Nonradiating and Minimum Energy Sources and Their Fields: Generalized Source Inversion Theory and Applications

Edwin A. Marengo, *Member, IEEE,* and Richard W. Ziolkowski, *Fellow, IEEE*

***Abstract*—A new general framework for characterizing scalar and electromagnetic (EM) nonradiating (NR) and minimum energy (ME) sources and their fields is developed that is of interest for both radiation and source reconstruction problems. NR sources are characterized in connection with the concept of reciprocity as nonreceptors. Localized ME sources are shown to be free fields truncated within the source's support. A new source analysis tool is developed that is based on the decomposition of a source and its field into their radiating and NR components. The individual radiating and reactive energy roles of the radiating and NR parts of a source are characterized. The general theory is illustrated with a time-harmonic EM example.**

***Index Terms*—Inverse problems.**

I. INTRODUCTION

THIS paper characterizes, in a unifying general theoretical framework, new and old results on nonradiating (NR) and minimum energy (ME) sources [1]–[3] and their fields, with applications to source analysis and reconstruction. We present this framework in the context of the inverse source problem (ISP) in which one seeks to reconstruct an unknown source from knowledge of its radiated field outside the source's support [2], [4]–[14]. The general NR and ME source-field descriptions derived in this paper apply to any source-field system (scalar or electromagnetic (EM), time-harmonic, or transient) in an arbitrary linear medium (homogeneous or inhomogeneous) and subjected to arbitrary boundary conditions. We also investigate the unique decomposition of a source and its field into their radiating and NR parts and the associated power budget.

In this paper, we build upon the lines of [13], [14], and formulate the ISP in a general linear operator framework in Hilbert space. The paper aims to develop a new physical picture of the ISP and, in particular, of both NR and ME sources and their fields rather than to address the associated computational aspects. The general source inversion and NR and ME source results presented in this work are relevant to antenna analysis and synthesis and to inverse scattering [15]. We also develop a new source analysis tool based on the decomposition of a source and its field into their radiating and NR components. The purpose is to isolate the field and energy roles of the radiating and NR parts of a source. In fact, a portion of this research was motivated by the question, "Is the NR component of a source spurious? Or, on the contrary, can it be useful to the overall performance of an antenna?" A question of much interest is whether a source can be modified by adding NR sources to it so as to minimize its reactive power.

In Section II, we formulate the general theory in an n-dimensional spatial or spatial–temporal coordinate space. We establish a new characterization of a NR source in terms of its interaction with fields produced outside the NR source's support. The new characterization relies physically on the concept of reciprocity and also leads to a number of new results on ME sources and their fields. We show that ME sources must be free-fields truncated within the source's support. Some of our results are generalizations to arbitrary sources and fields of results derived before, for special cases, by Friedlander [16], Kim and Wolf [17], and others [14], [18]. In Section III, we investigate the unique decomposition of a source and its field into their radiating and NR parts. A new form of power budget analysis is carried out which isolates the radiating and reactive energy roles of the radiating and NR source components. The general source-field decomposition and power budget analysis is illustrated for a time-harmonic current distribution in a one-dimensional (1-D) free-space. The 1-D results apply to transmission lines and also illustrate many of the general results of Section II. Section IV provides some concluding remarks.

II. GENERAL THEORY

A. Review of the General Linear Operator Formulation

Consider a general complex-valued scalar or vector source-field system (ρ, ψ) described by a linear scalar or vector partial differential equation (PDE)

$$(L\psi)(\mathbf{x}) = \rho(\mathbf{x}) \tag{1}$$

where L is a linear, scalar or dyadic partial differential (PD) operator, and ψ is the scalar or vector field produced by a scalar or

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E. A. Marengo is with the Department of Electrical and Computer Engineering, University of Arizona, Tucson, AZ 85721 USA (e-mail: emarengo@ece.arizona.edu). He is also with the Department of Electrical Engineering, Technological University of Panama, Panama City, Panama.

R. W. Ziolkowski is with the Department of Electrical and Computer Engineering, University of Arizona, Tucson, AZ 85721 USA.

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vector source distribution ρ of support D' in an n-dimensional $\mathbf{x}$ space where $\mathbf{x} \in \Re^n$ denotes the space or space–time coordinates of the problem at hand. Table I lists the relevant space or space-time coordinates ($\mathbf{r}$ or $(\mathbf{r}, t)$, respectively), field, source, and second-order PD operator associated with source-field systems of interest for EM and acoustic applications.

We formulate the inverse problem of finding an unknown source ρ, known *a priori* to be nonzero only within a certain space or space–time region D' (the source region) from knowledge of the field ψ in a certain field region D disjoint to D'. To address this problem on physically reasonable grounds, we define the Hilbert space X of square-integrable (L_2) sources ρ of support D' (the solution space) and the Hilbert space Y of L_2 data fields $\bar{\psi}$ that are nonzero only within D (the data space) and assign to them the respective inner products

$$\langle \rho \,|\, \rho' \rangle_X = \int_{D'} d^n x \, \rho^*(\mathbf{x}) \rho'(\mathbf{x}) \tag{2}$$

and

$$\langle \bar{\psi} \,|\, \bar{\psi}' \rangle_Y = \int_{D} d^n x \, \bar{\psi}^*(\mathbf{x}) \bar{\psi}'(\mathbf{x}) \tag{3}$$

where * denotes the complex conjugate. The data field $\bar{\psi}$ is related to ρ via

$$\bar{\psi}(\mathbf{x}) = (P\rho)(\mathbf{x}) = M_f(\mathbf{x}) \int_{D'} d^n x' \, \rho(\mathbf{x}') G(\mathbf{x}, \mathbf{x}') \tag{4}$$

where G is the scalar or dyadic Green function associated with the PD operator L and the given boundary conditions, whereas M_f is the field-masking function defined by

$$M_f(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in D \\ 0 & \text{else.} \end{cases} \tag{5}$$

We shall refer to the linear mapping $P: X \to Y$ defined by (4) as "the propagator."

The inverse problem in (4) admits a solution only if the data field $\bar{\psi}$ is in the range $R(P) = \{\bar{\psi} \in Y \,|\, \bar{\psi} = P\rho, \rho \in X\}$ of P as defined by the Picard conditions [13]. This problem does not, in general, admit a unique solution due to the possible presence of nontrivial sources ρ_N (to be referred to as "invisible sources" [19]) in the null space $N(P) = \{\rho \in X \,|\, P\rho = 0\}$ of P. Any L_2 NR source $\rho_{\rm NR}$ of support D' whose field vanishes entirely outside D' is classifiable as invisible. However, the converse is not always true since, depending on the problem considered, there might be sources generating vanishing fields in the relevant observation region D while generating nonvanishing fields outside D. An example arises in connection with the ISP with discrete far-field data in which the far fields are specified for certain discrete observation directions. In general, there might be sources $\rho_N \in N(P)$ generating nulls in the radiation pattern at those particular directions while generating nontrivial far fields at other directions.

As long as a solution to the ISP exists, one can always make it unique by enforcing the additional constraint of minimizing the source's L_2 norm $(\langle \rho \,|\, \rho \rangle_X)^{1/2}$. The solution in question is the usual ME solution $\rho_{\rm ME}$ [2], [9], and is given by the pseudo-inverse of P [13]

$$\rho_{\rm ME}(\mathbf{x}) = [P^\dagger (PP^\dagger)^{-1} \bar{\psi}](\mathbf{x}) \tag{6}$$

TABLE I
RELEVANT SPACE OR SPACE–TIME COORDINATES ($\mathbf{r}$ OR $(\mathbf{r}, t)$, RESPECTIVELY), FIELD, SOURCE, AND PD OPERATOR ASSOCIATED WITH SOURCE-FIELD SYSTEMS OF INTEREST FOR ELECTROMAGNETIC AND ACOUSTIC APPLICATIONS

Coordinates ($\mathbf{x}$)	Field (ψ)	Source (ρ)	PD Operator (L)
$(\mathbf{r}, t)$	$\psi(\mathbf{r}, t)$	$\rho(\mathbf{r}, t)$	$\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$
$\mathbf{r}$	$\psi(\mathbf{r})$	$\rho(\mathbf{r})$	$\nabla^2 + k^2$
$\mathbf{r}$	$\mathbf{E}(\mathbf{r})$	$\mathbf{J}(\mathbf{r})$	$(\nabla\times\nabla\times - k^2)/(i\omega\mu)$

where $P^\dagger$ is the adjoint of P, defined by

$$\langle \rho \,|\, P^\dagger \bar{\psi} \rangle_X = \langle P\rho \,|\, \bar{\psi} \rangle_Y. \tag{7}$$

We find by using (2)–(4) and (7) that

$$\begin{aligned}(P^\dagger \bar{\psi})(\mathbf{x}) &= M_s(\mathbf{x}) \int_D d^n x' \, \bar{\psi}(\mathbf{x}') G^*(\mathbf{x}', \mathbf{x}) \\ &= M_s(\mathbf{x}) \int_D d^n x' \, \bar{\psi}(\mathbf{x}') \tilde{G}^*(\mathbf{x}, \mathbf{x}')\end{aligned} \tag{8}$$

where $\tilde{G}$ is the Green function of the adjoint problem associated with (1) (see, e.g., [20, p. 870] and [21, ch. 9]) whereas M_s is the source masking function defined by

$$M_s(\mathbf{x}) = \begin{cases} 1, & \text{if } \mathbf{x} \in D' \\ 0, & \text{else.} \end{cases} \tag{9}$$

The second of (8) follows from the reciprocity condition $\tilde{G}(\mathbf{x}, \mathbf{x}') = G(\mathbf{x}', \mathbf{x})$ (see, e.g., [20, p. 883]). The linear mapping $P^\dagger: Y \to X$ backpropagates the data field into the source region whereas $P^\dagger P: X \to X$ and $PP^\dagger: Y \to Y$. Finally, to conclude this review, if the data field is not physically realizable from sources $\rho \in X$ (i.e., $\bar{\psi} \notin R(P)$), then one can seek approximate solutions, such as the usual least squares solution of minimum L_2 norm in the orthogonal complement $N(P)^\perp$ of $N(P)$ [22].

B. NR and ME Sources and Their Fields

Next, we derive a number of previously unknown fundamental results on NR and ME sources and their fields of interest for both direct and inverse problems. We also show how some of our general results lead to a number of previously known results corresponding to special cases. We focus primarily on L_2 localized sources, although some of the more general results can be obtained by treating the sources in the sense of distributions. In fact, we first consider general localized sources; later, we specialize our more general results to L_2, localized sources.

1) A New Characterization of a NR Source: We let $\psi_{\rm NR}$ be an arbitrary scalar or vector field that is completely contained in a certain bounded region D' (so that $\psi_{\rm NR}(\mathbf{x}) = 0$ if $\mathbf{x} \notin D'$). It follows immediately that $\rho_{\rm NR} = L\psi_{\rm NR}$ is a (perhaps distributional) NR source of support D' such that its generated field [1], [16]

$$\psi_{\rm NR}(\mathbf{x}) = \int_{D'} d^n x' \, \rho_{\rm NR}(\mathbf{x}') G(\mathbf{x}, \mathbf{x}') = 0 \quad \text{if } \mathbf{x} \notin D'. \quad (10)$$

Moreover, it has been known for some time that any NR source of support D' can be written in that form. Now, since $\psi_{\rm NR}$ resides entirely within the NR source region D', it then follows from the generalized Green theorem (see, e.g., [20, p. 870–877]) that

$$\int_{D'} d^n x v^*(\mathbf{x}) \rho_{\rm NR}(\mathbf{x}) = \int_{D'} d^n x v^*(\mathbf{x}) (L\psi_{\rm NR})(\mathbf{x}) = \int_{D'} d^n x (\tilde{L} v^*)(\mathbf{x}) \psi_{\rm NR}(\mathbf{x}) \quad (11)$$

for any function v, where $\tilde{L}$ is the adjoint of the PD operator L (as defined, e.g., in [20, ch. 7] and [21, ch. 9]). The last integral in (11) is seen to vanish if $(\tilde{L} v^*)(\mathbf{x}) = 0$ (or, equivalently, $(\tilde{L} v)(\mathbf{x}) = 0$) for $\mathbf{x} \in D'$. We have thus found NR sources of support D' to be orthogonal to all solutions (within the entire source region D', its boundary $\partial D'$ included) of the homogeneous form of the adjoint PDE of the corresponding source-field system. Thus, if $\rho_{\rm NR}$ is a NR source of support D' and

$$(\tilde{L} v)(\mathbf{x}) = 0 \quad \text{if } \mathbf{x} \in D' \quad (12)$$

then

$$\int_{D'} d^n x v^*(\mathbf{x}) \rho_{\rm NR}(\mathbf{x}) = 0. \quad (13)$$

If the PD operator L is *formally* self-adjoint so that $\tilde{L} = L$ (see, e.g., [23, p. 154]), as is the case for all the source-field systems listed in Table I, then (12) takes the special form

$$(Lv)(\mathbf{x}) = 0 \quad \text{if } \mathbf{x} \in D'. \quad (14)$$

Kim and Wolf [17] arrived, by means of a different procedure, at the orthogonality relation (13)–(14) for continuous scalar NR sources in the context of the inhomogeneous Helmholtz equation (where $L = \nabla^2 + k^2$). Here, we have put forth that result in a more general context (12)–(14), valid for any source-field system.

In addition, we can use (12)–(13) to generalize a result previously derived by Friedlander [16] in the context of the inhomogeneous scalar wave equation (where $L = \nabla^2 - c^{-2}\partial^2/\partial t^2$). By noting that $\tilde{G}^*(\mathbf{x}', \mathbf{x})$ obeys $(\tilde{L}\tilde{G}^*)(\mathbf{x}', \mathbf{x}) = 0$ if $\mathbf{x}' \notin D'$ and $\mathbf{x} \in D'$, one concludes from (12) and (13) that

$$\int_{D'} d^n x \rho_{\rm NR}(\mathbf{x}) \tilde{G}(\mathbf{x}', \mathbf{x}) = 0 \quad \text{if } \mathbf{x}' \notin D'.$$

For example, G and $\tilde{G}$ can be a retarded and an advanced Green function, respectively. One also deduces that if a source is NR relative to a retarded Green function, then it must also be NR relative to the corresponding advanced one and vice versa. This generalizes Friedlander's result [16, theorem 3.2] to any source-field system. This includes systems where L is not formally self adjoint.

The general orthogonality relation (12)–(13) or its formally self-adjoint PD operator version (13)–(14) can be shown to be both necessary and sufficient for a source of support D' to be NR. For the usual scalar and EM sources listed in Table I, this can be stated as follows: *A scalar or EM source of support D' is NR if and only if it obeys the orthogonality condition* (13) *with respect to all solutions of* (14). That this condition is necessary is precisely the statement made in connection with (12)–(14). To show sufficiency, we note that $G^*(\mathbf{x}', \mathbf{x})$ obeys $(LG^*)(\mathbf{x}', \mathbf{x}) = 0$ if $\mathbf{x}' \notin D'$ and $\mathbf{x} \in D'$ so that from (13)–(14)

$$\psi_{\rm NR}(\mathbf{x}') = \int_{D'} d^n x \rho_{\rm NR}(\mathbf{x}) G(\mathbf{x}', \mathbf{x}) = 0 \quad \text{if } \mathbf{x}' \notin D'$$

which is precisely the NR condition (10). The orthogonality condition (13) and (14) thus provides a new characterization of a scalar or EM NR source. An alternative proof of this result is given in the Appendix.

Next, we specialize the new NR source results (12)–(14) to L_2 NR sources of support D'. In particular, we show that the conditions contained in (12) and (14) can be put in a slightly different but less restrictive form if the NR sources considered are known to be not only localized but also L_2. To accomplish this goal, we note that if L is a second-order PD operator, then the vanishing for $\mathbf{x} \notin D'$ of the NR field $\psi_{\rm NR}$ associated with the NR source $\rho_{\rm NR} = L\psi_{\rm NR}$ automatically forces both $\psi_{\rm NR}$ and its normal derivatives to obey homogeneous boundary conditions on the boundary $\partial D'$ that bounds D' so long as the NR source lacks single-layer and higher order singularities (as defined, e.g., in [24, ch. 1]) on $\partial D'$, as is, in fact, the case, for $L_2(D')$ NR sources. The above-stated NR boundary conditions were derived first by Gamliel *et al.* [25] for bounded scalar sources to the inhomogeneous three-dimensional (3-D) Helmholtz equation and rederived later by Berry *et al.* [26] for the 1-D case. These NR boundary conditions have also appeared in two recent papers dealing with scalar NR sources and their fields [27], [28]. Now, the vanishing of $\psi_{\rm NR}$ on the boundary $\partial D'$ of D' enables us to relax the previous results which apply to any NR source confined within D'. In particular, when dealing with $L_2(D')$ sources, we need not require (12) and (14) to hold within the entire source region D' (its boundary $\partial D'$ included), but instead we require them to hold just in the interior of D', its boundary $\partial D'$ possibly excluded. Whether (12) and (14) hold or not on $\partial D'$ is then inconsequential, as can be deduced from (11), because of the above-stated guaranteed vanishing of $\psi_{\rm NR}$ on $\partial D'$. We have then arrived at the following L_2 version of (13) and (14). Let $v \in X \equiv L_2(D')$ be an homogeneous field solution in the interior of D' so that $(Lv)(\mathbf{x}) = 0$ if $\mathbf{x} \in D'$ (the boundary $\partial D'$ of D' excluded). Then, $\rho_{\rm NR} \in X$ is an L_2, scalar or EM NR source of support D', if and only if

$$\langle v \,|\, \rho_{\rm NR} \rangle_X = 0 \quad (15)$$

for all $v \in X$. The result (15) is consistent with the orthogonality relation $\langle \bar{\psi} \,|\, P\rho_{\rm NR} \rangle_Y = \langle P^\dagger \bar{\psi} \,|\, \rho_{\rm NR} \rangle_X = 0$ since scalar or EM backpropagated fields $P^\dagger \bar{\psi}$ produced outside the NR source's support D' must obey $(LP^\dagger \bar{\psi})(\mathbf{x}) = 0$, at least within the interior of D', as can be shown using (8) and (9). Because of the truncation, within D', of $(P^\dagger \bar{\psi})(\mathbf{x}), (LP^\dagger \bar{\psi})(\mathbf{x})$ will contain, in general, single- and/or double-layer singularities on $\partial D'$, which in no way affect the validity of (15).

2) The Wave Nature of ME Sources and Their Fields: L_2 ME sources $\rho_{\rm ME} \in X \equiv L_2(D')$ of support D' are orthogonal to all NR sources $\rho_{\rm NR} \in X$. It thus follows from (11) that any ME source $\rho_{\rm ME} \in X$ must obey, within the interior of the source region D', its boundary $\partial D'$ excluded, the homogeneous form of the adjoint PDE of the corresponding source-field system (since this and only this ensures the vanishing of the orthogonality integral in (11) with $v = \rho_{\rm ME}$ for arbitrary $\psi_{\rm NR}$). One reaches the same conclusion from (6), (8), and (9). Thus, for a formally self-adjoint PD operator L

$$(L\rho_{\rm ME})(\mathbf{x}) = 0 \quad \text{if } \mathbf{x} \in D' \text{ (excluding } \partial D'). \qquad (16)$$

ME sources are therefore free-fields truncated within the source's support. By referring to Table I, one finds the time-harmonic EM version of this previously unknown general result to be (see Fig. 1)

$$\nabla \times \nabla \times \mathbf{J}_{\rm ME}(\mathbf{r}) - k^2\mathbf{J}_{\rm ME}(\mathbf{r}) = 0 \quad \text{if } \mathbf{r} \in D' \text{ (excluding } \partial D'). \qquad (17)$$

The validity of (17) and of the scalar versions of (16) for $L = \nabla^2 - c^{-2}\partial^2/\partial t^2$ and $L = \nabla^2 + k^2$ has been corroborated, for special cases, in [14] and [18].

The result (16) has two interesting previously unknown consequences. In particular, it follows from (1) and (16) that the fields $\psi_{\rm ME}$ produced by ME sources must obey $(L^2\psi_{\rm ME})(\mathbf{x}) = 0$, where $L^2 \equiv LL$, in the interior of the source's support. For example, for time-harmonic EM problems, this means (see Fig. 1)

$$(\nabla \times \nabla \times -k^2)^2\mathbf{E}_{\rm ME}(\mathbf{r}) = 0 \quad \text{if } \mathbf{r} \in D' \text{ (excluding } \partial D'). \qquad (18)$$

We will use (18) in Section III in connection with source-field decompositions. The following result applies to second-order PD operators L: It can be shown by using (16) and standard Green function techniques that any nontrivial L_2 source $\rho \in X$ of compact support D' having compactly supported first partial derivatives within D' (so that ρ vanishes along with its first partial derivatives on the boundary $\partial D'$ of its support D') must possess a NR part in the Hilbert space $X \equiv L_2(D')$. In particular, no nontrivial solution of (16) exists that obeys the above-imposed continuity and differentiability properties on $\partial D'$ [the only solution of (16) that obeys both homogeneous boundary conditions simultaneously is $\rho_{\rm ME}(\mathbf{x}) = 0$]. Now, since the requirement (16) with $\rho_{\rm ME} = \rho$ must hold for a general source $\rho \in X$ to lack a NR part in the Hilbert space $X \equiv L_2(D')$, one concludes that any $L_2(D')$ source obeying the above-imposed properties must possess nontrivial projections $\langle \rho_{\rm NR} \,|\, \rho\rangle_X$ into the subspace of L_2 NR sources confined within D', i.e., any such source must possess a NR part. This result has been used and illustrated further in a recent treatment of the ISP with

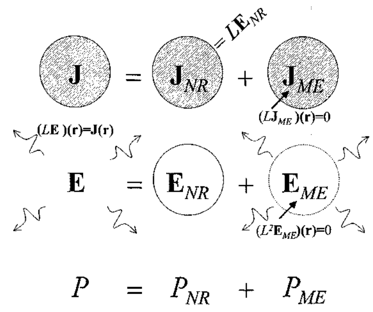

Fig. 1. Any $L_2(D')$ current distribution $\mathbf{J}(\mathbf{r})$ can be uniquely decomposed into the sum of a NR part $\mathbf{J}_{\rm NR}(\mathbf{r})$ that generates a NR field confined entirely within D' and a purely reactive power plus a radiating ME part $\mathbf{J}_{\rm ME}(\mathbf{r})$ that generates an identical field as the total field for $\mathbf{r} \notin D'$ and an identical real power as the total source's real power. The NR source $\mathbf{J}_{\rm NR}(\mathbf{r})$ is related to its confined NR field $\mathbf{E}_{\rm NR}(\mathbf{r})$ via $\mathbf{J}_{\rm NR}(\mathbf{r}) = (L\mathbf{E}_{\rm NR})(\mathbf{r})$ where $L = (\nabla \times \nabla \times -k^2)/(i\omega\mu)$. The ME source $\mathbf{J}_{\rm ME}(\mathbf{r})$ is a free-field truncated within the source region and its field obeys in the source region an iterated homogeneous vector wave equation $(L^2\mathbf{E}_{\rm ME})(\mathbf{r}) = 0$. The reactive portion of the ME power $\mathcal{P}_{\rm ME}$ is, in general, not trivial. It is due to energy storage associated with both its internal field (corresponding to the source region) and its external field.

regularity constraints (in addition to the usual localization constraints) [29].

3) A Reciprocity Relation for NR Sources: Next, we show how the result (15) and, in general, the more general result (13) and (14) for scalar or EM sources can be regarded physically as a manifestation of the well-known reciprocity property. In particular, NR sources are shown to be also nonreceptors from the points of view of both reaction and interaction.

The propagator describes the effect of a source located in a certain source region D' on test sources (receptors, e.g., receiving antennas) located in a certain field region D. In particular, the coupling between a scalar or vector source, say $\rho \in X$, confined within D' and a test source $\rho_t \in Y$ located in D, is described by

$$\mathcal{R}_{\rho\to\rho_t} = \langle \rho_t^* \,|\, P\rho\rangle_Y = \int_D d^n x\, \rho_t(\mathbf{x})(P\rho)(\mathbf{x}) = \int_{D'} d^n x (P^\dagger \rho_t^*)^*(\mathbf{x})\rho(\mathbf{x}). \qquad (19)$$

The quantity $\mathcal{R}_{\rho\to\rho_t}$ defined by (19) corresponds to what is known in EMs as "the reaction of ρ on ρ_t." This quantity appears in the usual Lorentz reciprocity theorem (see, e.g., [30, p. 326]). Another quantity of interest, the interaction power of the two-source system (ρ, ρ_t), is defined by an orthogonality integral of the form (see [30, pp. 20–22; 28–31] and [31, p. 440] for the EM case details)

$$\mathcal{P}_{\rho\to\rho_t} = \langle \rho_t \,|\, P\rho\rangle_Y = \int_D d^n x\, \rho_t^*(\mathbf{x})(P\rho)(\mathbf{x}) = \int_{D'} d^n x (P^\dagger \rho_t)^*(\mathbf{x})\rho(\mathbf{x}). \qquad (20)$$

To characterize NR sources in the role of receptors, we let $\rho_t \in X$ be a scalar or vector test source in the role of the radiator and $\rho_{\rm NR} \in Y$ a NR source in the role of the receptor. By noting that $(LP\rho_t)(\mathbf{x}) = 0$ if $\mathbf{x} \in D$ (the boundary ∂D of D excluded) as can be verified using (4) and (5), one finds from (15), (19), and (20) that for any $\rho_t \in X$

$$\mathcal{R}_{\rho_t\rightarrow\rho_{\rm NR}} = \langle \rho^*_{\rm NR} \,|\, P\rho_t\rangle_Y = 0$$
$$\mathcal{P}_{\rho_t\rightarrow\rho_{\rm NR}} = \langle \rho_{\rm NR} \,|\, P\rho_t\rangle_Y = 0. \qquad (21)$$

This means that NR sources are also nonreceptors, i.e., they do not interact energetically, with external sources and fields produced by external sources induce no reaction on them. The concept of reciprocity draws the physical picture for this previously unknown result. In particular, if a source does not radiate, then it does not receive either.

III. An Application to Source Analysis

Since ME sources are truncated free fields, the question arises, "Can one build a ME source?" Yes. For instance, one can build a wave-like source such as a traveling-wave antenna. Interestingly, sources of the form of a transient plane wave have received attention recently in connection with well-collimated ultrawideband radiation [32], [33]. The next question is then: Do ME sources outperform, perhaps by their lack of a NR part, non-ME sources in a physically meaningful way? Or, on the contrary, can one add NR components to a source so as to enhance its radiation performance? These questions apply to both radiation (transmission) and reception and, to our knowledge, have never been addressed satisfactorily. To address these questions, one must characterize the NR and radiating (ME) components of a source energetically. This section characterizes, for EM sources, the radiating and reactive energy properties of the NR and ME source components. This is accomplished by investigating the unique decomposition of a source and its field into their NR and ME components and the associated power budget. The general results are illustrated for the special case of a time-harmonic, homogeneous source in a 1-D space, which is applicable to transmission lines. The 1-D example also illustrates many of the general results of Section II. In the following, attention is restricted to L_2 sources in simply connected source regions.

A. General Power Budget Analysis

Consider an $L_2(D')$ current distribution $\mathbf{J}(\mathbf{r})$ in free space with a suppressed $e^{-i\omega t}$ time dependence spatially supported within a simply connected source volume D'. It is a well established fact that any $L_2(D')$ source, say a current distribution $\mathbf{J}(\mathbf{r})$, contained in a simply connected source region D', can be uniquely decomposed, in the Hilbert space $X \equiv L_2(D')$, into the sum of a NR and a radiating (ME) component (see, e.g., [13] and the projection theorem discussion in [34, p. 82]; see also Section III-B of this paper). Analogously, the associated total field can be decomposed into a NR and a ME part due to the source's NR and ME parts, respectively. These source-field decompositions are schematically illustrated in Fig. 1, which also shows some of the results derived in Section II-B. We find that the total (real plus reactive) input power $\mathcal{P}$ associated with an $L_2(D')$ source $\mathbf{J}(\mathbf{r})$ can be expressed as (see, e.g., [30, pp. 20–22; 28–31])

$$\begin{aligned}\mathcal{P} &= -\frac{1}{2}\int_{D'} d^3r\, \mathbf{J}^*(\mathbf{r})\cdot\mathbf{E}(\mathbf{r})\\ &= -\frac{1}{2}\int_{D'} d^3r[\mathbf{J}^*_{\rm NR}(\mathbf{r})\cdot\mathbf{E}_{\rm NR}(\mathbf{r}) + \mathbf{J}^*_{\rm ME}(\mathbf{r})\cdot\mathbf{E}_{\rm ME}(\mathbf{r})\\ &\quad + \mathbf{J}^*_{\rm NR}(\mathbf{r})\cdot\mathbf{E}_{\rm ME}(\mathbf{r}) + \mathbf{J}^*_{\rm ME}(\mathbf{r})\cdot\mathbf{E}_{\rm NR}(\mathbf{r})]\\ &= \mathcal{P}_{\rm NR} + \mathcal{P}_{\rm ME} - \frac{1}{2}\int_{D'} d^3r[\mathbf{J}^*_{\rm NR}(\mathbf{r})\cdot\mathbf{E}_{\rm ME}(\mathbf{r})\\ &\quad + \mathbf{J}^*_{\rm ME}(\mathbf{r})\cdot\mathbf{E}_{\rm NR}(\mathbf{r})]\end{aligned} \qquad (22)$$

where $(\mathbf{E}_{\rm NR}, \mathcal{P}_{\rm NR})$ and $(\mathbf{E}_{\rm ME}, \mathcal{P}_{\rm ME})$ are the individual field and power contributions due, respectively, to the NR and ME components $\mathbf{J}_{\rm NR}$ and $\mathbf{J}_{\rm ME}$ of the total source $\mathbf{J}$. Note that the ME power $\mathcal{P}_{\rm ME}$ contains, in general, both real and reactive components. Moreover, since the total and ME fields are identical outside the source region, one deduces that $\Re\{\mathcal{P}_{\rm ME}\} = \Re\{\mathcal{P}\}$, i.e., the real component of $\mathcal{P}_{\rm ME}$ equals the total real power exiting the source volume, also equal to the total real input power. The NR power $\mathcal{P}_{\rm NR}$ contributes nothing to the exiting power and is purely reactive ($\Re\{\mathcal{P}_{\rm NR}\} = 0$). The sum of the last two cross-term integrals in (22) equals zero. To show this, we express $\mathbf{J}_{\rm NR}(\mathbf{r})$ as $\mathbf{J}_{\rm NR}(\mathbf{r}) = (L\mathbf{E}_{\rm NR})(\mathbf{r})$, where $L = (\nabla\times\nabla\times -k^2)/(i\omega\mu)$ and substitute this expression into the last two integrals in (22). One deduces by manipulations similar to those used in Section II-B that

$$\begin{aligned}&\int_{D'} d^3r[\mathbf{J}^*_{\rm NR}(\mathbf{r})\cdot\mathbf{E}_{\rm ME}(\mathbf{r}) + \mathbf{J}^*_{\rm ME}(\mathbf{r})\cdot\mathbf{E}_{\rm NR}(\mathbf{r})]\\ &\quad = \int_{D'} d^3r[\mathbf{E}^*_{\rm NR}(\mathbf{r})\cdot\mathbf{J}_{\rm ME}(\mathbf{r}) + \mathbf{J}^*_{\rm ME}(\mathbf{r})\cdot\mathbf{E}_{\rm NR}(\mathbf{r})]\\ &\quad = 2\Re\left\{\int_{D'} d^3r\,\mathbf{E}^*_{\rm NR}(\mathbf{r})\cdot\mathbf{J}_{\rm ME}(\mathbf{r})\right\} = 0.\end{aligned} \qquad (23)$$

The sum of these cross terms is, thus, purely real; furthermore, it must be zero in order to comply with energy conservation as can be deduced from (22) since $\Re\{\mathcal{P}_{\rm ME}\} = \Re\{\mathcal{P}\}$ and $\Re\{\mathcal{P}_{\rm NR}\} = 0$. We have thus found that $\mathcal{P} = \mathcal{P}_{\rm NR} + \mathcal{P}_{\rm ME}$, i.e., the total source's input power equals the sum of the individual NR and ME powers. Fig. 1 summarizes this new form of power decomposition. To fully appreciate this result, one must bear in mind that fields can be superposed at will, whereas, in general, powers cannot. This is why we had to handle the cross-term contributions above carefully.

B. Example in 1-D Space: A Homogeneous Source

The space-dependent part of the electric field $\mathbf{E}(\mathbf{r}) = E(x)\hat{\mathbf{z}}$ produced by a 1-D current distribution $\mathbf{J}(\mathbf{r}) = J(x)\hat{\mathbf{z}}$ with a suppressed $e^{-i\omega t}$ time dependence, where

$$(d^2/dx^2 + k^2)E(x) = -i\omega\mu J(x) \qquad (24)$$

is given by (see, e.g., [21, p. 912])

$$E(x) = -\frac{\eta}{2}\int dx'\, J(x')e^{ik|x-x'|} \qquad (25)$$

where $\eta \equiv \omega\mu/k$. For a source $J(x)$ of spatial support $D' = [-a, a]$ (a strip of width $2a$ centered about the origin), (25) yields

$$E(x) = \begin{cases} F^+ e^{ikx}, & \text{if } x > a \\ F^- e^{-ikx}, & \text{if } x < -a \end{cases} \tag{26}$$

where

$$F^+ = -\frac{\eta}{2}\int_{-a}^{a} dx'\, J(x')e^{-ikx'}$$
$$F^- = -\frac{\eta}{2}\int_{-a}^{a} dx'\, J(x')e^{ikx'}. \tag{27}$$

For $|x| > a$, the electric field is then determined by the forward and backward plane wave amplitudes F^+ and F^-, respectively.

1) A 1-D Inverse Source Problem: We consider next the ISP of deducing an unknown source $J(x)$, of known support $D' = [-a, a]$, from knowledge of F^+ and F^-. To compute the associated ME solution $J_{\text{ME}}(x)$, we define the Hilbert space X of L_2 sources $J(x)$ localized within D' to which we assign the inner product

$$\langle J \,|\, J' \rangle_X = \int_{-a}^{a} dx\, J^*(x)J'(x). \tag{28}$$

Similarly, we define the data vectors $|F\rangle \equiv [F^+ \;\, F^-]$ obeying the square-summability requirement $|F^+|^2 + |F^-|^2 < \infty$. We also define the discrete Hilbert space Y of all such data vectors and assign to it the inner product

$$\langle F \,|\, F' \rangle_Y = (F^+)^* F'^+ + (F^-)^* F'^-. \tag{29}$$

The forward linear mapping $P : X \to Y$ (the propagator) is found from (27) to be defined by

$$F^\pm = (PJ)^\pm = -\frac{\eta}{2}\int_{-a}^{a} dx' J(x')e^{\mp ikx'}. \tag{30}$$

The class of invisible sources $\rho_N \in N(P) \equiv \{\rho \in X \,|\, P\rho = 0\}$ is seen to coincide with the class of L_2 NR sources of support D' since the vanishing of $|F\rangle$ automatically implies, in view of (26), the vanishing of $E(x)$ for $|x| > a$. It then follows from the projection theorem (see, e.g., [34, p. 82]) and well-known results derived in [13] that any source $J \in X$ can be uniquely decomposed into the sum of a radiating and a NR part $J_{\text{ME}}(x)$ and $J_{\text{NR}}(x)$, respectively, where $J_{\text{ME}}(x)$ and $J_{\text{NR}}(x)$ are the orthogonal projections of $J(x)$ into $N(P)^\perp$ and $N(P)$, respectively. We shall make use of this result next to carry out the unique decomposition of the homogeneous source $J(x) = M_s(x)$ into its radiating and NR parts.

The backpropagator $P^\dagger$, where $\langle J \,|\, P^\dagger F \rangle_X = \langle PJ \,|\, F \rangle_Y$, is found from (28)–(30) to be defined by

$$(P^\dagger |F\rangle)(x) = \frac{\eta}{2} M_s(x)(F^+ e^{ikx} + F^- e^{-ikx}) \tag{31}$$

where

$$M_s(x) = \begin{cases} 1, & \text{if } |x| \le a \\ 0, & \text{else.} \end{cases}$$

To determine the ME source $J_{\text{ME}}(x)$ consistent with a given data vector, we use (30) and (31) in addition to (6) with the substitutions $\rho_{\text{ME}} \to J_{\text{ME}}$ and $\bar{\psi} \to |F\rangle$. One obtains, after some manipulations, the result

$$J_{\text{ME}}(x) = \frac{M_s(x)}{\eta a[\text{sinc}^2(2ka) - 1]}\{[F^+ - F^- \,\text{sinc}(2ka)]e^{ikx} + [F^- - F^+ \,\text{sinc}(2ka)]e^{-ikx}\} \tag{32}$$

where $\text{sinc}(\cdot) \equiv \sin(\cdot)/(\cdot)$, which is identified to be a free-field plane wave expansion truncated within the source's support.

2) Special Case: A Homogeneous Source: The following analysis, based on (32), aims to isolate the wave properties and energetic roles of the radiating and NR components of the unit-amplitude homogeneous source $J(x) = M_s(x)$ and its field. One finds from (27) that for this source

$$F^+ = -\frac{\eta}{k}\sin ka. \tag{33}$$

The radiating component $J_{\text{ME}}(x)$ of $J(x) = M_s(x)$, corresponding to the Hilbert space X of L_2 sources of support $D' = [-a, a]$ is given from (32) and (33) by

$$J_{\text{ME}}(x) = \nu M_s(x)\cos kx \tag{34}$$

where

$$\nu = \frac{2\,\text{sinc}(ka)}{[\text{sinc}(2ka) + 1]}.$$

The corresponding NR part is then

$$J_{\text{NR}}(x) = M_s(x)(1 - \nu\cos kx). \tag{35}$$

The term ν, hence, the radiating part $J_{\text{ME}}(x)$ of $J(x) = M_s(x)$ defined by (34) vanishes if $ka = n\pi, n = 1, 2, \ldots$, i.e., a homogeneous source $J(x) = M_s(x)$ oscillating at those quantized frequencies is purely NR. Therefore, the extension of the smallest NR homogeneous source is $2a = \lambda$ for which the source's size coincides with the wavelength λ of the field. On the contrary, no frequencies $\omega > 0$ exist at which the homogeneous source $J(x) = M_s(x)$ lacks a NR part. This is not surprising since the homogeneous source $J(x) = M_s(x)$ does not obey the ME source necessary condition (16) with $L = (d^2/dx^2 + k^2)/(-i\omega\mu)$.

We have thus established the unique decomposition in the Hilbert space X of the homogeneous source $J(x) = M_s(x)$ into its radiating and NR parts, $J_{\text{ME}}(x)$ and $J_{\text{NR}}(x)$, respectively. We consider next the corresponding field decomposition. In carrying out the field decomposition, we shall also illustrate the use of the ME field necessary condition (18). Equation (25) yields the following expression for the total field $E(x)$ produced by the homogeneous source $J(x) = M_s(x)$ within the source's support:

$$E(x) = \frac{i\eta}{k}(e^{ika}\cos kx - 1), \quad \text{if } |x| \le a. \tag{36}$$

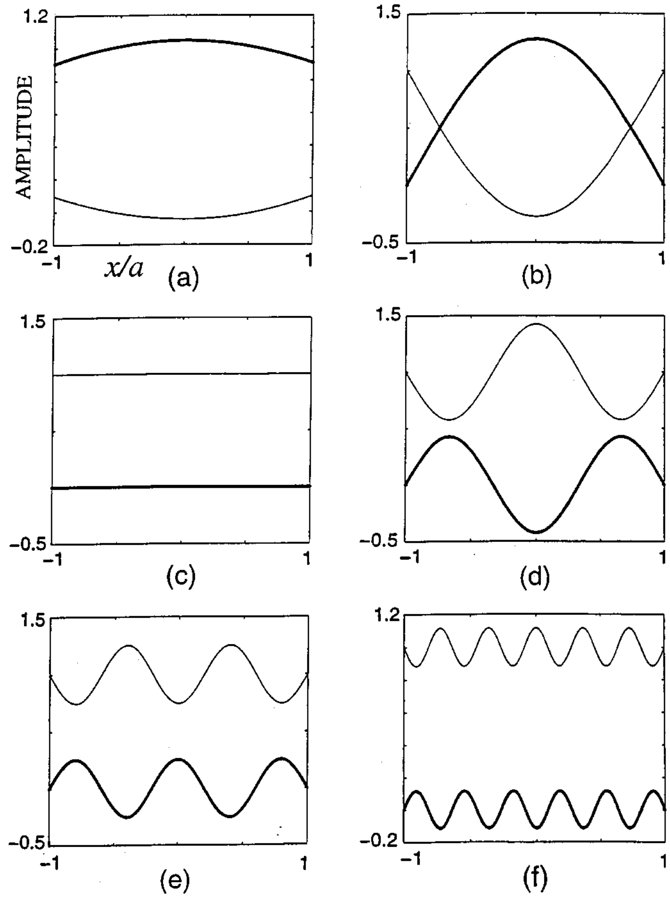

Fig. 2. The radiating (thick solid line) and NR (solid line) parts of the homogeneous source $J(x) = M_s(x)$ versus x/a, for (a) $ka = \pi/6$. (b) $ka = \pi/2$. (c) $ka = \pi$ (NR case). (d) $ka = 1.5\pi$. (e) $ka = 2.5\pi$. (f) $ka = 5.5\pi$.

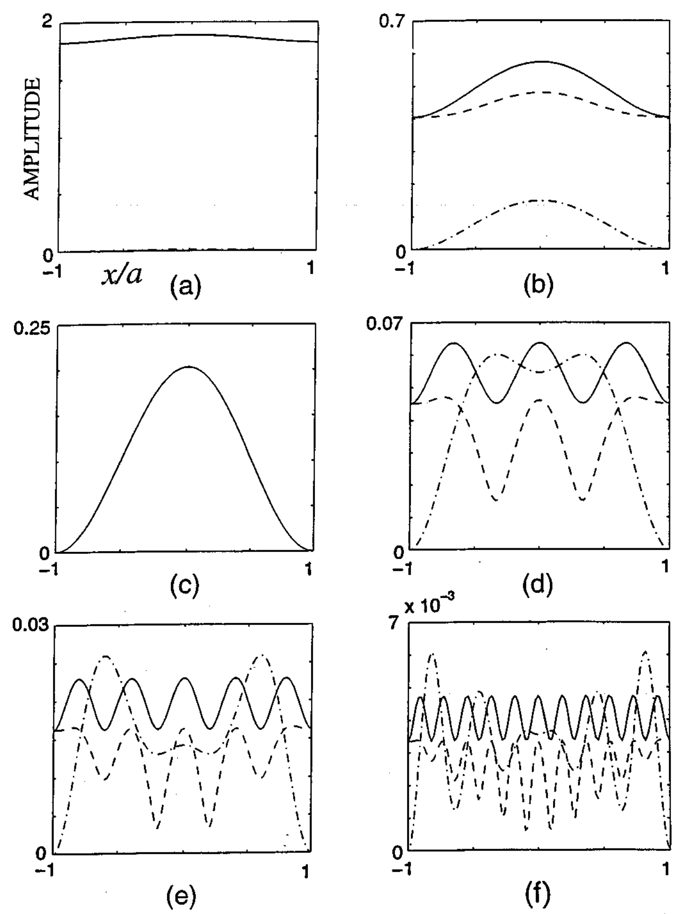

Fig. 3. The field magnitude for $|x| \leq a$ produced by the homogeneous source $J(x) = M_s(x)$ versus x/a (solid line). Also shown are the magnitudes of the radiating (dashed line) and NR (dashed-dotted line) parts of the total field for $|x| \leq a$. (a) $ka = \pi/6$. (b) $ka = \pi/2$. (c) $ka = \pi$ (NR case). (d) $ka = 1.5\pi$. (e) $ka = 2.5\pi$. (f) $ka = 5.5\pi$.

The field $E_{\rm ME}(x)$ produced by the radiating part $J_{\rm ME}(x)$ of the homogeneous source $J(x) = M_s(x)$ is defined by (25), with the substitution $J(x) \to J_{\rm ME}(x)$ and (34). We obtain, for $|x| \leq a$

$$E_{\rm ME}(x) = \frac{i\eta\nu}{4k}[(2ika + e^{2ika} - 1)\cos kx - 2kx \sin kx]. \tag{37}$$

The result (37) can be derived, alternatively, from (18) with the substitutions $\mathbf{E}_{\rm ME} \to E_{\rm ME}$ and $(\nabla \times \nabla \times -k^2) \to (d^2/dx^2 + k^2)$, i.e.,

$$(d^2/dx^2 + k^2)^2 E_{\rm ME}(x) = 0 \quad \text{if } |x| < a$$

so that

$$E_{\rm ME}(x) = A\cos(kx) + B\sin(kx) + Cx\sin(kx) + Dx\cos(kx) \quad \text{if } |x| < a. \tag{38}$$

To solve for the unknown coefficients A, B, C, D, we note that the ME source $J_{\rm ME}(x)$ defined by (34) is bounded and piecewise continuous. This immediately forces $E_{\rm ME}(x)$ to be everywhere continuous (see Appendix A of [25]). Therefore, the radiating field $E_{\rm ME}(x)$ must be defined by (24), (34) and (38) with the boundary conditions $E_{\rm ME}(a) = E_{\rm ME}(-a) = -(\eta \sin ka)e^{ika}/k$. One obtains, for $|x| \leq a$

$$E_{\rm ME}(x) = \frac{i\eta\nu}{2}\{ia[e^{ika}\,\mathrm{sinc}(ka) + 1]\cos kx - x\sin kx\}. \tag{39}$$

Equation (39) can be shown, after some algebra, to reduce to our previous result (37), as expected. The fields $E(x)$ and $E_{\rm ME}(x)$ for $|x| > a$ are given by (26) with $F^+ = F^- = -(\eta \sin ka)/k$. The NR field $E_{\rm NR}(x) = E(x) - E_{\rm ME}(x)$ produced by the NR part $J_{\rm NR}(x)$ of the homogeneous source $J(x) = M_s(x)$ vanishes for $|x| > a$. On the other hand, for $|x| \leq a$, the NR field $E_{\rm NR}(x)$ is explicitly defined by (36) and (39).

Fig. 2 shows plots of the spatial profile of the radiating and NR parts of $J(x) = M_s(x)$ for $|x| \leq a$ versus x/a, parameterized by ka. The radiating part $J_{\rm ME}(x)$ of $J(x)$ is a standing wave truncated within the source's support D', as expected. Fig. 2(a) illustrates the low-frequency nature of the source decomposition in which the homogeneous source is mostly radiating (as can be shown from (34) and L'Hopital's rule). Fig. 2(b) corresponds to the $ka = \pi/2$ case. The radiating part of the homogeneous source now vanishes on the

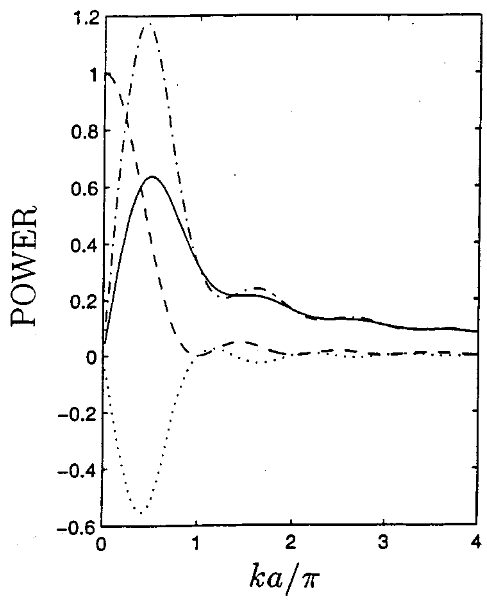

Fig. 4. Dashed line: Real power radiated by the homogeneous source $J(x) = M_s(x)$ versus ka/π (the value at $ka \simeq 0$ is used as a reference for normalization). This is also the real input power of this source. Solid line: total reactive power due to $J(x)$. Dashed-dotted line: reactive power due to $J_{\rm NR}(x)$. Dotted line: reactive power due to $J_{\rm ME}(x)$.

boundaries $x = a$ and $x = -a$ and, therefore, has compact support in $[-a, a]$. This behavior holds, in general, only if $ka = (n + 1/2)\pi, n = 0, 1, \ldots,$ as can be verified from (34). Fig. 2(c) corresponds to the smallest NR homogeneous source wherein $ka = \pi$. Fig. 2(d)–(f) corresponds to $ka = 1.5\pi, 2.5\pi$ and 5.5π, respectively. A gradual increase of the NR component of the homogeneous source is observed as ka increases. This was to be expected since $J_{\rm ME}(x)$ in (34) tends to zero as $ka \to \infty$. Fig. 3 shows plots of the spatial profile of the magnitude of the radiating and NR fields associated with the radiating and NR source components in Fig. 2. Also shown are the total fields (magnitude only). As expected, the latter are seen to coincide with the radiating fields $E_{\rm ME}(x)$ on the boundaries $x = a$ and $x = -a$. Fig. 4 shows plots of the real and reactive power of the J-E self interaction as a function of ka/π. Also shown are plots of the radiating and NR contributions to the reactive power. The real power of the J-E self interaction equals the time-averaged radiated power and is contributed only by the radiating part $J_{\rm ME}(x)$ of $J(x)$, as expected. On the other hand, the reactive energy is contributed by both the radiating and NR parts. The reactive contribution of the radiating part decays rapidly for $ka \gtrsim \pi$. This is not surprising in light of the observations above since $J_{\rm ME}(x)$ itself decays rapidly for $ka \gtrsim \pi$. Interestingly, the radiating and NR reactive power plots in Fig. 4 suggest the possibility of modifying a given source by adding a NR source to it, so as to reduce the source's overall reactive power. NR source additions have no effect whatsoever neither on the original source's exterior field nor on its exiting power but can reduce, through their reactive power contributions, the overall reactive input power (as seen, e.g., at an antenna's terminals). Ideally, one would want the added NR source to fully suppress the ME source's reactive power contribution. These ideas are illustrated below.

Consider a modified source $J_M(x)$ of the form

$$J_M(x) = \chi J_{\rm NR}(x) + J_{\rm ME}(x) \tag{40}$$

where $J_{\rm NR}(x)$ and $J_{\rm ME}(x)$ are the NR and ME sources defined in the previous developments and χ a parameter that needs to be chosen so as to minimize the source's overall reactive power. Note that independently of the value assigned to χ, the modified source $J_M(x)$ in (40) is "equivalent" to the homogeneous source $J(x) = M_s(x)$ from which it was derived in the sense that both produce the same external fields. Fig. 5 shows the value of χ for which the imaginary part $\Im\{\mathcal{P}_{\rm NR}\} = -\Im\{\mathcal{P}_{\rm ME}\}$, i.e., a source of the form (40) whose value of χ is chosen according to Fig. 5 possesses no reactive input power. This shows that NR sources can be useful from an antenna design point of view. In the example considered, the addition of a NR component tailored to the given homogeneous source was found to fully suppress the original source's reactive power. The same result also shows that ME sources are not necessarily optimal from a perfomance point of view. It seems that, instead, sources of optimal performance will contain, in general, both NR and ME contributions.

IV. Conclusion

This paper characterized in novel ways NR and ME sources and their fields in the context of a general ISP formulation, with applications to source analysis and reconstruction. The general results developed in the paper apply to any source-field system described in a spatial or spatial–temporal coordinate space by a linear scalar or vector PDE, such as the usual scalar and EM systems in a linear medium. Central to the general theory was the reciprocity principle, which we used in different ways. This principle has played a significant role in illuminating a variety of problems in EMs and acoustics [35]. Our use of this principle in Sections II-A and II-B can be regarded as yet a new application of this powerful tool. In Section II-B, we derived several previously unknown orthogonality relations among NR and ME sources and homogeneous field solutions of the associated scalar or vector source-field system. The reciprocity principle was pivotal then in establishing the physical nature of those results. In particular, it was concluded that NR sources are also nonreceptors, and vice versa (NR sources do not absorb power from nor react to incident fields). The orthogonality of a NR source relative to all homogeneous field solutions in its support is then a manifestation of a NR source's null receptive nature. A NR source's nonreceptive behavior and its orthogonality to homogeneous field solutions are then, respectively, a physical and a mathematical statement of the same property. We also conclude that if a source is NR relative to a given Green function, then it must also be NR relative to its associated adjoint and, in general, any Green function of the governing scalar or vector PDE. It was also shown that ME sources take the form of free-fields truncated within the source's support, i.e., the ultimate sources of wave radiation are themselves waves. Throughout the paper, an effort was made to show how some of the new general results reduce to known results derived before under special conditions.

Many of the general ISP concepts developed in the paper were illustrated with a time-harmonic EM example. We also

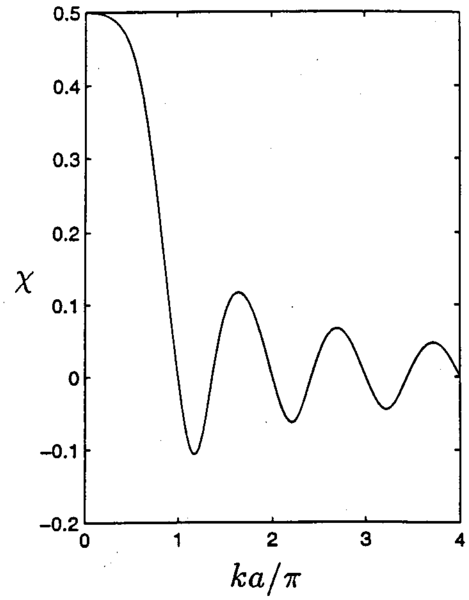

Fig. 5. Values of χ for which the imaginary part $\Im\{\mathcal{P}_{\rm NR}\} = -\Im\{\mathcal{P}_{\rm ME}\}$.

developed a new source analysis tool based on the decomposition of a source and its field into their radiating and NR parts. The associated power budget analysis revealed previously unknown aspects of the radiating and reactive energy properties of a source's radiating and NR components. It was found that the total input power associated with a given source can be decomposed into the sum of the individual power contributions due to the source's radiating and NR parts. This previously unknown result isolates the energy roles of the radiating and NR source components. It thereby opens the possibility of using NR sources as a tool for antenna optimization. The latter possibility was illustrated for a 1-D source by means of a simple source construction procedure. We plan to investigate this possibility further in the future in connection with antenna dipoles and arrays.

APPENDIX

We develop here a proof of the EM form of (13) and (14). In particular, we show that a localized current distribution $\mathbf{J}_{\rm NR}(\mathbf{r})$ with a suppressed $e^{-i\omega t}$ time-dependence and spatial support D' is NR if and only if

$$\int_{D'} d^3r\, \mathbf{v}^*(\mathbf{r})\cdot\mathbf{J}(\mathbf{r}) = 0 \tag{A.1}$$

where $\mathbf{v}(\mathbf{r})$ is any solution of the homogeneous vector wave equation

$$\nabla\times\nabla\times\mathbf{v}(\mathbf{r}) - k^2\mathbf{v}(\mathbf{r}) = 0 \quad \text{if } \mathbf{r}\in D'. \tag{A.2}$$

Let $\mathbf{J}_R$ and $\mathbf{J}_{\rm NR}$ be a radiating and a NR current distribution, respectively, localized in disjoint bounded spatial regions D'_R and D'. It is not hard to show by following a procedure analogous to that employed in [30, pp. 324–325] to derive the Lorentz reciprocity theorem that

$$\begin{aligned}\int_{D''} d^3r\, \mathbf{E}_R^*(\mathbf{r})\cdot\mathbf{J}_{\rm NR}(\mathbf{r}) &= -\int_{D''} d^3r\, \mathbf{E}_{\rm NR}(\mathbf{r})\cdot\mathbf{J}_R^*(\mathbf{r}) \\ &\quad -\int_{\partial D''} dS\,\hat{\mathbf{n}}\cdot[\mathbf{E}_{\rm NR}(\mathbf{r})\times\mathbf{H}_R^*(\mathbf{r}) \\ &\quad +\mathbf{E}_R^*(\mathbf{r})\times\mathbf{H}_{\rm NR}(\mathbf{r})]\end{aligned} \tag{A.3}$$

where

$\mathbf{E}_R$ and $\mathbf{H}_R$	electric and magnetic fields produced by $\mathbf{J}_R$, respectively;
$\mathbf{E}_{\rm NR}$ and $\mathbf{H}_{\rm NR}$	fields produced by $\mathbf{J}_{\rm NR}$;
D''	bounded spatial region enclosing both D'_R and D';
$\partial D''$	surface that bounds D'';
dS	denotes a surface differential element over $\partial D''$;
$\hat{\mathbf{n}}$	unit vector in the direction of the outward-normal to $\partial D'$.

One arrives at the same result by using the vector analog of Green's second identity (see [36, p. 250]) and the representation $\mathbf{J}_{\rm NR}(\mathbf{r}) = (\nabla\times\nabla\times - k^2)\mathbf{E}_{\rm NR}(\mathbf{r})/(i\omega\mu)$. By noting that the regions D'_R and D' are disjoint and $\mathbf{E}_{\rm NR}(\mathbf{r}) = 0$ if $\mathbf{r}\notin D'$, one finds from (A.3) that

$$\int_{D'} d^3r\, \mathbf{E}_R^*(\mathbf{r})\cdot\mathbf{J}_{\rm NR}(\mathbf{r}) = 0. \tag{A.4}$$

The necessary portion of the proof is completed by noting that the field $\mathbf{E}_R$ in (A.4) represents the most general solution of (A.2) inside D', i.e., the most general solution of (A.2) can be synthesized via sources external to D' such as the current distributions $\mathbf{J}_R$ used in proving this result. To show sufficiency, we note that any dyadic Green function of (A.2), say the outgoing dyadic Green function $\bar{\mathbf{G}}_0(\mathbf{r},\mathbf{r}')$, must obey $\nabla\times\nabla\times\bar{\mathbf{G}}^*(\mathbf{r}',\mathbf{r}) - k^2\,\bar{\mathbf{G}}^*(\mathbf{r}',\mathbf{r}) = 0$ if $\mathbf{r}'\notin D'$ and $\mathbf{r}\in D'$ so that from (A.1) $\mathbf{E}_{\rm NR}(\mathbf{r}') = \int_{D'} d^3r\,\bar{\mathbf{G}}(\mathbf{r}',\mathbf{r})\cdot\mathbf{J}_{\rm NR}(\mathbf{r}) = 0$ if $\mathbf{r}'\notin D'$, which completes the proof.

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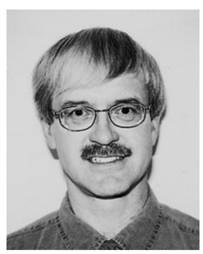
Edwin A. Marengo (S'89–M'97) was born in Panama City, Panama, in 1967. He received the Bachelor's degree in electromechanical engineering (*summa cum laude*) from the Technological University of Panama, Panama City, in 1990, and the M.S.E.E. and Ph.D. degrees from Northeastern University, Boston, MA, in 1994, and 1997, respectively.

From 1989 to 1990, he was a Teaching Assistant at the Department of Electrical Engineering, Technological University of Panama. From 1991 to 1992, he held positions as Mechanical Engineer at Asesores Tecnicos e Industriales, Panama City, as a Systems Engineering Trainee at IBM, Panama City, and as an Instructor of Applied Dynamics and Thermal Machines at the Technological University of Panama and of Electronics Laboratory and Electric Machinery, Universidad Santa Maria La Antigua, Panama City. From 1992 to 1997, he was with the Department of Electrical and Computer Engineering, Northeastern University, first as a Fulbright Scholar (1992–1994), sponsored by the United States government, and later as a Stipended Graduate Research Assistant. From September 1997 to April 1998, he did postdoctoral research in computational electromagnetics and inverse problems at the Center for Electromagnetics Research, Department of Electrical and Computer Engineering, Northeastern University. During that period he also taught an Electric Energy Devices course at Northeastern University. From April 1998 to August 2000, he did postdoctoral research in inverse problems, nonradiating sources, and quantum information at the Department of Electrical and Computer Engineering, The University of Arizona, Tucson, AZ. He is currently a Research Associate at the Technological University of Panama, where he also teaches a graduate Applied Probability and Stochastic Processes course. His main research interests include wave inverse problems, nonradiating sources, and classical and quantum wave aspects of information.

Dr. Marengo is a member of the IEEE Antennas and Propagation Society, the Optical Society of America, the American Physical Society, the American Association for the Advancement of Science, Phi Kappa Phi, and Eta Kappa Nu.



Richard W. Ziolkowski (M'87–SM'91–F'94) received the Sc.B. degree in physics (*magna cum laude* with honors) from Brown University, Providence, RI, in 1974 and the M.S. and Ph.D. degrees in physics from the University of Illinois at Urbana-Champaign, in 1975 and 1980, respectively.

He was a member of the Engineering Research Division at the Lawrence Livermore National Laboratory, CA, from 1981 to 1990 and served as the Leader of the Computational Electronics and Electromagnetics Thrust Area for the Engineering Directorate from 1984 to 1990. He joined the Department of Electrical and Computer Engineering at the University of Arizona as an Associate Professor in 1990, and was promoted to Full Professor in 1996. He was a CoGuest Editor of the 1998 feature issue of the *Journal of Optical Society of America A on Mathematics and Modeling in Modern Optics*. His research interests include the application of new mathematical and numerical methods to linear and nonlinear problems dealing with the interaction of acoustic and electromagnetic waves with realistic materials and structures.

Dr. Ziolkowski is a member of Tau Beta Pi, Sigma Xi, Phi Kappa Phi, the American Physical Society, the Optical Society of America, the Acoustical Society of America, and Commissions B (Fields and Waves) and D (Electronics and Photonics) of URSI (International Union of Radio Science). He was an Associate Editor for the IEEE TRANSACTIONS ON ANTENNAS AND PROPAGATION from 1993 to 1998. He served as the Vice Chairman of the 1989 IEEE/AP-S and URSI Symposium in San Jose and as the Technical Program Chairperson for the 1998 IEEE Conference on Electromagnetic Field Computation in Tucson. He is currently serving as a member of the IEEE AP-S Administrative Committee (ADCOM). For the US URSI Commission B he served as Secretary from 1993 to 1996 and as Chairperson of the Technical Activities Committee from 1997 to 1999. He is currently serving as a Member-at-Large of the U.S. National Committee (USNC) of URSI. He was a co-organizer of the Photonics Nanostructures Special Symposia at the 1998, 1999, and 2000 OSA Integrated Photonics Research Topical Meetings. He was awarded the Tau Beta Pi Professor of the Year Award in 1993 and the IEEE and Eta Kappa Nu Outstanding Teaching Award in 1993 and 1998.