Impedance Boundary Conditions in Ultrasonics

John D. Shumpert, *Student Member, IEEE,* and Thomas B. A. Senior, *Fellow, IEEE*

***Abstract*—A generalized impedance boundary condition (GIBC) is developed to approximate the scattering of a plane acoustic wave from a bone structure such as a rib. In particular, the rib and surrounding tissue are modeled as a viscoelastic cylinder of infinite length immersed in an infinite, inviscid fluid medium. In order to determine the scattered pressure wave, appropriate boundary conditions are imposed on the relevant differential equations at the fluid-solid surface. The exact solution is then used to develop first- and second-order impedance boundary conditions applicable at the surface of the cylinder. Numerical results demonstrate the improved accuracy of the second order condition.**

***Index Terms*—Acoustic applications, impedance boundary conditions.**

I. Introduction

IMPEDANCE boundary conditions have a rich history in electromagnetics. The simplest form is the first-order (or standard) impedance boundary condition (SIBC) and its application to a lossy material surface is generally attributed to Leontovich [1]. It first appeared in the Russian literature during World War II, but the first North American scientist to recognize its utility was J. R. Wait. In a sequence of papers starting in the 1950s, Wait used the boundary condition to simulate the land in studies of ground wave propagation over the earth, and it was a feature of many of his papers throughout his life (see, for example, [2]). In recent years more general versions of these boundary conditions have been developed [3], but the application of impedance boundary conditions is not limited to electromagnetics.

A problem of interest in bioengineering is the scattering of acoustic waves from bone structures such as the rib cage. For biomedical applications, a particular concern is the effect of pressure waves propagating from one fluid to another and the effect of shear wave formation in the more solid regions of the body, such as bone. Even though bone and other calcified materials are neither solid nor homogeneous, they are capable of supporting both shear and compressional waves. This is in contrast to the background tissue surrounding the rib cage. This tissue is soft and reasonably homogeneous but does not permit a shear wave to develop. From a consideration of the material properties of bone and soft tissue at the frequencies of concern, it is clear that the porous bone structure may be modeled as a viscoelastic medium and the soft tissue as a fluid.

TABLE I
VARIOUS BIOLOGICAL PARAMETERS AT 1 MHz

Tissue	Speed (c) [m s^{-1}]	Density (ρ) [g cm^{-3}]	Impedance (Z) [MPa s m^{-1}]	Absorption (α) [dB cm^{-1} MHz^{-1}]
Blood	1570	1.06	1.61–1.62	0.18
Bone	3360–4080	1.38–1.81	3.75–7.80	13
Brain	1541	1.03	1.55–1.66	0.85
Fat	1450–1476	0.92	1.35–1.38	0.63
Kidney	–	1.04	1.62	–
Liver	1570-1585	1.06	1.64–1.68	–
Lung	650	0.40	0.26	41
Muscle	1568-1585	1.07	1.64–1.74	1.3–3.3
Spleen	–	1.06	1.65–1.67	–
Water	1480	1.00	1.48–1.52	0.0022
Air	343	–	0.000415	12
HST (mean)	1540	–	1.63	0.81

In order to appropriately model the bone and soft tissue structures, various parameter estimates for the speed of sound, material density, characteristic impedance, and absorption, are listed in Table I [4]–[7]. For an operating frequency of 1 MHz, the corresponding acoustic wavelength inside the rib is 4 mm. Human soft tissue, (HST) is an ensemble parameter average of the different soft tissues for the body. For most soft tissues, the HST value is an acceptable approximation. As can be seen, acoustic absorption in human tissue occurring through natural viscosity, heat conduction, and/or molecular exchanges can be significant and must be accounted for.

The reflection and transmission of acoustic waves at a fluid-fluid interface is analogous to that of electromagnetic waves interacting at a dielectric interface. It is a straightforward procedure to compute the reflection and transmission coefficients for a variety of simple shapes [8]. However, if we now concern ourselves with the reflection and transmission of pressure waves from an elastic solid, two types of elastic waves are propagated, longitudinal bulk waves traveling in one direction and transverse shear waves traveling at a lower speed and in a different direction. Elastic media can be described using the Lamé constants, λ and μ, and the density ρ, or alternatively, using Young's Modulus E and Poisson's ratio σ along with the density. An important parameter that is used in defining acoustic media is the specific acoustic impedance Z, defined as the ratio of acoustic pressure to the normal particle velocity at the surface. For our purposes, however, it is more convenient to employ the reciprocal of the impedance, i.e., the specific acoustic admittance Y. For plane acoustic waves, the impedance of the wave can be found from the product of the density ρ and the wave speed c in the medium.

One of the first complete treatments of elastic scattering from cylinders and spheres was carried out by Faran [9], but no attention was paid to viscous losses. Vogt *et al.* [10], [11] were one

Manuscript received September 9, 1999; revised March 1, 2000. The work of J. D. Shumpert was supported by by the National Science Foundation through its Graduate Fellowship Program.

J. D. Shumpert was with the Radiation Laboratory, Department of Electrical Engineering and Computer Science, University of Michigan, Ann Arbor, Michigan 48109-2122 USA. He is now with the Northrop Grumman Corporation, Baltimore, MD 21203 USA (e-mail: john_d_shumpert@mail.northgrum.com).

T. B. A. Senior is with the Radiation Laboratory, Department of Electrical Engineering and Computer Science, University of Michigan, Ann Arbor, Michigan 48109-2122 USA (e-mail: senior@eecs.umich.edu).

Publisher Item Identifier S 0018-926X(00)09383-2.

of the first groups to determine the reflection of a plane wave from an absorbing sphere. Flax and Neubauer [12] derived a solution for a layered cylindrical shell with absorption. Many of the early solutions [13] assumed the specific acoustic impedance was constant for all partial wave modes of the body, equivalent to using a first-order impedance boundary condition. Parametric studies were carried out to determine an appropriate value for this impedance, but as was noted by Ayres and Gaunaurd [14], the assumption of a constant impedance is inadequate. The specific acoustic impedance is frequency dependent and depends not only on the material parameters but is also a function of the partial wave index n.

The simplest canonical shape that bears a resemblance to a rib is a circular cylinder and the problem considered here is the scattering of a plane acoustic wave incident on a homogeneous viscoelastic cylinder of infinite length immersed in an inviscid fluid. The exact solution is obtained in the form of an eigenfunction expansion, and the specific acoustic admittance of each partial mode is determined. The solution is then used to develop first and second-order impedance boundary conditions applicable at the surface of the cylinder. It would appear that this is the first time that second-order conditions have been considered in the context of viscoelastic media. An SIBC is developed in Section III-A and a second-order generalized impedance boundary condition (GIBC) in Section III-B, and their accuracy is determined by comparison with the exact solution. Such boundary conditions have the advantage of converting a two media problem into a single medium one, and since an actual rib is not circular in cross section, they may be helpful in the numerical solution for a more realistic geometry.

II. Formulation of Soultion

In the straightforward solution of this problem, one treats the body as a scatterer and determines the total field as the sum of the incident field and the scattered field contributed by the body. Following Faran [9], we consider an infinite elastic circular cylinder of radius a, Young's modulus E, Poisson's ratio σ, and density ρ immersed in an inviscid fluid medium with density ρ_0. The cylinder (see Fig. 1) is excited by a time-harmonic ($e^{j\omega t}$) incident acoustic plane wave of the form

$$\begin{aligned} p^i &= P_0 e^{-jk_0 x} \\ &= P_0 e^{-jk_0 r \cos\phi} \\ &= P_0 \sum_{n=0}^{\infty} \varepsilon_n j^{-n} J_n(k_0 r) \cos n\phi \end{aligned} \tag{1}$$

producing an outward going scattered acoustic wave

$$p^s = P_0 \sum_{n=0}^{\infty} \varepsilon_n j^{-n} c_n H_n^{(2)}(k_0 r) \cos n\phi \tag{2}$$

where c_n is an unknown coefficient to be determined.

The equation of motion of a solid elastic medium is given by Love [15]

$$\frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)} \nabla(\nabla\cdot\mathbf{u}) - \frac{E}{2(1+\sigma)} \nabla\times(\nabla\times\mathbf{u}) = \rho \frac{\partial^2 \mathbf{u}}{\partial^2 t} \tag{3}$$

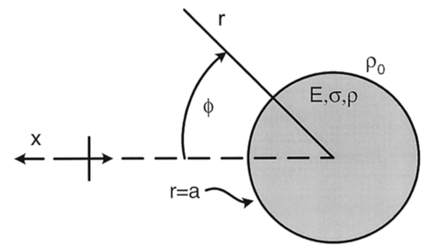

Fig. 1. Cross-sectional view of viscoelastic circular cylinder of radius a excited by plane acoustic wave

where (3) has been written in terms of E, σ and the particle displacement u. Taking the divergence of both sides of (3) yields the differential equation that the compressional wave solution satisfies; taking the curl of both sides of (3) yields the differential equation that the shear wave solution satisfies. The displacement can be expressed as the sum of the gradient of a scalar potential function U and the curl of a vector potential function $\mathbf{A}$ [16]

$$\mathbf{u} = -\nabla U + \nabla \times \mathbf{A} \tag{4}$$

where U represents the compression wave solution and A the shear wave solution. For two-dimensional (2-D) problems, where A has only a z component, the potentials satisfy the scalar wave equations

$$\left(\nabla^2 - \frac{1}{c_1^2}\frac{\partial^2}{\partial^2 t}\right) U = 0 \tag{5a}$$

$$\left(\nabla^2 - \frac{1}{c_2^2}\frac{\partial^2}{\partial^2 t}\right) A_z = 0 \tag{5b}$$

where the longitudinal and transverse wave velocities in the elastic body are expressed, respectively, as

$$c_1 = \sqrt{\frac{E(1-\sigma)}{\rho(1+\sigma)(1-2\sigma)}} = \sqrt{\frac{\lambda+2\mu}{\rho}}$$

$$c_2 = \sqrt{\frac{E}{2\rho(1+\sigma)}} = \sqrt{\frac{\mu}{\rho}}.$$

Solutions in cylindrical coordinates (r, ϕ, z) for U and A_z are

$$U = P_0 \sum_{n=0}^{\infty} \varepsilon_n j^{-n} a_n J_n(k_1 r) \cos n\phi \tag{6a}$$

$$A_z = P_0 \sum_{n=0}^{\infty} \varepsilon_n j^{-n} b_n J_n(k_2 r) \sin n\phi \tag{6b}$$

where

- k_p — ω/c_p for $p = 0$, 1, or 2;
- ε_n — Neumann factor;
- a_n and b_n — unknown coefficients to be determined.

Throughout this work, the subscript "0" refers to the external fluid, and subscripts "1" and "2" to the longitudinal and transverse components of the viscoelastic medium, respectively.

A. Elastic Media

Three boundary conditions must be satisfied at the surface of the cylinder:

- the pressure in the fluid must be equal to the normal component of stress e_{rr} on the solid at the surface $r = a$

$$p^i + p^s = -e_{rr} \tag{7a}$$

where

$$e_{rr} = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}\nabla\cdot\mathbf{u} + \frac{E}{1+\sigma}\frac{\partial u_r}{\partial r} \tag{7b}$$

with

$$\nabla\cdot\mathbf{u} = P_0 k_1^2 \sum_{n=0}^{\infty} \varepsilon_n j^{-n} a_n J_n(k_1 r)\cos n\phi \tag{7c}$$

- the normal component of displacement of the fluid must be equal to the normal component of displacement of the solid at the surface $r = a$

$$u_r^i + u_r^s = u_r \tag{8}$$

where the radial components of displacement of the incident and scattered fields are

$$u_r^i = \frac{1}{\rho_0\omega^2}\frac{\partial p_i}{\partial r} = \frac{P_0}{\rho_0\omega^2}\sum_{n=0}^{\infty}\varepsilon_n j^{-n}\frac{d}{dr}J_n(k_0 r)\cos n\phi \tag{9a}$$

$$u_r^s = \frac{1}{\rho_0\omega^2}\frac{\partial p_s}{\partial r} = \frac{P_0}{\rho_0\omega^2}\sum_{n=0}^{\infty}\varepsilon_n j^{-n}\frac{d}{dr}H_n^{(2)}(k_0 r)\cos n\phi \tag{9b}$$

and the radial component of displacement inside the cylinder is

$$\begin{aligned} u_r &= -\frac{\partial U}{\partial r} + \frac{1}{r}\frac{\partial A_z}{\partial \phi} \\ &= P_0 \sum_{n=0}^{\infty}\varepsilon_n j^{-n}\left[\frac{n b_n}{r}J_n(k_2 r) - a_n\frac{d}{dr}J_n(k_1 r)\right]\cos n\phi \end{aligned} \tag{10}$$

where U and A_z are defined in (6).

- the tangential components of shearing stress must vanish at the surface $r = a$

$$e_{r\phi} = e_{rz} = 0 \tag{11a}$$

where

$$e_{r\phi} = \frac{E}{2(1+\sigma)}\left[\frac{1}{r}\frac{\partial u_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right)\right] \tag{11b}$$

$$e_{rz} = \frac{E}{2(1+\sigma)}\left[\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right]. \tag{11c}$$

Inside the cylinder, the radial component of displacement is given in (10) and the angular component of displacement is

$$\begin{aligned} u_\phi &= -\frac{1}{r}\frac{\partial U}{\partial \phi} - \frac{\partial A_z}{\partial r} \\ &= P_0 \sum_{n=0}^{\infty}\varepsilon_n j^{-n}\left[\frac{n a_n}{r}J_n(k_1 r) - b_n\frac{d}{dr}J_n(k_2 r)\right]\sin n\phi. \end{aligned} \tag{12}$$

By symmetry, $e_{rz} = 0$ everywhere.

Using (1)–(12), one can solve for c_n to obtain

$$c_n = -\frac{J_n(x_0) + jY_n J_n'(x_0)}{H_n^{(2)}(x_0) + jY_n H_n'^{(2)}(x_0)} \tag{13}$$

where the specific acoustic admittance Y_n of the nth partial wave is shown in (14) at the bottom of the page and

$$\Gamma_n = \frac{2n x_1 J_n'(x_1) - 2n J_n(x_1)}{(2n^2 - x_2^2)J_n(x_2) - 2x_2 J_n'(x_2)} \tag{15}$$

with $x_p = k_p a$ for $p = 0$, 1, or 2. Inserting the expression for Γ_n and rearranging terms yields (16) as shown at the bottom of the page where

$$C = \frac{2x_0 x_1}{x_2^2}\frac{\rho_0}{\rho} = \frac{2k_0 k_1}{k_2^2}\frac{\rho_0}{\rho} \tag{17}$$

independent of a and $A_i = J_n'(x_i)/J_n(x_i)$ for $i = 1$, 2. Equation (16) can be written as $Y_n = CN/D$, where N and D represent the numerator and denominator, respectively.

$$Y_n = \frac{2\rho x_0}{\rho_0 x_2^2}\,\frac{\left(\frac{\sigma}{1-2\sigma} + 1 - \frac{n^2}{x_1^2}\right)x_1^2 J_n(x_1) + x_1 J_n'(x_1) - n\Gamma_n J_n(x_2) + n\Gamma_n x_2 J_n'(x_2)}{n\Gamma_n J_n(x_2) - x_1 J_n'(x_1)} \tag{14}$$

$$Y_n = jC\,\frac{\left[\frac{1-\sigma}{1-2\sigma} + \frac{A_1}{x_1} - \frac{n^2}{x_1^2}\right]\left[1 + \frac{2A_2}{x_2} - \frac{2n^2}{x_2^2}\right] - \frac{2n^2}{x_1 x_2}\left(A_1 - \frac{1}{x_1}\right)\left(A_2 - \frac{1}{x_2}\right)}{A_1\left[1 + \frac{2A_2}{x_2} - \frac{2n^2}{x_1 x_2^2 A_1}\right]}. \tag{16}$$

In the limiting case of an acoustically hard medium, $\mu \to \infty$ implying $Y_n \to \infty$ and then

$$c_n = -\frac{J_n'(x_0)}{H_n'^{(2)}(x_0)} \tag{18}$$

whereas for a soft medium $\mu \to 0$ implying $Y_n \to 0$, giving

$$c_n = -\frac{J_n(x_0)}{H_n^{(2)}(x_0)}. \tag{19}$$

III. Impedance Boundary Conditions

Having found the scattered field, we now examine the accuracy to which the field can be determined using impedance boundary conditions of different orders. The simplest (first order) impedance boundary condition can be written as

$$\frac{\partial p}{\partial n} + jk_0 Y p = 0 \tag{20}$$

where Y is a normalized admittance and n is in the direction of the outward normal to the surface. More generally, a boundary condition of order $M > 1$ has the form

$$\frac{\partial p}{\partial n} = -jk_0 \left\{ \beta_0 + \sum_{m=1}^{M/2} (-1)^m \cdot \left[\frac{\partial^m}{\partial s^m} \left(\beta_{11}^{(m)} \frac{\partial^m}{\partial s^m} + \beta_{12}^{(m)} \frac{\partial^m}{\partial t^m} \right) + \frac{\partial^m}{\partial t^m} \left(\beta_{21}^{(m)} \frac{\partial^m}{\partial s^m} + \beta_{22}^{(m)} \frac{\partial^m}{\partial t^m} \right) \right] \right\} p \tag{21}$$

where s and t are tangential variables [3], [17]. In the particular case of a two-dimensional body with no t dependence, this reduces to

$$\frac{\partial p}{\partial n} = -jk_0 \left\{ \beta_0 + \sum_{m=1}^{M/2} (-1)^m \beta_{11}^{(m)} \frac{\partial^{2m}}{\partial s^{2m}} \right\} p \tag{22}$$

and for a circular cylinder with $n = r$ and $s = r\phi$

$$\frac{\partial p}{\partial n} = -jk_0 \left\{ \beta_0 - \beta_{11}^{(1)} \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \beta_{11}^{(2)} \frac{1}{r^4} \frac{\partial^4}{\partial \phi^4} - \cdots \right\} p. \tag{23}$$

When applied to an eigenfunction expansion of p in the form $\sum_n a(n) \cos n\phi$, the corresponding admittance is

$$Y = \beta_0 + \beta_{11}^{(1)} \left(\frac{n}{r} \right)^2 + \beta_{11}^{(2)} \left(\frac{n}{r} \right)^4 + \cdots. \tag{24}$$

A. First-Order IBC

Let $x_i = N_i x_0$ $(i = 1, 2)$ where N_1 and N_2 are the complex refractive indexes of the viscoelastic medium. If $|x_i|$ are so large that terms $\mathcal{O}(x_i^{-2})$ can be neglected, A_i can be approximated by the first two terms of the series given in (A.3), i.e.,

$$A_i \simeq j \left(1 - \frac{1}{2N_i j x_0} \right). \tag{25}$$

For a first-order solution, N and D can be approximated by polynomials of order $1/x$

$$N \simeq Z + \frac{K}{x}, \quad D \simeq 1 + \frac{P}{x}$$

where Z, K, and P, the coefficients of the inverse power series, are determined in [18] to be

$$Z = \frac{1-\sigma}{1-2\sigma} \tag{26a}$$

$$K = -\frac{1}{N_1 j} - \frac{1-\sigma}{1-2\sigma} \frac{2}{N_2 j} \tag{26b}$$

$$P = \frac{1}{2N_1 j} + \frac{2}{N_2 j}. \tag{26c}$$

Substituting the above expressions for Z, K, and P, and neglecting terms of order $\mathcal{O}(x^{-2})$ yields

$$Y \simeq C \left\{ \frac{1-\sigma}{1-2\sigma} + \frac{j}{x_1} \left[1 - \frac{1}{2} \left(\frac{1-\sigma}{1-2\sigma} \right) \right] \right\}. \tag{27}$$

Thus, for the first-order solution β_0 of (24) is

$$\beta_0 = C \left\{ \frac{1-\sigma}{1-2\sigma} \left(1 + \frac{1}{2N_1 j x_0} \right) - \frac{1}{N_1 j x_0} \right\}. \tag{28}$$

To this order the admittance is mode *independent*.

B. Second-Order IBC

If $|x_i|$ are somewhat smaller, but still large enough to allow terms $\mathcal{O}(x_i^{-4})$ to be neglected, A_i can be approximated by the first four terms of the series given in (A.3)

$$A_i \simeq j \left(1 - \frac{1}{2N_i j x_0} - \frac{4n^2-1}{8N_i^2 x_0^2} + \frac{4n^2-1}{8N_i^3 j x_0^3} \right).$$

For a second-order solution, N and D in (14) can be approximated by polynomials of order $1/x^3$

$$N \simeq \left(Z + \frac{K}{x} + \frac{L}{x^2} + \frac{M}{x^3} \right)$$

$$D \simeq \left(1 + \frac{P}{x} + \frac{Q}{x^2} + \frac{R}{x^3} \right)$$

where Z, K, and P are defined in (26) and L, M, Q, and R, are [18]

$$L = -\frac{2n^2+1}{2N_1^2} + \frac{2n^2-2}{N_1 N_2} - \frac{2n^2+1}{N_2^2} \frac{1-\sigma}{1-2\sigma} \tag{29a}$$

$$M = \frac{4n^2-1}{8N_1^3 j} + \frac{4n^2-1}{4N_2^3 j} \frac{1-\sigma}{1-2\sigma} - \frac{1}{N_1^2 N_2 j}(n^2-1) - \frac{1}{N_1 N_2^2 j}(n^2-1) \tag{29b}$$

$$Q = \frac{4n^2-3}{8N_1^2} - \frac{1}{N_1 N_2} - \frac{3}{N_2^2} \tag{29c}$$

$$R = -\frac{1}{8N_1^3 j} + \frac{4n^2-3}{4N_1^2 N_2 j} + \frac{4n^2-3}{2N_1 N_2^2 j} - \frac{4n^2+15}{4N_2^3 j}. \tag{29d}$$

Thus, for the second-order boundary condition, β_0 and $\beta_{11}^{(1)}$ of (24) are

$$\beta_0 = C\left\{\frac{1-\sigma}{1-2\sigma}\left(1+\frac{1}{2N_1jx_0}-\frac{3}{8N_1^2x_0^2}-\frac{1}{8N_1^3jx_0^3}\right) - \frac{1}{N_1jx_0}\right\} \tag{30}$$

$$\beta_{11}^{(1)} = \frac{C}{2k_0^2}\left\{\frac{1-\sigma}{1-2\sigma}\left(\frac{1}{N_1^2}-\frac{4}{N_2^2}+\frac{2}{N_1^2N_2k_0j}-\frac{8}{N_2^3k_0j}\right) - \frac{2}{N_1^2}+\frac{4}{N_1N_2}-\frac{2}{N_1^3jx_0}-\frac{4}{N_1^2N_2jx_0} + \frac{6}{N_1N_2^2jx_0}\right\}. \tag{31}$$

Note that the second-order solution has the requisite mode dependence.

IV. Validation

In the far-field where $r \gg a$, $H_n^{(2)}(k_0r)$ may be written in its asymptotic form

$$H_n^{(2)}(k_0r) \sim \sqrt{\frac{2j}{\pi k_0 r}}j^n e^{-jk_0r} \tag{32}$$

and from (2) the far field pressure then becomes

$$p^s = -P_0 e^{-jk_0r}\sqrt{\frac{2j}{\pi k_0 r}}\sum_{n=0}^{\infty}\varepsilon_n c_n \cos n\phi. \tag{33}$$

The scattering width (SW), or alternatively, the bistatic acoustic cross section per unit length defined in [8] as

$$SW(\phi) = \lim_{r\to\infty}\left[2\pi r\left|\frac{p^s}{p^i}\right|^2\right] \tag{34}$$

is now

$$SW(\phi) = \frac{2\lambda}{\pi}\left|\sum_{n=0}^{\infty}\varepsilon_n c_n \cos n\phi\right|^2 \tag{35}$$

with c_n given by (13).

Fig. 2 shows the normalized monostatic scattering width $SW(\phi = 0°)/\lambda$ as a function of k_0a for a lossy cylinder ($\sigma = 1/3$, $E = 20$ MPa, $N_1 = 2 - j1.5$, $N_2 = 2 - j4$, and $\rho = 1.81$ g/cm^3) immersed in an inviscid fluid medium ($c_0 = 1540$ m/s and $\rho_0 = 1.06$ g/cm^3) for the SIBC, the second-order GIBC, and the exact normal-mode series solution computed using 31 modes. The parameter values approximate a rib embedded in a soft tissue background. The exact solution requires the evaluation of the specific acoustic admittance, and consequently, computation of many cylindrical functions for each nth partial wave solution at each frequency. In contrast, the SIBC and GIBC are calculated using the relatively simple inverse power series shown in (24) in conjunction with (13). Not surprisingly, neither boundary condition is accurate for small values of k_0a and do not pick up the effects corresponding to the first two shear wave resonances across the diameter of the cylinder. The higher order resonances are invisible because of the absorption, and as the absorption increases, the second and then the first resonances disappear.

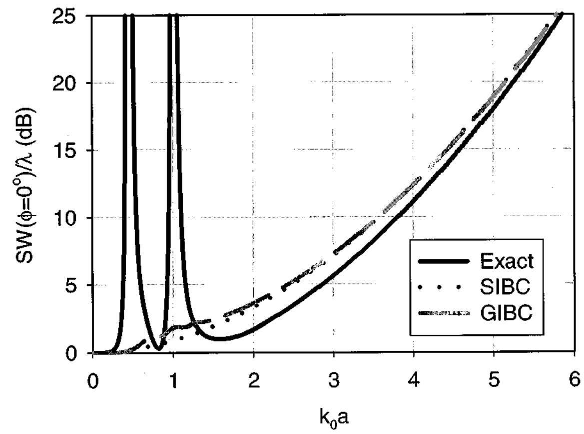

Fig. 2. Normalized monostatic scattering width as a function of k_0a, with $N_1 = 2 - j1.5$, $N_2 = 2 - j4$.

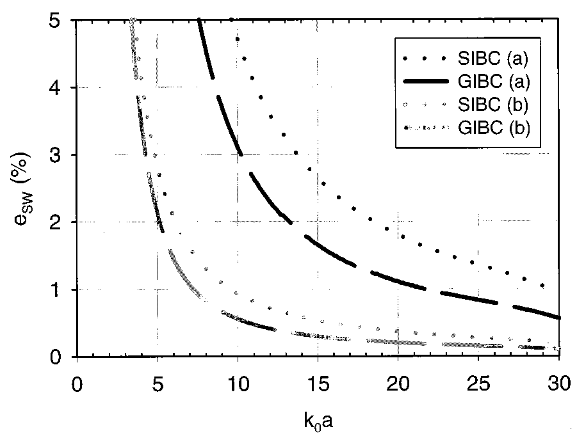

Fig. 3. Normalized monostatic scattering width percent error as a function of k_0a for (a) $N_1 = 2 - j1.5$, $N_2 = 2 - j4$, (b) $N_1 = 2 - j2$, $N_2 = 2 - j8$.

The normalized monostatic scattering width percent error e_{SW} is plotted as a function of k_0a in Fig. 3 for (a) $N_1 = 2 - j1.5$, $N_2 = 2 - j4$ and (b) $N_1 = 2 - j2$, $N_2 = 2 - j8$. In case (a), for an allowable error of 3%, the SIBC approximation is valid for $k_0a > 13.7$ and the GIBC for the somewhat smaller values $k_0a > 10.2$. For an allowable error of 1%, the SIBC is valid for $k_0a > 29.6$ and the GIBC for $k_0a > 21.8$. If the loss is increased as shown for case (b), the SIBC can be seen to be accurate to within 3% for $k_0a > 4.85$ and the GIBC for $k_0a > 4.28$ and to within 1% for $k_0a > 9.59$ and the GIBC for $k_0a > 7.34$.

The normalized bistatic scattering width $SW(\phi)/\lambda$ of a cylinder with $k_0a = 8$, $N_1 = 2 - j8$ and $N_2 = 2 - j1$ for the SIBC, GIBC, and exact solution is shown in Fig. 4. Clearly, the GIBC more accurately models the angular variation of the scattered field than the SIBC, and this is particularly noticeable as the scattering angle ϕ increases. The accuracy of the GIBC can be attributed to its ability to model some of the partial mode dependence of the specific acoustic admittance and to the inclusion of more terms representing the surface curvature. The

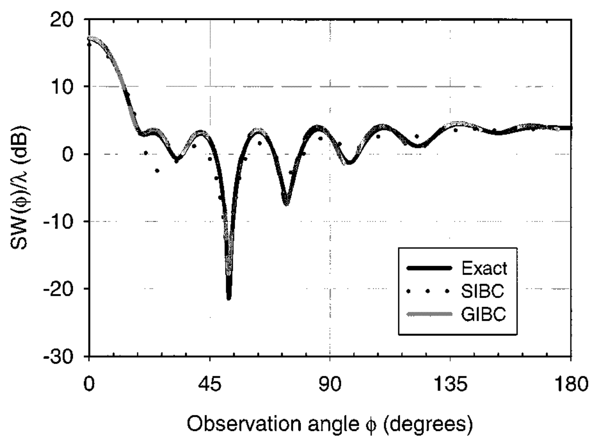

Fig. 4. Normalized bistatic scattering width of a cylinder with $k_0a = 8, N_1 = 2 - j8$, and $N_2 = 2 - j1$.

former is the dominant effect, and the accuracy of the SIBC is unchanged if the β_0 of (30) is used in place of (28).

V. Conclusion

In ultrasonic scattering by bone-like structures, it is necessary to include the effect of the shear waves, and these significantly complicate the solution of a problem as fundamental as the scattering from an infinite circular cylinder. Since the structures are lossy, one way to simplify the problem is to invoke an impedance boundary condition and this could make tractable the analytical or numerical determination of the scattering from other geometries. From the exact expression for the partial mode impedances for a circular cylinder, first and second-order impedance boundary conditions were derived by asymptotic expansion for large k_0a. With each boundary condition the accuracy improves as k_0a increases, or as the absorption increases for a fixed k_0a and, for many purposes, the accuracy of the SIBC may be adequate. Under all circumstances, however, the GIBC is more accurate primarily because of the mode dependence that it provides.

Appendix

The Bessel function $J_n(x)$ of order n can be expanded in an asymptotic series for large x as shown in [3]

$$J_n(x) = e^{j(x-n\pi/2)}\sqrt{\frac{2j}{\pi x}} \cdot \left\{1 - \frac{4n^2-1}{8jx} - \frac{(4n^2-1)(4n^2-9)}{128x^2} + \frac{(4n^2-1)(4n^2-9)(4n^2-25)}{128\cdot 24jx^3} + \mathcal{O}(x^{-4})\right\} \tag{A.1}$$

provided $\mathrm{Im}\{x\} < 0$. Accordingly,

$$J_n'(x) = je^{j(x-n\pi/2)}\sqrt{\frac{2j}{\pi x}} \cdot \left\{1 - \frac{4n^2+3}{8jx} - \frac{(4n^2-1)(4n^2+15)}{128x^2} + \frac{(4n^2-1)(4n^2-9)(4n^2+35)}{128\cdot 24jx^3} + \mathcal{O}(x^{-4})\right\} \tag{A.2}$$

and, therefore,

$$\frac{J_n'(x)}{J_n(x)} = j\left\{1 - \frac{1}{2jx} - \frac{4n^2-1}{8x^2} + \frac{4n^2-1}{8jx^3} + \mathcal{O}(x^{-4})\right\}. \tag{A.3}$$

Acknowledgment

The first author would like to thank S. R. Legault for conversations regarding this work.

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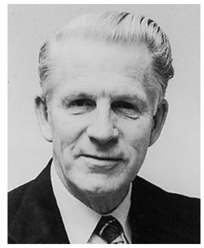
John D. Shumpert (S'89) was born in Columbus, MS, in 1970. He received the B.S.E.E. and M.S.E.E. degrees from Clemson University, Clemson, SC, in 1993 and 1995, respectively, and the Ph.D. degree from the University of Michigan, Ann Arbor, in 2000.

From 1993 to 1996, he was a National Science Foundation Graduate Fellow working on electromagnetic field penetration through small apertures in conducting surfaces and acoustic scattering from elastic cylinders. From 1996 to 2000, he was employed as a Graduate Research Assistant in the Radiation Laboratory at the University of Michigan, Ann Arbor, designing artificial dielectric structures for microwave and millimeter-wave applications. He is currently employed in the Electronic Sensors and System Sector of the Northrop Grumman Corporation in Baltimore, MD. His current research interests include frequency-selective surfaces, artificial dielectric structures for microwave and millimeter-wave applications, and phased-array antennas.

Dr. Shumpert is a member of the IEEE Antennas and Propagation Society (AP-S), IEEE Microwave Theory and Techniques Society (MTT-S), Phi Kappa Phi, Tau Beta Pi, Eta Kappa Nu, and Sigma Xi.



Thomas B. A. Senior (SM'66–F'72) received the M.Sc. degree in applied mathematics from the University of Manchester, Manchester, U.K., in 1950, and the Ph.D. degree from Cambridge University, Cambridge, U.K., in 1954.

He is now an Emeritus Professor of Electrical Engineering and Computer Science in the Electrical Engineering and Computer Science Department, University of Michigan. From 1952 to 1957, he was at the Radar Research and Development Establishment at Malvern, U.K., and joined the Radiation Laboratory of the University of Michigan in 1957. He was appointed Professor in 1969. He served as Director of the Laboratory from 1975 to 1986, acting Department Chair in 1987, and served as Associate Chair for the Electrical Science and Engineering Division and for Academic Affairs in the Department from 1986 until his retirement in 1998. He has been as Associate Editor and then Editor of *Radio Science* from 1973 to 1978. His primary research interests are in the study of the diffraction and propagation of electromagnetic waves, with application to physical problems. He is the author or coauthor of three books and over 200 journal articles.

Dr. Senior is a former Chair of Commission B of the International Union of Radio Science (URSI) and of the U.S. National Committee for URSI and served as President of URSI from 1996 to 1999. He is now the Immediate Past President and a member of the Board of Officers. He is a member of Sigma Xi, Tau Beta Pi, and Eta Kappa Nu. He received several awards for research and teaching including the Stephen S. Attwood Award in 1983 and a State of Michigan Teaching Excellence Award in 1991. He was named Arthur F. Thurnau Professor in that year, and was the URSI Van der Pol Gold Medal winner in 1993.