics.

II. Preliminaries

We shall consider our problem in a complex Hilbert space $\mathcal{H}$. Let

$$La = b; \quad L \neq 0 \tag{1}$$

where L is compact and selfadjoint. Then, the principal results from the spectral theory for compact self-adjoint operators [22], [23] are as follows:

1) all eigenvalues of L are real;
2) L has at least one nonzero eigenvalue and at most a countable set of eigenvalues with accumulation point only at zero;
3) all eigenspaces for nonzero eigenvalues have finite dimension;
4) eigenspaces for different eigenvalues are orthogonal;
5) eigenfunctions of L corresponding to nonzero eigenvalues form an orthonormal basis for $\mathcal{R}_L$ (range of L), *viz.*

$$La = \sum_{k=1}^{\infty}\langle La,\, e_k\rangle e_k = \sum_{k=1}^{\infty}\langle a,\, Le_k\rangle e_k = \sum_{k=1}^{\infty} \mu_k\langle a,\, e_k\rangle e_k \tag{2}$$

6) an arbitrary $a \in \mathcal{H}$ can be decomposed

$$a = a_0 + \sum_{k=1}^{\infty}\langle a,\, e_k\rangle e_k \tag{3}$$

where $a_0 \in \mathcal{N}_L$ (null space of L), *viz*:

$$La_0 = 0. \tag{4}$$

We remark that our sums above are countably infinite in length; this will be the usual case in our problems of interest, however, the theorems do not rule out finite sums in the above or in the sequel.

We shall be particularly interested in cases, where $\mathcal{N}_L$ is not empty. In addition, the operators we shall consider are in general nonselfadjoint. We, therefore, must relax the self-adjoint requirement. If L is compact, we may use a classic procedure and preoperate on both sides of (1) with the adjoint operator L^* to give

$$L^*La = \hat{b} \tag{5}$$

where

$$\hat{b} = L^*b \tag{6}$$

with the well-known characteristics:

1) L^*L is nonnegative. If $La = 0$ implies that $a = 0$, then L^*L is positive;
2) L^*L is selfadjoint.

Implicit is $b \in \mathcal{D}_{L^*}$ (domain of L^*). Since L^*L is selfadjoint, we have the associated eigenproblem

$$L^*Lv_n = \lambda_n^2 v_n, \tag{7}$$

where the eigenvalues λ_n^2 are real and nonnegative and $\{v_n\}$ is an orthonormal sequence. We remark that L being compact implies that L^*L is compact. By the singular-value decomposition theorem [24], [25]), there exists $\{u_n\}$ such that

$$Lv_n = \lambda_n u_n, \qquad \{u_n\} \text{ orthonormal} \tag{8}$$

$$L^*u_n = \lambda_n v_n. \tag{9}$$

In (8) and (9), the numbers λ_n are the *singular values*. Next, from the spectral theorem for compact self-adjoint operators applied to L^*L, it follows that

$$a = \sum_{n=1}^{\infty}\langle a,\, v_n\rangle v_n + a_0, \qquad a_0 \in \mathcal{N}_{L^*L}. \tag{10}$$

Furthermore, since $a_0 \in \mathcal{N}_{L^*L}$,

$$\langle a_0,\, L^*La_0\rangle = 0 = \langle La_0,\, La_0\rangle \tag{11}$$

which implies $La_0 = 0$ and, therefore, the important result

$$\mathcal{N}_{L^*L} = \mathcal{N}_L. \tag{12}$$

We, therefore, have a stronger result in (10), *viz.*

$$a = \sum_{n=1}^{\infty}\langle a,\, v_n\rangle v_n + a_0, \qquad a_0 \in \mathcal{N}_L. \tag{13}$$

We may now apply the operator L to (13) with the result

$$La = \sum_{n=1}^{\infty}\langle a,\, v_n\rangle Lv_n = \sum_{n=1}^{\infty}\langle a,\, v_n\rangle \lambda_n u_n. \tag{14}$$

We note that the passing of L inside the sum is legitimate, since L is compact and thus bounded.

Some comments are in order.

1) a_0 is an eigenfunction of L^*L with zero eigenvalue. The eigenfunction a_0 has the important property that it is orthogonal to all v_n.
2) The set of all eigenfunctions, including those with zero eigenvalues, forms a basis for $\mathcal{H}$.
3) The set $\{v_n\}$ forms a basis if and only if $L^*La = 0$ implies $a = 0$.
4) The maximum eigenvalue $\max(\lambda_n^2) = ||L^*L||$.

We note that the adjoint operator enters into all of this, as is apparent in (5) and (9). We may multiply both sides of (9) by L and substitute (8) to obtain

$$LL^*u_n = \lambda_n^2 u_n. \tag{15}$$

Therefore, we now have the eigenfunctions v_n of L^*L given by (7) and the eigenfunctions u_n of LL^* given by (15); in addition, we have *complete* expansions for both the domain and the range given by (13) and (14). We emphasize that one has to include the null-space eigenfunctions in (13), if any.

III. First-Kind Integral Equations

We consider the following integral equation in Hilbert space $\mathcal{L}_2(c, d)$:

$$\int_c^d a(z)g(x, z)\,dz = b(x). \tag{16}$$

We refer to $a(z)$ as the "solution" and to $b(x)$ as the "observables" or "data." We write (16) in operator form as

$$La = b \tag{17}$$

where

$$L = \int_c^d (\cdot)g(x, z)\,dz. \tag{18}$$

We refer to the integral operator in (18) with a specific $g(x, z)$ as the "model." We shall assume that L is compact (and, therefore, bounded). An example of a class of compact operators of potential interest is the Hilbert–Schmidt operator, with Hilbert–Schmidt kernel defined by [26]

$$\int_c^d \int_c^d |g(x, z)|^2\,dx\,dz < \infty. \tag{19}$$

For L nonselfadjoint, we find that

$$\begin{aligned}\langle La, s\rangle &= \int_c^d \int_c^d a(z)g(x, z)\,dz\overline{s}(x)\,dx\\ &= \int_c^d a(z)\int_c^d \overline{s}(x)g(x, z)\,dx\,dz\\ &= \int_c^d a(x)\left(\overline{\int_c^d s(z)\overline{g}(z, x)\,dz}\right)dx\\ &= \langle a, L^*s\rangle\end{aligned} \tag{20}$$

where

$$L^* = \int_c^d (\cdot)\overline{g}(z, x)\,dz. \tag{21}$$

Although L is nonselfadjoint, L^*L is selfadjoint, where, in this case

$$L^*L = \int_c^d \left[\int_c^d (\cdot)g(y, z)\,dz\right]\overline{g}(y, x)\,dy = \int_c^d (\cdot)h(x, z)\,dz \tag{22}$$

where

$$h(x, z) = \int_c^d \overline{g}(y, x)g(y, z)\,dy. \tag{23}$$

We note that

$$\overline{h}(z, x) = h(x, z). \tag{24}$$

Since L compact implies that L^*L is self adjoint and compact, the singular-value decomposition theorem applies. We, therefore, can use (7)–(9) and (13)–(15) for decomposition of the integral equation in (16). We find that

$$LL^* = \int_c^d \left[\int_c^d (\cdot)\overline{g}(z, y)\,dz\right]g(x, y)\,dy = \int_c^d (\cdot)f(x, z)\,dz \tag{25}$$

where

$$f(x, z) = \int_c^d g(x, y)\overline{g}(z, y)\,dy. \tag{26}$$

We note that

$$\overline{f}(z, x) = f(x, z). \tag{27}$$

We may manipulate (14) to give

$$\begin{aligned}La &= \sum_{n=1}^{\infty} \lambda_n \left[\int_c^d a(z)\overline{v}_n(z)\,dz\right]u_n(x)\\ &= \int_c^d a(z)\left[\sum_{n=1}^{\infty} \lambda_n\overline{v}_n(z)u_n(x)\right]dz\\ &= \int_c^d a(z)g(x, z)\,dz\end{aligned} \tag{28}$$

where we identify the following expansion for the kernel $g(x, z)$:

$$g(x, z) = \sum_{n=1}^{\infty} \lambda_n\overline{v}_n(z)u_n(x). \tag{29}$$

Substitution of (29) into (23) and (26) and use of orthogonality gives

$$\begin{aligned}h(x, z) &= \int_c^d \left[\sum_{n=1}^{\infty} \lambda_n v_n(x)\overline{u}_n(y)\right]\left[\sum_{n=1}^{\infty} \lambda_n\overline{v}_n(z)u_n(y)\right]dy\\ &= \sum_{n=1}^{\infty} \lambda_n^2 v_n(x)\overline{v}_n(z)\end{aligned} \tag{30}$$

$$\begin{aligned}f(x, z) &= \int_c^d \left[\sum_{n=1}^{\infty} \lambda_n u_n(x)\overline{v}_n(y)\right]\left[\sum_{n=1}^{\infty} \lambda_n\overline{u}_n(z)v_n(y)\right]dy\\ &= \sum_{n=1}^{\infty} \lambda_n^2 u_n(x)\overline{u}_n(z).\end{aligned} \tag{31}$$

IV. Completeness Relationship

From the singular value decomposition theorem, we know that $a(x) \in \mathcal{L}_2(c, d)$ can be expanded in the complete expansion

$$\begin{aligned}a(x) &= \sum_{n=1}^{\infty} \gamma_n v_n + \sum_{m=1}^{M} \tilde{\gamma}_m\tilde{v}_m\\ &= \sum_{n=1}^{\infty} \langle a, v_n\rangle v_n + \sum_{m=1}^{M} \langle a, \tilde{v}_m\rangle\tilde{v}_m\end{aligned} \tag{32}$$

where the sequence $\{\tilde{v}_m\}$ consists of all the eigenfunctions in the eigenspace $\mathcal{N}_L$, typified by zero eigenvalue. If the null space is empty, the second summation in (32) vanishes. If there are a countably infinite number of eigenfunctions in $\mathcal{N}_L$, the index M runs to infinity. We may manipulate (32) to give

$$a(x) = \int_c^d a(y) \left[\sum_{n=1}^{\infty} \overline{v}_n(y) v_n(x) + \sum_{m=1}^{M} \overline{\tilde{v}}_m(y) \tilde{v}_m(x) \right] dy. \tag{33}$$

We recognize the expression in brackets under the integral sign as $\delta(x-y)$ and write

$$\delta(x-y) = \sum_{n=1}^{\infty} v_n(x) \overline{v}_n(y) + \sum_{m=1}^{M} \tilde{v}_m(x) \overline{\tilde{v}}_m(y). \tag{34}$$

Equation (34) is the *spectral representation of the delta function* [27] for the operator L. It is the *completeness relationship* for expansions in the Hilbert space. This specific representation of the delta function has an important application in analysis of the solution to (16) under the condition in this paper: the null space $\mathcal{N}_L$ is not empty.

V. Solution Properties

We consider some of the properties of the solution(s) to the nonself-adjoint integral equation described in the previous section. We begin by writing (13) as

$$a = \hat{a} + a_0, \qquad a_0 \in \mathcal{N}_L \tag{35}$$

where

$$\hat{a} = \sum_{n=1}^{\infty} \gamma_n v_n \tag{36}$$

$$\gamma_n = \langle a, v_n \rangle. \tag{37}$$

We operate from the left with L on both sides of (35) and substitute the result into (17) to obtain

$$La = L\hat{a} = b. \tag{38}$$

Substituting (14), we find that

$$\sum_{n=1}^{\infty} \lambda_n \gamma_n u_n = b. \tag{39}$$

In general, the set of eigenfunctions u_n does not constitute a complete set of basis functions that are sufficient to represent an arbitrary function. If the data function b cannot be expressed as a linear combination of the set of eigenfunctions u_n, as implied by (39), then the integral equation (16) is incompatible and does not have a solution. On the other hand, if the data function b can be expressed as a linear combination of the eigenfunctions u_n, then the integral equation (16) is compatible and has a solution (or solutions).

Taking the inner product from the right with u_m on both sides of (39), we obtain the classic solution for the coefficients γ_n, *viz.*

$$\gamma_n = \frac{1}{\lambda_n} \langle b, u_n \rangle. \tag{40}$$

Substitution of (40) into (36) and (36) into (35) yields

$$\hat{a} = \sum_{n=1}^{\infty} \frac{\langle b, u_n \rangle}{\lambda_n} v_n \tag{41}$$

and

$$a = \sum_{n=1}^{\infty} \frac{\langle b, u_n \rangle}{\lambda_n} v_n + a_0 \tag{42}$$

where by Picard's theorem [25], it is necessary that

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} |\langle b, u_n \rangle|^2 < \infty. \tag{43}$$

If the sum is finite, satisfaction of (43) is trivial. If the sum is countably infinite, the limit point of the λ_n at zero necessitates rapid fall-off of the magnitude of the Fourier coefficients $\langle b, u_n \rangle$. We emphasize that $\hat{a}$, the first term in (42), has been specifically and uniquely determined, whereas a_0, the portion of the solution in $\mathcal{N}_L$ remains unspecified. From (38), it is clear that our knowledge of the observables $b(x)$ in (16) does not tell us anything about the parts of $a(z)$ that belong to the null-space $\mathcal{N}_L$. Indeed, the addition of an arbitrary weighted sum of the components of the null space $\mathcal{N}_L$ to any solution of the integral equation (16), would still make the resulting function satisfy (16), and therefore these components must be recovered from information other than that contained in the data b. This recovery is the essential purpose of this paper.

Even though a_0 remains unspecified, we may show a "size" comparison between the two terms in (42) by calculating norms. The norm induced by the inner product is given as follows:

$$||a|| = \sqrt{\langle a, a \rangle} = \sqrt{\int_c^d |a|^2 \, dz}. \tag{44}$$

We calculate $||a||^2$ in (35) and obtain

$$||a||^2 = \langle a, a \rangle = ||\hat{a}||^2 + ||a_0||^2 \tag{45}$$

where we have used the fact that $a_0 \perp \hat{a}$ in eliminating cross-terms. We may show that $||\hat{a}||$ has an important special property. Indeed, from (45),

$$||\hat{a}|| \le ||a|| \tag{46}$$

with equality if and only if

$$\mathcal{N}_L = 0. \tag{47}$$

Because of the inequality in (46), $\hat{a}$ is the unique *minimum norm* solution to the integral equation in (16).

For use in the sequel, we also exhibit the solution to (17) by projection on a closed linear manifold $\mathcal{M}$ in Hilbert space $\mathcal{H}$. We seek an approximation $a_N \in \mathcal{M}$ to the solution $a \in \mathcal{H}$ by writing

$$a_N = \sum_{n=1}^{N} \gamma_n v_n. \tag{48}$$

Operating from the left with L, we obtain an estimate $\hat{b}$ of the data b, *viz.*

$$La_N = \hat{b} = b + e \tag{49}$$

where e is the data error. We seek to minimize $||e||$ given by

$$||e|| = ||b - La_N|| = \left\| b - \sum_{n=1}^{N} \gamma_n \lambda_n u_n \right\|. \qquad (50)$$

By the projection theorem, the unique minimizing vector is given by [28]

$$\langle b - La_N,\, u_m \rangle = 0, \qquad m = 1, 2, \cdots, N. \qquad (51)$$

Substituting (48) and using the orthnormality of the v_n gives

$$\gamma_n = \frac{\langle b,\, u_n \rangle}{\lambda_n} \qquad (52)$$

which is identical to the result in (40). Further, in the limit as $N \to \infty$, we recover the minimum norm solution $\hat{a}$. We emphasize that minimizing the data error allows us to obtain the minimum norm solution. This situation changes in later sections, where we consider the use of weighting functions.

VI. Solution Accuracy and Sensitivity

We seek measures for the accuracy of the solution(s) to the integral equation and the sensitivity to errors in the forcing function. We have

$$La = b \qquad (53)$$

and the estimate of the solution

$$\hat{a} = \sum_{n=1}^{\infty} \frac{\langle b,\, u_n \rangle}{\lambda_n} v_n. \qquad (54)$$

We manipulate (54) to give

$$\hat{a} = \int_c^d b(z) \sum_{n=1}^{\infty} \frac{v_n(x)\overline{u}_n(z)}{\lambda_n}\, dz = \int_b^c b(z) K(x, z)\, dz \qquad (55)$$

where

$$K(x, z) = \sum_{n=1}^{\infty} \frac{v_n(x)\overline{u}_n(z)}{\lambda_n}. \qquad (56)$$

We write (55) as

$$\hat{a} = L^{\dagger} b \qquad (57)$$

where

$$L^{\dagger} = \int_c^d (\cdot) K(x, z)\, dz. \qquad (58)$$

The operator $L^{\dagger}$ is a *generalized inverse* to the operator L. Indeed, it is the specific generalized inverse that returns minimum norm solution $\hat{a}$. Substitution of (53) into (57) gives

$$\hat{a} = L^{\dagger} L a = L_R a \qquad (59)$$

where L_R is the *model resolution operator* given by

$$L_R = L^{\dagger} L. \qquad (60)$$

The model resolution operator is a continuous version of the discrete model resolution matrix discussed in [29]. Specifically, using (18) and (58), we obtain

$$\begin{aligned} L_R &= \int_c^d \left[\int_c^d (\cdot) g(z, y)\, dy \right] K(x, z)\, dz \\ &= \int_c^d (\cdot) R(x, y)\, dy \end{aligned} \qquad (61)$$

where $R(x, y)$ is the *model resolution kernel* given by

$$R(x, y) = \int_c^d g(z, y) K(x, z)\, dz. \qquad (62)$$

The significance of the model resolution kernel can be dramatized by the substitution of (29) and (56), *viz.*

$$\begin{aligned} R(x, y) &= \int_c^d \left[\sum_{n=1}^{\infty} \lambda_n \overline{v}_n(y) u_n(z) \right] \left[\sum_{n=1}^{\infty} \frac{v_n(x)\overline{u}_n(z)}{\lambda_n} \right] dz \\ &= \sum_{n=1}^{\infty} v_n(x) \overline{v}_n(y) \\ &= \delta(x - y) - \sum_{n=1}^{M} \tilde{v}_n(x) \overline{\tilde{v}}_n(y) \end{aligned} \qquad (63)$$

where we have used (34). To appreciate the significance of model resolution, consider (59) with (61) substituted, *viz.*

$$\hat{a} = \int_c^d a(y) R(x, y)\, dy. \qquad (64)$$

The model resolution operator in (64) resolves the true solution a into the solution estimate $\hat{a}$. If the model resolution kernel in (63) consists solely of the delta function, the solution and the estimate coincide and the resolution is perfect. If, on the other hand, the model is imperfect, the model resolution kernel $R(x, y)$ contains, in addition to the delta function, a function with other than point support. This added term in (63) is caused by a nonempty null space $\mathcal{N}_L$. In this case, $R(x, y)$ produces a "smearing" away from the true solution a. How significant this smearing is depends on whether or not $R(x, y)$ is sharply peaked around $x = y$. In general, the less the peaking, the more perturbed the solution estimate. We note the important fact that the model resolution kernel $R(x, y)$ is independent of the forcing function b, which supplies the data to the integral equation. The study of $R(x, y)$ can therefore be undertaken without running any experiments or simulations with data. Indeed, $R(x, y)$ is a valuable tool in experiment design [30], the result of which is the "model" for the process we are studying. In this case, the "model" is the integral equation. Once a particular model is adopted, $R(x, y)$ is fixed. If the second term in (63) produces smearing in solutions beyond acceptable limits, the model should either be modified or discarded.

We next turn to a consideration of the data b. We solve, the integral equation and produce an estimate $\hat{a}$ of the solution. We then feed the solution estimate through the system and obtain an estimate of the data $\hat{b}$, *viz.*

$$\hat{b} = L\hat{a}. \qquad (65)$$

Substituting (57), we obtain

$$\hat{b} = LL^{\dagger}b = L_F b \tag{66}$$

where L_F is the *data fit operator* given by

$$L_F = LL^{\dagger}. \tag{67}$$

The data fit operator is a continuous version of the data resolution matrix discussed in [31]. Specifically, using (18) and (58), we obtain

$$\begin{aligned} L_F &= \int_c^d \left[\int_c^d (\cdot) K(x, y)\, dy \right] g(x, z)\, dz \\ &= \int_c^d (\cdot) F(x, y)\, dy \end{aligned} \tag{68}$$

where $F(x, y)$ is the *data fit kernel* given by

$$F(x, y) = \int_c^d K(z, y) g(x, z)\, dz. \tag{69}$$

Substituting (29) and (56), we obtain

$$\begin{aligned} F(x, y) &= \int_c^d \left[\sum_{n=1}^{\infty} \frac{v_n(z)\overline{u}_n(y)}{\lambda_n} \right] \left[\sum_{n=1}^{\infty} \lambda_n \overline{v}_n(z) u_n(x) \right] dz \\ &= \sum_{n=1}^{\infty} u_n(x)\overline{u}_n(y). \end{aligned} \tag{70}$$

Consider equation (15). Its solution consists of a sequence of eigenfunctions $\{u_n\}$ with nonzero eigenvalues, plus a sequence of eigenfunctions $\{\tilde{u}_n\}$ with zero eigenvalue. These eigenfunctions are subject to the following completeness relation:

$$\delta(x - y) = \sum_{n=1}^{\infty} u_n(x)\overline{u}_n(y) + \sum_{m=1}^{N} \tilde{u}_n(x)\overline{\tilde{u}}_n(y). \tag{71}$$

Using this relation in (70) gives

$$F(x, y) = \delta(x - y) - \sum_{m=1}^{N} \tilde{u}_n(x)\overline{\tilde{u}}_n(y). \tag{72}$$

To appreciate the significance of data fit, consider (66) with (68) substituted, *viz.*

$$\hat{b} = \int_c^d b(y) F(x, y)\, dy. \tag{73}$$

The data fit operator in (73) changes the observed data b into the data estimated by passing b through the model. Similar to the case of model resolution, if the data fit kernel contains only the delta function, the data transformation is perfect. Otherwise the model corrupts the data. Again, we note the important fact that the data fit matrix $F(x, y)$ is independent of the data and therefore is another important tool in experiment design [30].

We may form a useful relationship between input and output error by using correlation techniques. We assume errors $\Delta b(x)$ in the data that cause errors $\Delta a(x)$ in the output through the integral equation. We further assume $\Delta a(x)$ and $\Delta b(x)$ are stochastic processes over the deterministic variable x. We have

$$\Delta a(x) = \int_c^d \Delta b(z) K(x, z)\, dz. \tag{74}$$

Taking the expectation, we obtain

$$\begin{aligned} E\{\Delta a(x)\} &= E\left\{ \int_c^d \Delta b(z) K(x, z)\, dz \right\} \\ &= \int_c^d E\{\Delta b(z)\} K(x, z)\, dz. \end{aligned} \tag{75}$$

If we assume that the error in the noise has zero mean for all $x \in (c, d)$, then

$$E\{\Delta b(z)\} = 0 \Rightarrow E\{\Delta a(x)\} = 0. \tag{76}$$

Therefore, the autocovariance $C_{\Delta a}(x_1, x_2)$ is equal to the autocorrelation $R_{\Delta a}(x_1, x_2)$ and we have

$$\begin{aligned} C_{\Delta a}(x_1, x_2) &= E\{\Delta a(x_1)\overline{\Delta a}(x_2)\} \\ &= E\left\{ \int_c^d \Delta b(z) K(x_1, z)\, dz \right. \\ &\qquad \left. \times \overline{\int_c^d \Delta b(z) K(x_2, z)\, dz} \right\} \\ &= \int_c^d \int_c^d E\{\Delta b(z_1)\overline{\Delta b}(z_2)\} \\ &\qquad \times K(x_1, z_1)\overline{K}(x_2, z_2)\, dz_1\, dz_2 \\ &= \int_c^d \int_c^d C_{\Delta b}(z_1, z_2) \\ &\qquad \times K(x_1, z_1)\overline{K}(x_2, z_2)\, dz_1\, dz_2. \end{aligned} \tag{77}$$

We shall assume that Δb is a mean-zero white noise process so that [31]

$$C_{\Delta b}(z_1, z_2) = q(z_1)\delta(z_1 - z_2), \qquad q(t_1) \geq 0. \tag{78}$$

If, in addition, the process is assumed Gaussian, then $q(z_1) = \sigma_{\Delta b}^2$ and

$$C_{\Delta b}(z_1, z_2) = \sigma_{\Delta b}^2 \delta(z_1 - z_2). \tag{79}$$

Substituting into (77), we obtain

$$C_{\Delta a}(x_1, x_2) = \sigma_{\Delta b}^2 \int_c^d K(x_1, z)\overline{K}(x_2, z)\, dz. \tag{80}$$

If we substitute (56) into (80) and perform the indicated integration, we produce the following result:

$$C_{\Delta a}(x_1, x_2) = \sigma_{\Delta b}^2 \sum_{n=1}^{\infty} \frac{v_n(x_1)\overline{v}_n(x_2)}{\lambda_n^2}. \tag{81}$$

The reader may wish to compare this result with a similar result for discrete systems in [32]. In the great majority of problems of practical interest, the eigenvalues λ_n^2 are infinite in number. They are bounded from above by $||L^* L||$ and have an accumulation point at zero. The autocovariance result in (81) points up the possibility of serious error magnification, caused by the presence of the eigenvalues λ_n^2 in the denominator, especially the small eigenvalues.

It is obvious from (81) that the autocovariance becomes larger when λ_n is small. Thus, in order to limit the noise magnification to within a certain level, eigenfunctions with small eigenvalues have to be excluded. This, however, reduces the number of eigenfunctions that are used in the expansion of $\hat{a}$ [see (41)], degrading the resolution in both the minimum-norm solution and the recovered data [see (63) and (70), (72)]. An appropriate number of eigenfunctions can be selected by studying the trade-offs between the resolution (of both model and data) and the autocovariance of the solution due to noise in data. In order to limit the effects of this error magnification, one normally uses some form of regularization technique applied to the minimum norm solution $\hat{a}$ in (41). These techniques are well known [3]–[11] and are outside the scope of this paper.

VII. Solutions Using Weighting Functions

We shall be interested in the possibility of using weighting operators to weight certain characteristics in our solution. We again consider solutions to the linear operator equation

$$La = b. \tag{82}$$

We begin with some general concepts and then specialize to the first-kind integral equations under study. We adopt a Hilbert space $\mathcal{H}_W$ [33] with inner product

$$[f, g] = \langle Wf, g\rangle \tag{83}$$

and norm

$$\|f\|_W = \sqrt{[f, f]} = \sqrt{\langle Wf, f\rangle} \tag{84}$$

where W is called a *weighting operator*. To produce a legitimate inner product, W must be positive, *viz.*

$$\langle Wf, f\rangle \geq 0 \tag{85}$$

with equality if and only if $f = 0$. Under such conditions W is selfadjoint. We remark that

$$\mathcal{H} \subset \mathcal{H}_W. \tag{86}$$

Indeed, $\mathcal{H}$ is a particular subspace of $\mathcal{H}_W$ with weighting operator $W = I$, the identity operator. As an example, in first-kind integral equations, a typical weighting operator W_b is given by

$$W_b = \int_c^d (\cdot) w_b(x, z)\, dz \tag{87}$$

where

$$w_b(x, z) = \overline{w}_b(z, x) \tag{88}$$

and where $w_b(x, z)$ is positive.

We again seek an estimate a_N to the solution a, where

$$a_N = \sum_{n=1}^{N} \beta_n v_n \tag{89}$$

so that the solution is given with an error e, *viz.*

$$e = b - La_N. \tag{90}$$

We form

$$\|e\|_{W_b} = \|b - La_N\|_{W_b} = \left\| b - \sum_{n=1}^{N} \beta_n \lambda_n u_n \right\|_{W_b} \tag{91}$$

where we have used (14). We require $b \in \mathcal{H}_{W_b}$ and remark that $La_N \in \mathcal{M}_{W_b}$, a closed linear manifold in $\mathcal{H}_{W_b}$.

The weighting operator W_b can be determined by the data covariance which describes the estimated uncertainties in the available data set (due to noise contamination). It describes not only the estimated variance for each particular data point, but also the estimated correlation between the various data misfits. It also provides a point by point weighting of the input data according to a prescribed criterion. In the case when the measurement noise is stationary and uncorrelated, then

$$w_b(x, z) = \frac{1}{\sigma_b^2(x)}\, \delta(x - z) \tag{92}$$

where $\sigma_b(x)$ is the root mean square (rms) deviation of the noise.

By the projection theorem, the unique minimizing vector in (91) is given by

$$[b - La_N, u_m]_b = 0, \qquad m = 1, 2, \cdots, N \tag{93}$$

so that

$$\sum_{n=1}^{N} \beta_n \lambda_n [u_n, u_m]_b = [b, u_m]_b \tag{94}$$

where

$$[f, g]_b = \langle W_b f, g\rangle. \tag{95}$$

This relation can be written in matrix form as

$$U\mathbf{r} = \mathbf{s} \tag{96}$$

where U is the Gram matrix

$$U = \begin{bmatrix} [u_1, u_1]_b & [u_2, u_1]_b & \cdots & [u_N, u_1]_b \\ [u_1, u_2]_b & [u_2, u_2]_b & \cdots & [u_N, u_2]_b \\ \vdots & \vdots & \ddots & \vdots \\ [u_1, u_N]_b & [u_2, u_N]_b & \cdots & [u_N, u_N]_b \end{bmatrix} \tag{97}$$

and where

$$\mathbf{r} = \begin{bmatrix} \lambda_1\beta_1 \\ \lambda_2\beta_2 \\ \vdots \\ \lambda_N\beta_N \end{bmatrix} \tag{98}$$

and

$$\mathbf{s} = \begin{bmatrix} [b, u_1]_b \\ [b, u_2]_b \\ \vdots \\ [b, u_N]_b \end{bmatrix}. \tag{99}$$

Explicitly, if c_{ij} are the elements of U^{-1}, then

$$\beta_n = \lambda_n^{-1} \sum_{m=1}^{N} c_{nm}[b, u_m]_b \tag{100}$$

and, in (89)

$$a_N = \sum_{n=1}^{N} \lambda_n^{-1} \sum_{m=1}^{N} c_{nm}[b, u_m]_b v_n. \tag{101}$$

Note that although the u_n are orthonormal on $\mathcal{H}$, they do not have this property on $\mathcal{H}_{W_b}$. In addition, note that the coefficients recovered by solving (94) produce a solution in $\mathcal{H}_{W_b}$, which is different from the one in $\mathcal{H}$. By "solution" we mean the following:

$$\begin{aligned}\tilde{a} &= \lim_{N\to\infty} a_N = \lim_{N\to\infty} \sum_{n=1}^{N} \beta_n v_n \\ &= \sum_{n=1}^{\infty} \lambda_n^{-1} \sum_{m=1}^{\infty} c_{nm}[b, u_m]_b v_n. \end{aligned}\tag{102}$$

This solution, in general, is distinct from that in (41). However, the minimum norm result in $\mathcal{H}$ is contained in $\mathcal{H}_{W_b}$ as a special case. Indeed, when $W_b = I$, $\mathcal{H}_{W_b} \to \mathcal{H}$ and the inner product $[f, g]_b$ reverts to the usual inner product $\langle f, g\rangle$. The u_n are orthonormal on this space, U diagonalizes into the identity matrix, and (100) gives the expected result, *viz.*

$$\beta_n = \frac{\langle b, u_n\rangle}{\lambda_n} = \gamma_n \tag{103}$$

which is identical to the result in (40).

VIII. Solution Using Weighted Singular System

In the previous section, we have discussed the weighting operator W_b. In experiment design, it can be used to weight certain characteristics of the data. In this section, we force the weighting of the data on Hilbert space $\mathcal{H}$ by premultiplying the original integral equation in (17) by a weighting operator W_b, *viz.*

$$W_b L a = W_b b. \tag{104}$$

We define the weighted data by

$$b_W = W_b b \tag{105}$$

and a new weighted operator L_W by

$$L_W = W_b L \tag{106}$$

and obtain the operator equation

$$L_W a = b_W. \tag{107}$$

We note in (107) that the new operator equation contains weighted data b_W as a forcing function. From previous considerations, there exists a singular system r_n, s_n, λ_n^W, where the r_n and s_n are each orthonormal and where

$$L_W^* L_W s_n = (\lambda_n^W)^2 s_n \tag{108}$$

$$L_W L_W^* r_n = (\lambda_n^W)^2 r_n \tag{109}$$

$$L_W s_n = \lambda_n^W r_n \tag{110}$$

$$L_W^* r_n = \lambda_n^W s_n. \tag{111}$$

By the singular value decomposition theorem, the solution to (107) is given by

$$a = \hat{a}_W + a_0^W, \qquad a_0^W \in \mathcal{N}_{L_W} \tag{112}$$

where

$$\hat{a}_W = \sum_{n=1}^{\infty} \gamma_n^W s_n. \tag{113}$$

As before, we evaluate the coefficients γ_n^W by seeking the least squares solution. Let

$$\hat{a}_N^W = \sum_{n=1}^{N} \gamma_n^W s_n. \tag{114}$$

Operating from the left with L_W, we obtain an estimate $\hat{b}_W$ of b_W, *viz.*

$$L_W \hat{a}_N^W = \hat{b}_W = b_W + e_W. \tag{115}$$

We wish to minimize the norm of e_W given by

$$\|e_W\| = \|b_W - L_W \hat{a}_N^W\| = \left\| b_W - \sum_{n=1}^{N} \gamma_n^W \lambda_n^W r_n \right\|. \tag{116}$$

This minimization is given by the following projection:

$$\langle b_W - L_W \hat{a}_N^W, r_n\rangle = 0. \tag{117}$$

Solving, we obtain the expected result

$$\hat{a}_N^W = \sum_{n=1}^{N} \frac{\langle b_W, r_n\rangle}{\lambda_n^W} s_n \tag{118}$$

and in the limit

$$\hat{a}_W = \sum_{n=1}^{\infty} \frac{\langle b_W, r_n\rangle}{\lambda_n^W} s_n. \tag{119}$$

We may make these results specific to the first-kind integral equation in (16) as follows. Let the weighting operator W_b be given by (87) and (88), *viz.*

$$W_b = \int_c^d (\cdot) w_b(x, z)\, dz \tag{120}$$

$$w_b(x, z) = \overline{w}_b(z, x). \tag{121}$$

Then

$$\begin{aligned} W_b L a &= \int_c^d w_b(x, y) \int_c^d a(z) g(y, z)\, dz\, dy \\ &= \int_c^d a(z) \int_c^d w_b(x, y) g(y, z)\, dy\, dz \\ &= \int_c^d a(z) g_w(x, z)\, dz = W_b b \end{aligned}\tag{122}$$

where

$$g_w(x, z) = \int_c^d w_b(x, y) g(y, z)\, dy \tag{123}$$

and

$$W_b b = \int_c^d b(z) w_b(x, z)\, dz. \tag{124}$$

IX. A Solution Including Null-Space Vectors

We now come to the crucial point in our development where we seek a solution in the Hilbert space $\mathcal{H}_{W_b}$ that includes the null-space vectors; that is, we seek a_W such that

$$a_W = \hat{a} + a_0 \tag{125}$$

where $a_0 \in \mathcal{N}_L$ is given by

$$a_0 = \sum_{n=1}^{M} \tilde{\gamma}_n \tilde{v}_n \tag{126}$$

and where we recall that $\hat{a}$ is minimum norm on $\mathcal{H}$. We minimize a_W by projection, *viz.*

$$[(\hat{a} + a_0), \tilde{v}_m]_a = 0, \qquad m = 1, 2, \cdots, M \tag{127}$$

or

$$\sum_{n=1}^{M} \tilde{\gamma}_n [\tilde{v}_n, \tilde{v}_m]_a = -[\hat{a}, \tilde{v}_m]_a \tag{128}$$

where

$$[f, g]_a = \langle W_a f, g \rangle \tag{129}$$

W_a can be determined by the model covariance representing the degree of confidence in the model $a(x)$. As we have previously pointed out, such considerations are independent of the data. Therefore, W_a is an important tool in *a priori* experiment design, where it provides a point by point weighting of the model according to a prescribed criterion. Substituting (36) and (40), we obtain

$$\sum_{n=1}^{M} \tilde{\gamma}_n [\tilde{v}_n, \tilde{v}_m]_a = -\sum_{n=1}^{\infty} \lambda_n^{-1} \langle b, u_n \rangle [v_n, \tilde{v}_m]_a. \tag{130}$$

This matrix equation can be inverted to produce the coefficients $\tilde{\gamma}_n$. These coefficients can then be substituted into (126) to complete the solution in (125). We note that this solution contains the elements of the null space $\mathcal{N}_L$. Explicitly, we write the inversion to (130) in matrix form, *viz.*

$$\boldsymbol{\gamma} = A^{-1}\mathbf{b} = B\mathbf{b} \tag{131}$$

or

$$\begin{bmatrix} \tilde{\gamma}_1 \\ \tilde{\gamma}_2 \\ \vdots \\ \tilde{\gamma}_M \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1M} \\ B_{21} & B_{22} & \cdots & B_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ B_{M1} & B_{M2} & \cdots & B_{MM} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_M \end{bmatrix} \tag{132}$$

where B is inverse to the $M \times M$ A matrix, whose elements are $[\tilde{v}_n, \tilde{v}_m]_a$. We may write the mth column explicitly as

$$\tilde{\gamma}_m = \sum_{\ell=1}^{M} B_{m\ell} b_\ell \tag{133}$$

where

$$b_\ell = -\sum_{n=1}^{\infty} \lambda_n^{-1} \langle b, u_n \rangle [v_n, \tilde{v}_\ell]_a. \tag{134}$$

Substituting (134) into (133) and (133) into (126), we obtain

$$\begin{aligned} a_0 &= -\sum_{m=1}^{M} \sum_{\ell=1}^{M} B_{m\ell} \sum_{n=1}^{\infty} \lambda_n^{-1} \langle b, u_n \rangle [v_n, \tilde{v}_\ell]_a \tilde{v}_m \\ &= \sum_{n=1}^{\infty} q_n \lambda_n^{-1} \langle b, u_n \rangle \end{aligned} \tag{135}$$

where

$$q_n = \sum_{m=1}^{M} \alpha_{mn} \tilde{v}_m \tag{136}$$

$$\alpha_{mn} = -\sum_{\ell=1}^{M} B_{m\ell} [v_n, \tilde{v}_\ell]_a. \tag{137}$$

Substituting (135) and (41) into (125), we have our principal result, *viz.*

$$a_W = \sum_{n=1}^{\infty} \lambda_n^{-1} \langle b, u_n \rangle (v_n + q_n). \tag{138}$$

We may interpret the solution a_W as follows. From (125), we write

$$\hat{a} = a_W - a_0. \tag{139}$$

By the projection theorem (127), we determine the "best" a_0 and produce a_W such that the vectors $-a_0$ and a_W are orthogonal. Taking the weighted norm in (139), we produce

$$||\hat{a}||_W^2 = ||a_W||_W^2 + ||a_0||_W^2 \tag{140}$$

from which we conclude that

$$||a_W||_W^2 \ \leq \ ||\hat{a}||_W^2. \tag{141}$$

We reach two conclusions as follows.

1) We have produced a solution containing the null-space vectors.
2) The weighted norm of this solution is less than the weighted norm of our previous solution $\hat{a}$.

We note that if we were to consider a sequence of projection problems, where the weighting operator W approaches unity, the two vectors $\hat{a}$ and $-a_0$ must approach orthogonality, a_0 must approach zero and the solution a_W approaches the minimum norm solution $\hat{a}$ required in the unweighted Hilbert space.

We next consider the model resolution operator on the weighted Hilbert space. Writing the inner product in (138) in integral form, we produce

$$a_W(x) = \int_c^d b(z) K_W(x, z)\, dz \tag{142}$$

where

$$\begin{aligned} K_W(x, z) &= \sum_{n=1}^{\infty} \gamma_n^{-1}[v_n(x) + q_n(x)]\overline{u}_n(z) \\ &= \sum_{n=1}^{\infty} \gamma_n^{-1}\overline{u}_n(z) v_n(x) \\ &\quad + \sum_{m=1}^{M} \left[\sum_{n=1}^{\infty} \alpha_{mn}\lambda_n^{-1}\overline{u}_n(z)\right] \tilde{v}_m(x). \end{aligned} \tag{143}$$

The reader should compare this result with that in (55) and (56) for the unweighted Hilbert space. Paralleling the development there, we write (142) as

$$a_W = L_W^{\dagger} b \tag{144}$$

where

$$L_W^{\dagger} = \int_c^d (\cdot) K_W(x, z)\, dz. \tag{145}$$

We note that $L_W^{\dagger}$ is the specific generalized inverse that returns the weighted norm solution a_W. Substitution of (53) into (144) gives

$$a_W = L_W^{\dagger} L a = L_W^R a \tag{146}$$

where L_W^R is the weighted model resolution operator. Paralleling the development in (61) and (62), this operator is given by

$$L_W^R = \int_c^d (\cdot) R_W(x, y)\, dy \tag{147}$$

where R_W is the weighted model resolution kernel given by

$$R_W(x, y) = \int_c^d g(z, y) K_W(x, z)\, dz. \tag{148}$$

Similar to the development in (63), we may expand the kernel and the Green's function in (148) to give

$$\begin{aligned} R(x, y) &= \int_c^d \left[\sum_{n=1}^{\infty} \lambda_n \overline{v}_n(y) u_n(z)\right] \\ &\quad \cdot \left[\sum_{n=1}^{\infty} \lambda_n^{-1}\overline{u}_n(z) v_n(x) \right.\\ &\quad \left. + \sum_{m=1}^{M}\sum_{n=1}^{\infty} \alpha_{mn}\lambda_n^{-1}\overline{u}_n(z)\tilde{v}_m(x)\right] dz \\ &= \sum_{n=1}^{\infty} \overline{v}_n(y) v_n(x) + \sum_{n=1}^{\infty} \overline{v}_n(y) \sum_{m=1}^{M} \alpha_{mn}\tilde{v}_m(x) \\ &= \delta(x-y) - \sum_{n=1}^{M} \tilde{v}_n(x)\overline{\tilde{v}}_n(y) \\ &\quad + \sum_{n=1}^{\infty} \overline{v}_n(y) \sum_{m=1}^{M} \alpha_{mn}\tilde{v}_m(x) \\ &= \delta(x-y) - \sum_{m=1}^{M} \left[\overline{\tilde{v}}_m(y) - \sum_{n=1}^{\infty} \alpha_{mn}\overline{v}_n(y)\right] \tilde{v}_m(x). \end{aligned} \tag{149}$$

This result gives us a concrete expression for model resolution. To obtain the "best" model resolution, we may minimize a weighted norm of the function $d(y)$, where

$$d(y) = \overline{\tilde{v}}_m(y) - \sum_{n=1}^{\infty} \alpha_{mn}\overline{v}_n(y). \tag{150}$$

This minimization depends on the coefficients α_{mn} which, from (137) and the elements of the A matrix, are controlled by the choice of the weighting function W_a.

X. Conclusion

One of the most intriguing problems in inverse scattering theory concerns methods for reducing the size of the null space of the relevant operator. Such considerations involve somehow including in the total solution eigenfunctions with zero eigenvalue. In this paper, we have presented a theory whereby such eigenfunctions are included by transferring the minimum norm solution obtained in one Hilbert space into another Hilbert space containing a weighting operator. In this new Hilbert space, we have been able, in principle, to append solutions from the null space to the minimum norm solution. Our principal result is given in (138), with model resolution given by (149).

The ultimate success of the method must await trials by example. On our part, we plan to produce examples where we shall select weighting operators to take into account additional information based on the problem physics.

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Dr. Wolf is the recipient of numerous awards for his scientific contributions and is an honorary member of the Optical Society of America, of which he was the President in 1978. He is also an honorary member of the Optical Societies of India and Australia and is the recipient of seven honorary degrees from Universities in The Netherlands, Great Britain, the Czech Republic, Canada, Denmark, and France.