Radiation of a Hertzian Dipole Immersed in a Dissipative Medium

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***Abstract*—The general radiation formula for a Hertzian dipole immersed in an isotropic dissipative medium of infinite extent has been derived. As a boundary condition of the source, it is assumed that the dipole moment is a given quantity. When the conductivity of the medium is finite, the total radiating power is found to be infinite. Thus, in order to define a finite physically meaningful quantity, the dipole must be "insulated." The total radiating power is then a function of the thickness of the insulator and the constants of the media. When the radius of the spherical insulator is large compared to a wavelength, the reflection coefficient of the wave traveling from the dielectric to the dissipative medium with the dipole as a source reduces to that of a plane wave as derived from Fresnel's equations. The similarity between this and the problem by Weyl is discussed in this paper.**

***Index Terms*—Absorbing media, dipole antenna, electromagnetic (EM) radiation.**

I. Preface

TWO years ago, the first author of this paper received a letter from Dr. J. R. Wait, our beloved friend whom we are honoring in this special issue. In the letter, he mentioned a discussion they had had in 1953 about the physics of electromagnetic (EM) radiation in a lossy medium. It happened that in 1947, Tai wrote a technical report issued by Cruft Laboratory, Harvard University, Cambridge, MA, addressing the same subject. At that time, they had a very delightful discussion on this work and became very close professional friends. Over the years, he sent Tai most of the reprints of his papers. He was certainly one of the most productive scientist in EMs and geophysics.

The abovementioned report was never published nor presented in a meeting, so it seemed that a paper based on this report would be an appropriate remembrance to Dr. Wait. The reviewing board found some errors in the paper based on the original work (which had been submitted to the Antennas and Propagation Society Transactions Editor). Because of the historical interest in the work the reviewing board asked the second author to revise the paper and make the necessary corrections. This paper is thus a joint work and becomes a fine remembrance to our mutual friend Dr. J. R. Wait.

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II. Introduction

In investigating the radiation characteristic of an antenna placed in a semi-infinite dissipative medium, the simplest problem is to consider a Hertzian dipole. The problem is then similar to Weyl's problem [1] except that the role played by the two media is interchanged. Mathematically, the formulation seems to have been well established. What one has to do is to interchange k_1 and k_2, the propagation constants of the two media, in Weyl's solution. Before this is done, however, we should like to know whether this simple mathematical manipulation has a real physical significance. To give a thorough understanding of the problem, this paper will first investigate the power relationship of an oscillating dipole placed in a dissipative medium of infinite extent. After it has been established that an infinite amount of power is required to sustain the oscillation of a dipole in a dissipative medium, the problem of an "insulated" dipole will next be considered. Finally, the restriction upon Sommerfeld's reciprocal theorem as a result of this investigation is discussed.

III. Hertzian Dipole Placed in an Arbitrary Medium of Infinite Extent

If we use $e^{j\omega t}$ as the harmonic time factor for all the quantities defined in Maxwell's equations, then the complex amplitude of $\mathbf{E}$ and $\mathbf{H}$ satisfy the following equations in a simple medium:

$$\begin{aligned} \operatorname{curl}\mathbf{E} &= -j\omega\mu\mathbf{H} \\ \operatorname{div}\mathbf{H} &= (\sigma + j\omega\varepsilon)\mathbf{E} + \mathbf{J} \\ \operatorname{div}(\varepsilon\mathbf{E}) &= \rho \\ \operatorname{div}(\mu\mathbf{H}) &= 0 \end{aligned} \tag{1}$$

where $\mathbf{J}$ is the impressed current; σ, ε, μ are the constitutive constants of the medium. It is well known that the EM field generated by a Hertzian dipole immersed in a simple medium of infinite extent can be derived from a simple vector function Π, known as Hertzian potential function given by

$$\Pi = \frac{-j}{4\pi\omega\underline{\varepsilon}} \iiint \frac{\mathbf{J}e^{-jkr}}{r}\,dr = \hat{z}\,\frac{pe^{-jkr}}{4\pi\underline{\varepsilon}r} \tag{2}$$

where p is the dipole moment; $\underline{\varepsilon}$ and k are two complex constants defined in the following:

$$\underline{\varepsilon} = \varepsilon\left(1 - j\,\frac{\sigma}{\omega\varepsilon}\right) \tag{3}$$

$$k = \beta - j\alpha = \omega\sqrt{\underline{\varepsilon}\mu}. \tag{4}$$

The relations between $\mathbf{E}$, $\mathbf{H}$ and Π are

$$\mathbf{E} = \operatorname{grad}\operatorname{div}\Pi + k^2\Pi \tag{5}$$

$$\mathbf{H} = j\omega\underline{\varepsilon}\operatorname{curl}\Pi. \tag{6}$$

By substituting (2) in (5)–(6), we can easily derive the explicit solutions for $\mathbf{E}$ and $\mathbf{H}$. Expressed in spherical coordinates r, θ, and ϕ, they can be written as

$$\begin{aligned}\mathbf{E} =& \frac{p}{4\pi\underline{\varepsilon}}\cos\theta\left(\frac{2}{r^3}+\frac{2jk}{r^2}\right)e^{-jkr}\hat{r} \\ &+ \frac{p}{4\pi\underline{\varepsilon}}\sin\theta\left(\frac{1}{r^3}+\frac{jk}{r^2}-\frac{k^2}{r}\right)e^{-jkr}\hat{\theta}\end{aligned} \tag{7}$$

$$\mathbf{H} = \frac{jk^2}{\omega\mu}\,\frac{p}{4\pi\underline{\varepsilon}}\sin\theta\left(\frac{1}{r^2}+\frac{jk}{r}\right)e^{-jkr}\hat{\phi}. \tag{8}$$

The complex Poynting's vector is then given by

$$\begin{aligned}\underline{S} &= \tfrac{1}{2}E_\theta H_\phi^* \\ &= \frac{-j\omega p^2\sin^2\theta}{32\pi^2\underline{\varepsilon}}\left[\frac{1}{r^5}+\frac{j(k-k^*)}{r^4}-\frac{k(k-k^*)}{r^3}\right. \\ &\qquad \left.+\frac{jk^2k^*}{r^2}\right]e^{-j(k-k^*)r}.\end{aligned} \tag{9}$$

Conjugate quantities are denoted by an asterisk in (9). The power radiated outward measured from a sphere of radius R is then equal to

$$\begin{aligned}P &= \operatorname{Real}\left\{\int_0^{2\pi}\int_0^{\pi}\underline{S}R^2\sin\theta\,d\theta\,d\phi\right\} \\ &= \frac{\omega^3p^2\mu}{12\pi(\alpha^2+\beta^2)^2}\left[\frac{2\alpha\beta}{R^3}+\frac{4\alpha^2\beta}{R^2}+\frac{2\alpha\beta(\alpha^2+\beta^2)}{R}\right. \\ &\qquad \left.+\beta(\alpha^2+\beta^2)^2\right]e^{-2\alpha R}.\end{aligned} \tag{10}$$

It can be shown that P is also equal to the volume integral of $\sigma|\mathbf{E}|^2$ taken over the entire region outside the sphere with radius equal to R.

If the medium is a perfect dielectric, σ is equal to zero. Equation (10) then reduces to the well-known expression of the power radiated by a Hertzian dipole immersed in a nonconducting medium, namely

$$P_0 = \frac{\omega^3p^2\mu\beta}{12\pi} = \frac{\omega^4p^2}{12\pi\varepsilon V^3} \tag{11}$$

where V is the velocity of propagation in that medium, being equal to $(\mu\varepsilon)^{-1/2}$. The value of P_0 defined by (11) is then a constant, independent of the surface upon which the integral of $\underline{S}$ is taken.

To find the total power radiated by a Hertzian dipole placed in a dissipative medium, one must let R approach zero in (10). The quantity then becomes infinite. Since in a physical world we can have only finite quantities, it is concluded that a Hertzian dipole cannot sustain its oscillation in a dissipative medium to maintain a finite field at a distance. Consequently, it is impossible to speak of the total radiating *power* of a Hertzian dipole when the latter is in direct contact with a dissipative medium. This fact seems to have been overlooked by Sommerfeld in formulating his reciprocal theorem [2].

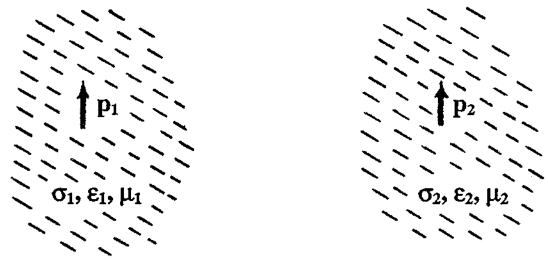

Fig. 1. Two Hertzian dipoles immersed in two arbitrary media.

To give a better understanding of the problem, we may reiterate Sommerfeld's result. Referring to Fig. 1, we have two Hertzian dipoles, with dipole moments p_1 and p_2, respectively, immersed in two simple media. The constitutive constants in the neighborhood of the dipoles are assumed to be quite arbitrary. Then, according to Sommerfeld, if an equal amount of *power* is supplied to the dipoles, the following reciprocal relation can be derived:

$$\left(\varepsilon_1 - j\,\frac{\sigma_1}{\omega}\right)\frac{V_1^{3/2}}{\sqrt{\varepsilon_1}}\,E_{2z} = \left(\varepsilon_2 - j\,\frac{\sigma_2}{\omega}\right)\frac{V_2^{3/2}}{\sqrt{\varepsilon_2}}\,E_{1z}$$

[2, eq. (18a)].

Sommerfeld uses $e^{-i\omega t}$ system in his work. His i corresponds to our $-j$. E_{2z} is the component of the electrical field in the direction of $\mathbf{p}_1$ measured at P_1, the site of the dipole $\mathbf{p}_1$, due to the distant radiation of $\mathbf{p}_2$ and E_{1z} is that measured at P_2, the site of dipole $\mathbf{p}_2$, in the direction of $\mathbf{p}_2$, due to the radiation of $\mathbf{p}_1$; V_1 and V_2 are the velocity of propagation in these two media. If one follows closely the derivation of [2, eq. (18a)] it will be seen that this equation is derived under the condition that (11) is used to define the power supplied to a dipole regardless of the kind of medium in which the dipole is placed. This is, of course, not true according to what we have shown. For this reason, Sommerfeld's formula seems to have no real physical significance unless the media under consideration are perfect dielectrics. With this restriction, his reciprocal theorem should read

$$\varepsilon_1^{1/2}V_1^{3/2}E_{2z} = \varepsilon_2^{1/2}V_2^{3/2}E_{1z}. \tag{12}$$

Returning now to our main subject, we have thus seen that whenever the medium has finite conductivity, we cannot even define a radiating power for a Hertzian dipole. This difficulty, however, can be overcome with the dipole "insulated." Physically, this implies that the tremendous ohmic loss as a result of the large induction field in the neighborhood of the otherwise uninsulated dipole can be avoided.

IV. Insulated Hertzian Dipole

The configuration of an insulated Hertzian dipole placed in a dissipative medium is shown in Fig. 2. For mathematical simplicity, the insulator is assumed to be of spherical shape with radius equal to R_1. The propagation constants of the media are given by

$$\begin{aligned}k_1 &= \beta_1 = \omega(\varepsilon_1\mu_1)^{1/2} \\ k_2 &= \beta_2 - j\alpha_2 = \omega(\underline{\varepsilon}_2\mu_2)^{1/2}.\end{aligned} \tag{13}$$

It is understood that k_1 is real, while k_2 is complex. As a boundary condition for the source, we shall treat the dipole moment $\mathbf{p}$ as a given quantity. It should be pointed out here that the setup of this problem is very similar to Weyl's [1], which is an extension of Sommerfeld's famous work [3], on the effect of a finitely conducting plane upon the radiation of an oscillating dipole. The similarity between our problem and the previous ones is shown in Fig. 3. Instead of having the dipole placed above a finitely conducting plane, we have now a dipole surrounded by a finitely conducting spherical body. By Kirchhoff's method of integration, the Hertzian potential inside the dielectric can be written as

$$\Pi_{z1} = \frac{p}{4\pi\varepsilon_1}\frac{e^{-jk_1 r}}{r} + \frac{1}{4\pi}\iint_S \left(\frac{\Pi'_{z1}}{r'^2}\frac{\partial r'}{\partial n} + \frac{1}{r'}\frac{\partial \Pi'_{z1}}{\partial n} + \frac{jk_1\Pi'_{z1}}{r'}\frac{\partial r'}{\partial n}\right) d\sigma \quad (14)$$

where r, r' are, respectively, the distances measured from the point of observation to the dipole and to the surface of contact of the two media, n is the normal to S. The first term on the right side of (14) is designated by Sommerfeld as the "primary excitation" term, while the term represented by the surface integral simply as a reflected wave from the spherical boundary. Thus, we shall assume Π_z to satisfy the following equations for the two different regions under consideration

$$\Pi_{z1} = (A - B)\frac{e^{-jk_1 r}}{r} + B\frac{e^{jk_1 r}}{r} \qquad r < R_1 \quad (15)$$

$$\Pi_{z2} = C\frac{e^{-jk_2 r}}{r} \qquad r > R_1 \quad (16)$$

where A is identically equal to $p/4\pi\varepsilon_1$ in order to satisfy the singularity condition at the source and p is the dipole moment. It can be shown that when Π_z has only one component in the z direction, (5)–(6) can be written as

$$E_r = \cos\theta(\Pi''_z + k^2\Pi_z) \quad (17)$$

$$E_\theta = -\sin\theta\left(\frac{1}{r}\Pi'_z + k^2\Pi_z\right) \quad (18)$$

$$H_\phi = \frac{-jk^2}{\omega\mu}\sin\theta\Pi'_z \quad (19)$$

where Π'_z denotes the derivative of Π_z with respect to r and Π''_z the second derivative.

The boundary condition at $r = R_1$ requires that the tangential components of $\mathbf{E}$ and $\mathbf{H}$ be continuous for all values of θ or

$$k_1^2\Pi'_{z1} = k_2^2\Pi'_{z2} \quad (20)$$

$$\frac{\Pi'_{z1}}{r} + k_1^2\Pi_{z1} = \frac{\Pi'_{z2}}{r} + k_2^2\Pi_{z2} \quad (21)$$

$\Bigg\}$ at $r = R_1$

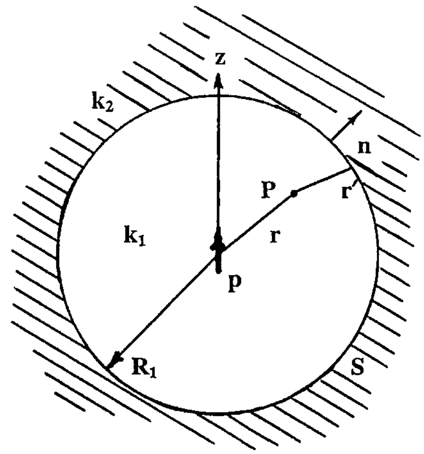

Fig. 2. An insulated dipole.

where we have assumed that $\mu_1 = \mu_2 = \mu_0$. Substituting the values of Π_{z1} and Π_{z2} given by (15) and (16) into (20) and (21), we can solve for the coefficients B and C in terms of A', where, for convenience, we have set $A - B = A'$. They are shown in (22) and (23) at the bottom of the page. It will now be convenient to introduce the reflection and transmission coefficients r_s and t_s for the Hertzian potentials by means of the following relations:

$$r_s = \frac{B}{A'} \quad (24a)$$

$$t_s = \frac{C}{A'} \quad (24b)$$

where r_s and t_s are the factors that multiply A' in (22) and (23). In terms of these parameters we find that

$$B = \frac{r_s}{1 + r_s}A \quad (25a)$$

$$A' = \frac{1}{1 + r_s}A = \frac{p}{4\pi\varepsilon_1(1 + r_s)}. \quad (25b)$$

V. Total Power Radiated by an Insulated Dipole

The total power P_i radiated by an insulated dipole can be computed using the same formulas given by (7)–(10), except that the amplitude $A = p/4\pi\underline{\varepsilon}$ for the outward propagating spherical wave is now replaced by the new amplitude $C = t_s p/4\pi\varepsilon_1(1 + r_s)$. In addition, α, β, and R are replaced by α_2, β_2, and R_1. We can express r_s, the factor multiplying A' in (22)

$$B = \frac{-(k_1 - k_2)[(k_1 + k_2)(1 + jk_1R_1)(1 + jk_2R_1) - jk_1^2k_2^2R_1^3]}{(k_1 + k_2)[(k_1 - k_2)(1 - jk_1R_1)(1 + jk_2R_1) + jk_1^2k_2^2R_1^3]}e^{-2jk_1R_1}A' \quad (22)$$

$$C = \frac{2jk_1^5R_1^3e^{-j(k_1 - k_2)R_1}}{(k_1 + k_2)[(k_1 - k_2)(1 - jk_1R_1)(1 + jk_2R_1) + jk_1^2k_2^2R_1^3]}A' \quad (23)$$

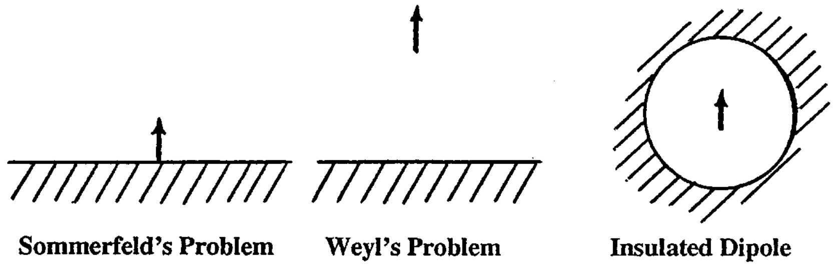

Fig. 3. Three boundary value problems.

in the form $(r_1+jr_2)\,e^{-2jk_1R_1}$. After a laborious algebraic manipulation, we find, as shown in (26) at the bottom of the page, where

$$\begin{aligned} D &= k_1R_1 = \beta_1R_1 \\ \frac{k_2}{k_1} &= \frac{\beta_2 - j\alpha_2}{\beta_1} = b - ja \\ A_0 &= (1-b)^2 + a^2 \\ A_1 &= 2a[(1-b)^2 + a^2] \\ A_2 &= (1+a^2+b^2)[(1-b)^2+a^2] \\ A_3 &= 2a \\ A_4 &= 4a^2 - (a^2+b^2)[(1-b)^2+a^2] \\ A_5 &= 2a(a^2+b^2) \\ A_6 &= (a^2+b^2)^2. \end{aligned} \tag{27}$$

Note that the effect of the standing wave in the insulating cavity is to cause the radiated power to vary in a cyclical manner with changing radius of the spherical cavity. The period of the cyclical variations is $\lambda_1/2$, where λ_1 is the wavelength in the insulating medium. This phenomena is similar to that occurring on transmission lines when the source and load impedances are not matched to the characteristic impedance of the line. There is also a cyclical variation in radiated power from a dipole located above the earth, but for this case, the variations diminish as the dipole is moved farther away from the interface because the reflected wave is dispersed and, hence, becomes weaker at the location of the dipole. For a dipole located within a spherical cavity the reflected wave from the interface always converges toward the origin where the dipole is located.

The denominator of the first factor in (26) can be expressed in the form

$$\left(1+\sqrt{r_1^2+r_2^2}\right)^2 - 2\sqrt{r_1^2+r_2^2}[1-\cos(2k_1R_1-\eta)]$$

where $\tan\eta = r_2/r_1$. When k_1R_1 becomes very large it is easy to show that

$$r_1 = \frac{1-b^2-a^2}{(1+b)^2+a^2} \tag{28a}$$

$$r_2 = \frac{2a}{(1+b)^2+a^2}. \tag{28b}$$

The maximum and minimum values of the first factor in (26) are

$$\frac{1}{\left(1-\sqrt{r_1^2+r_2^2}\right)^2} \quad \text{and} \quad \frac{1}{\left(1+\sqrt{r_1^2+r_2^2}\right)^2}.$$

As a typical case, let $a = b = 60$ for which the maximum and minimum values are found to be 3661 and 0.25. Thus, the radiated power shows a very large variation with changing radius of the insulating sphere when the contrast between the two media is large. The insulating sphere is functioning like a spherical cavity and results in a large field within the cavity and a large amount of radiated power at the resonant frequencies of the spherical cavity. The lowest resonance occurs when the diameter of the insulating sphere is approximately one wavelength in medium one.

By comparison with (11), that is, the power radiated by the same dipole when immersed in the dielectric medium of infinite extent, we find that the ratio of P_i to P_0 is shown in (29) at the bottom of the next page, where the coefficient A's are the same as defined in (27). Equation (29) also defines the ratio of the radiation resistance of an insulated dipole immersed in a dissipative medium to that of the same dipole when placed in the insulated dielectric of infinite extent.

It can be further demonstrated that if we identify the two terms defined in (15) as a forward and a backward propagating spherical wave, then the power reflection coefficient measured at the interface of the two media is

$$\begin{aligned} \Gamma_s &= \frac{\mathrm{Real}\left(\frac{1}{2}E_BH_B^*\right)}{\mathrm{Real}\left(\frac{1}{2}E_AH_A^*\right)} \\ &= \frac{|B|^2}{|A'|^2} = \frac{(1-b)^2+a^2}{(1+b)^2+a^2}\,\frac{P_B}{P_A} \end{aligned} \tag{30}$$

$$\begin{aligned} P_i &= \left[\frac{1}{1+r_1^2+r_2^2+2(r_1\cos 2k_1R_1 + r_2\sin 2k_1R_1)}\right] \\ &\times \frac{k_1^3D^3p^2b[2a+4a^2D+2a(a^2+b^2)D^2+(a^2+b^2)^2D^3]}{3\pi\varepsilon_1[(1+b)^2+a^2][A_0+A_1D+A_2D^2+A_3D^3+A_4D^4+A_5D^5+A_6D^6]} \end{aligned} \tag{26}$$

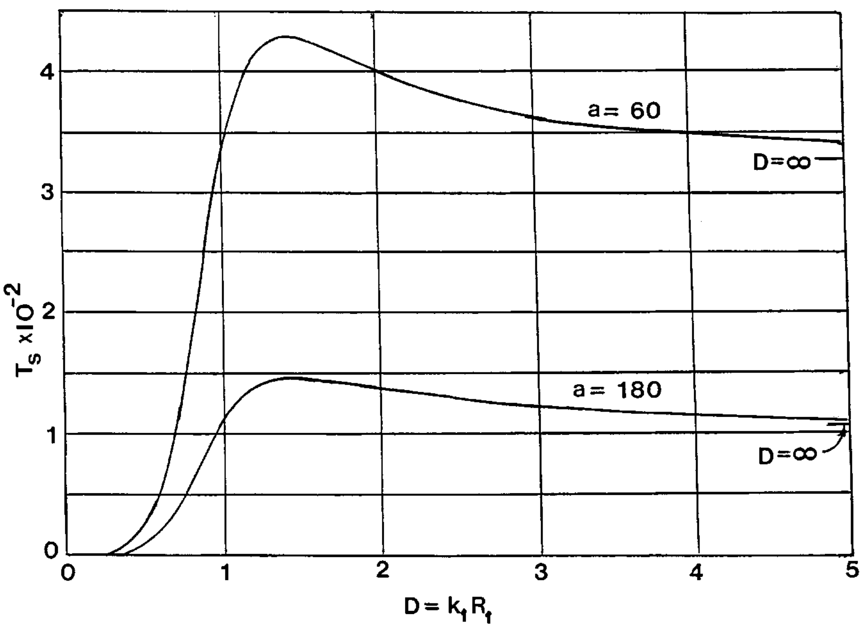

Fig. 4. Power transmission coefficient of the waves generated by an insulated dipole.

where P_A is the same sixth degree polynomial of D defined in the denominator of (26), and P_B is similar to P_A except that b is replaced by $-b$ in all the coefficients, A_0, A_1, etc. For large values of D, Γ_s reduces to

$$\frac{\Gamma_s}{\lim . D \to \infty} = \frac{(1-b)^2 + a^2}{(1+b)^2 + a^2} \tag{31}$$

which is the value of the reflection coefficient at normal incidence of plane waves traveling from the dielectric medium to the dissipative medium [4]. The power transmission coefficient at the interface is given by

$$T_s = \frac{P_i}{P_{\text{inc.}}} = \frac{P_i}{P_0}\frac{|A|^2}{|A'|^2} = |1 + r_s|^2 T_0 \tag{32}$$

which is also equal to the second factor that appears in (29). In (32) $P_{\text{inc.}}$ is the incident power on the interface for the insulated dipole case. It can be verified that, in general, we have

$$1 - \Gamma_s = T_s. \tag{33}$$

Thus, T_s is also equal to the refraction coefficient of the spherical waves generated by the Hertzian dipole. For a highly dissipative medium

$$a \approx b \approx \sqrt{\frac{\sigma}{2\omega\varepsilon_1}}. \tag{34}$$

Values of T_s for $a = b =$ 60 and 180 are plotted in Fig. 4. From the figure, it can be seen that the power transmission coefficient at the interface increases rapidly until $D = k_1 R_1$ becomes somewhat greater than unity. It then decreases to an asymptotic value as $k_1 R_1$ becomes large. The power transmission coefficient does not manifest any cyclical variation. The cyclical variation in radiated power is caused by the cyclical variation in the amplitude of the forward propagating spherical wave in the insulating cavity, which is a result of the varying phase angle of the reflected wave at the dipole location as the size of the cavity changes. The changing amplitude and phase of the reflected wave has a pronounced effect on the radiation impedance seen by the dipole.

VI. Reciprocal Theorem for Two Insulated Dipoles

Extension of Sommerfeld's reciprocal relation to two insulated dipoles can now be formulated based upon the result we have obtained concerning the EM radiation of an insulated dipole. We shall not give here the details of what Sommerfeld has used in his original work. Instead of using (2), we shall use (15) as the Hertzian potential defined in the neighborhood of a dipole. It is assumed, as in the original work, that reflected waves other than that produced at the interface between the insulator and the dissipative medium has a negligible effect upon the radiation of either dipole.

In order to obtain an expression that is not too complicated, we make the following assumptions: R_1 is large enough so that the Fresnel reflection coefficients may be used and only the part of the electric field that varies as the inverse first power of r needs to be retained. For the first situation, we consider a dipole

$$T_0 = \frac{P_i}{P_0} = \left[\frac{1}{1 + r_1^2 + r_2^2 + 2(r_1 \cos 2k_1R_1 + r_2 \sin 2k_1R_1)}\right] \times \frac{4D^3 b[2a + 4a^2 D + 2a(a^2+b^2)D^2 + (a^2+b^2)^2 D^3]}{[(1+b)^2 + a^2][A_0 + A_1 D + A_2 D^2 + A_3 D^3 + A_4 D^4 + A_5 D^5 + A_6 D^6]} \tag{29}$$

located within an insulating sphere of radius R_1 and with permittivity ε_1 and that radiates into a medium a with parameters ε_a, σ_a. Dipole 2 is located in medium a a distance r from the origin. The equations given earlier in the paper apply if we replace the subscript 2 by a new subscript a. E_a is the electric field radiated by dipole 1 at the location of dipole 2. It is also assumed that the dipoles are located and oriented for maximum interaction. For the second situation, we consider that dipole 2 is enclosed in an insulating sphere with the same radius but with a permittivity ε_2 and radiates a field E_b into medium b with parameters ε_b, σ_b. The last assumption is the same as that which Sommerfeld made, namely, that both dipoles radiate the same amount of power. Corrersponding to [2, eq. (18a)], we find a new reciprocal relationship which is

$$\frac{E_a}{E_b} = \frac{\varepsilon_2^{1/4}}{\varepsilon_1^{1/4}} \frac{\sqrt{1-\Gamma_b}}{\sqrt{1-\Gamma_a}} \frac{\left(1+\sqrt{\Gamma_a}e^{j\theta_a}\right)}{\left(1+\sqrt{\Gamma_b}e^{j\theta_b}\right)} \frac{e^{-jk_1R_1-jk_a(r-R_1)}}{e^{-jk_2R_1-jk_b(r-R_1)}} \tag{35}$$

Γ_a and Γ_b are the reflection coefficients defined by (30) and Θ_a and Θ_b are the phase angles of (B/A') without the factor $e^{-2jk_1R_1}$ defined in (22). Equation (35) is derived under the condition that $\mu_1 = \mu_2 = \mu_0$. If we consider the special case where $\varepsilon_1 = \varepsilon_a$ and $\varepsilon_2 = \varepsilon_b$ and make the additional assumption that the losses in media a and b are so small that the reflection coefficients can be set equal to zero then (35) gives

$$\frac{E_a}{E_b} = \frac{\varepsilon_2^{1/4}}{\varepsilon_1^{1/4}} \frac{e^{-jk_1R_1-jk_a(r-R_1)}}{e^{-jk_2R_1-jk_b(r-R_1)}}. \tag{36}$$

This expression is the same as Sommerfeld's with the exception of the propagation factors provided the losses are very small [see (12)], but is quite different from the expression given by Sommerfeld by [2, eq. (18a)]. Our conclusion is that Sommerfeld's reciprocal relationship is very limited and that the new one given by (35) is considerably more general.

VII. Conclusion

In this work, it was demostrated that a Hertzian dipole radiating in a lossy medium must be supplied with infinite power to overcome the loss in the surrounding medium. This limitation in the simple theory of a radiating dipole is removed by considering a dipole within an insulating sphere. Sommerfeld's reciprocity theorem in dissipative media is reformulated using the models of two insulated Hertzian dipoles. The treatment removes the deficiency found in Sommerfeld's formulation.

Acknowledgment

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