Letters

An Accurate Closed-Form Approximate Representation for the Hankel Function of the Second Kind

Judd Gardner and R. E. Collin

***Abstract*—A second-order asymptotic evaluation of the Hankel function is presented and its numerical evaluation for the case of large orders and large arguments is compared with the numerical results obtained using forward recursion. It is shown that the second-order results are very accurate as long as the argument is a few percent larger than the order of the function.**

***Index Terms*—Cylinder, Hankel function, numerical analysis, orthogonal functions, scattering.**

I. Introduction

IN his book, *Electromagnetic Radiation from Cylindrical Structures* (New York: Pergamon, 1959), Prof. Wait attributed a second-order approximation for the derivative of the Hankel function to Sommerfeld which he employed to enable solving for the far field pattern of an axial slot on a circular conducting cylinder [1]. A recent study of the scattering of a Gaussian laser beam from a large perfectly conducting cylinder produced a solution, which, as might be anticipated by the geometry of the problem, included an integral where the integrand contained the derivative of the Hankel function of the second kind [2]. An integrable form for the Hankel function derivative was needed to enable integration. In addition, the integral was summed over the order of the Hankel function so that significant contributions were made by the integral as the order of the Hankel function closely approached the argument value of the Hankel function. These two stipulations precluded the use of the standard asymptotic approximations for the Hankel function given by Gradshteyn [3] and Watson [4]. Therefore, in accordance with Prof. Wait, a second-order approximation for the Hankel function was obtained from Sommerfeld's integral [5] representation of the Hankel function, which allowed integration and was surprisingly accurate for values where the orders approach the argument.

II. Asymptotic Expansion

The asymptotic expansion of the Hankel function of the second kind representative of outgoing waves for the choice of $e^{j\omega t}$ time variation will be considered, though the Hankel function of the first kind representative of incoming waves for the choice of $e^{j\omega t}$ time variation follows the same procedure. Sommerfeld's integral representation of the Hankel function of the second kind is given by [5]

$$H_n^{(2)}[z] = \frac{e^{-jn\pi/2}}{\pi} \int_{(\pi/2)-j\infty}^{(3\pi/2)+j\infty} e^{jz\cos(\phi)+jn\phi}\, d\phi \qquad (1)$$

where the contour of integration is shown in Fig. 1. The method of steepest descent may be applied to evaluate the integral in (1). Let ϕ equal the complex variable, $\sigma + j\eta$. The contour for the integral as shown in Fig. 1 may be deformed into any suitable path providing that the end points of the original contour are maintained, and no singularities are incurred. The saddle point is located by differentiating the exponent of the integrand, setting it equal to zero, and solving for the values of ϕ

$$\frac{d}{d\phi}(z\cos(\phi) + n\phi) = -z\sin(\phi) + n = 0 \qquad (2a)$$

$$\phi = \sigma_0 = \begin{cases} \arcsin(n/z), & 0 < \sigma_0 < \pi/2 \\ \pi\ \arcsin(n/z), & \pi/2 < \sigma_0 < \pi \end{cases} \qquad (2b)$$

where σ_0 is the location of the saddle point on the σ axis. Selecting the above relation for ϕ in the range from $\pi/2$ to π will locate the saddle point within that range. To evaluate the integral in (1), a Taylor Series expansion of the exponent of the integrand is found

$$z\cos(\phi) + n\phi \approx z\cos(\sigma_0) + n\sigma_0 - (z\sin(\sigma_0) - n)(\phi - \sigma_0) - z\,\frac{\cos(\sigma_0)}{2}(\phi - \sigma_0)^2 + \cdots.$$

The selected expression for σ_0 from (2) may be substituted to produce the second-order expression

$$z\cos(\phi) + n\phi = -\sqrt{z^2 - n^2} + n\pi - n\arcsin(n/z) + \frac{\sqrt{z^2 - n^2}}{2}(\phi - \sigma_0)^2.$$

This expansion may be substituted into (1) to give the approximation

$$H_n^{(2)}[z] \approx \frac{e^{-jn\pi/2}}{\pi}\, e^{-j\sqrt{z^2-n^2}+jn\pi-jn\arcsin(n/z)} \cdot \int_{(\pi/2)-j\infty}^{(3\pi/2)+j\infty} e^{j\left((\sqrt{z^2-n^2})/2\right)(\phi-\sigma_0)^2}\, d\phi \qquad (3)$$

where the constant part of the integrand was removed from the integral. The procedure for the method of steepest descent requires the contour to be deformed into the path of steepest descent. The steepest descent path passes through the saddle point along a line making an angle of $\pi/4$ with respect to the real axis. In order to proceed, the following representation for the expression $(\phi - \sigma_0)$ is used

$$(\phi - \sigma_0) = \rho\, e^{j\theta}. \qquad (4)$$

Using (4), the integrand of (3) becomes

$$e^{j\left((\sqrt{z^2-n^2})/2\right)\rho^2 e^{2j\theta}}.$$

From this, it is seen that choosing $\theta = \pi/4$ eliminates the imaginary part of the exponent, ensures the exponent is negative, and maximizes its magnitude. The new contour is displayed in Fig. 2, which shows the contour makes an angle of $\theta = \pi/4$ near the saddle point: A change of variables for $\theta = \pi/4$ produces:

$$d\phi = e^{j\theta}\, d\rho = e^{j\pi/4}\, d\rho$$

and the new limits of integration may be obtained from (4). As long as the exponent is large, the major contribution to the integral occurs for small values of ρ so that the error made in replacing the steepest

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R. E. Collin is with Case Western Reserve University, Cleveland, OH 44106 USA.
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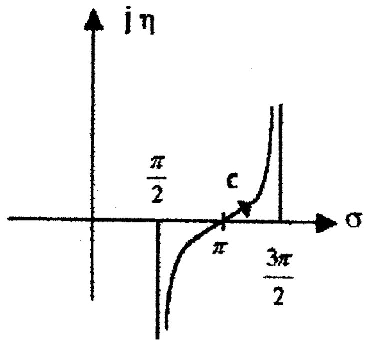

Fig. 1. Integration contour for Sommerfeld's integral representation of the Hankel function of the second kind.

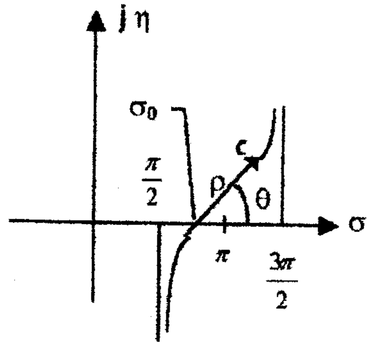

Fig. 2. New contour path of integration.

TABLE I
COMPARISON OF THE AMPLITUDE OF $H_n^{(2)}(z)$ USING FORWARD RECURSION AND THE APPROXIMATION IN (6)

		z = 6000	z = 5250	z = 5025
n = 4990	Recurs	0.013823	0.019753	0.032663
	Approx	0.013823	0.019752	0.032791
n = 4995	Recurs	0.013838	0.019845	0.033877
	Approx	0.013839	0.019846	0.034076
n = 5000	Recurs	0.013854	0.019943	0.035332
	Approx	0.013855	0.019942	0.035660
n = 5005	Recurs	0.013870	0.020040	0.037122
	Approx	0.013870	0.020041	0.037701
n = 5010	Recurs	0.013886	0.020142	0.039391
	Approx	0.013886	0.020142	0.040508

TABLE II
COMPARISON OF THE PHASE FOR $H_n^{(2)}(z)$ USING FORWARD RECURSION AND THE APPROXIMATION IN (6)

		z = 6000	z = 5250	z = 5025
n = 4990	Recurs	2.55710	2.64780	4.33741
	Approx	2.55774	2.64755	4.31389
n = 4995	Recurs	5.49681	4.22030	4.91171
	Approx	5.49817	4.21994	4.88304
n = 5000	Recurs	2.14580	5.77737	5.44262
	Approx	2.14664	5.77720	5.40636
n = 5005	Recurs	5.07044	1.03558	5.92719
	Approx	5.07193	1.03575	5.87950
n = 5010	Recurs	1.70434	2.56128	0.07880
	Approx	1.70477	2.56090	0.01324

descent path by the straight line extending from minus to plus infinity along the $\pi/4$ line is negligible. The equation in (3) now becomes

$$H_n^{(2)}[z] \approx \frac{e^{-jn\pi/2}}{\pi} e^{-j\sqrt{z^2-n^2}+jn\pi-jn\arcsin(n/z)} \cdot \int_{-\infty}^{\infty} e^{-\left((\sqrt{z^2-n^2})/2\right)\rho^2} e^{j\pi/4}\, d\rho. \tag{5}$$

The integral in (5) may be obtained from standard integration tables and is given by

$$\int_{-\infty}^{\infty} e^{-\left((\sqrt{z^2-n^2})/2\right)\rho^2}\, d\rho = \frac{\sqrt{2\pi}}{(z^2-n^2)^{1/4}}.$$

This result may be substituted into (5) and rearranged to give a closed form asymptotic approximation for the Hankel function of the second kind

$$H_n^{(2)}[z] \approx \sqrt{\frac{2}{\pi\sqrt{z^2-n^2}}} \cdot e^{-j\left(\sqrt{z^2-n^2}-\pi/4-n\pi/2+n\arctan(n/\sqrt{z^2-n^2})\right)}. \tag{6}$$

When $z \gg n$, the standard first order asymptotic expression for the Hankel function of the second kind is attained. The advantage of the above approximation is made evident when $\sqrt{z^2-n^2}$ is very large and as n approaches z. Then the above second-order approximation is much more accurate than the standard first order asymptotic expansion. This second-order asymptotic expansion is attributed to Sommerfeld by Wait [1]. This asymptotic expansion may also be found in [4] where the book by Watson [5] is listed as the reference source, and in a book by Wong [6]. The formula given in [4] includes correction factors as well.

Several tables of values for $H_n^{(2)}(z)$ have been compiled using forward recursion starting with $H_0^{(2)}(z)$ and $H_1^{(2)}(z)$, and were compared to the results obtained by the asymptotic expression given in (6). For large n, and z greater than $1.01n$, these values are in excellent agreement. Tables I and II display the magnitude and phase, respectively, for several pertinent excerpts from the original tables given in [2] in order to highlight the accuracy achieved by the approximation for the Hankel function of the second kind given by (6).

III. CONCLUSION

A second-order asymptotic expansion of the Hankel function of the second kind that is accurate when both the order and argument are large, but with the argument being at least a few percent larger than the order was presented. Some numerical results obtained from this second-order asymptotic expansion were compared with numerical results obtained using forward recursion and shows that the expansion is surprisingly accurate, even when the argument is only about 1% larger than the order.

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