The Radiation Operator

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***Abstract*—The concept of the radiation operator is introduced to assist in the analysis of various problems involving sources and their radiation fields. It gives the field outside the source region as operating on the field of a point source. Because there is a simple connection between the radiation vector describing the far-field and the radiation operator, it can be used to define fields anywhere outside the source region from their values in the far-field zone. Another important property of the radiation operator is its ability to express sources of fields given their radiation pattern and polarization in the far zone. The source of such a field can be written in the form of radiation operator operating on a current element, the delta function source. To interpret this in terms of computable functions, existing tables of operational rules for different classes of operators can be applied. Examples of radiation operators corresponding to different sources are given together with examples of sources corresponding to given radiation field patterns. Finally, it is shown that the radiation operator allows a considerable simplification to the derivation of the multipole expansion theory when compared to the classical recursion-formula derivation through spherical harmonic eigenfunctions.**

***Index Terms*—Antenna theory, electromagnetic (EM) radiation, operator methods.**

I. INTRODUCTION

IN BOOKS on antennas and radiation problems (see, e.g., [1]–[4]) the far-field expressions for radiated fields are typically derived after approximating the distance function as

$$
\begin{aligned}
D(\mathbf{r}-\mathbf{r}') &= \sqrt{r^2 - 2\mathbf{r}\cdot\mathbf{r}' + r'^2}\\
&\approx r\sqrt{1-2\mathbf{u}_r\cdot\mathbf{r}'/r}\\
&\approx r - \mathbf{u}_r\cdot\mathbf{r}'
\end{aligned}
\tag{1}
$$

where

- $\mathbf{r}$ field point,
- $\mathbf{r}'$ point in the source region, and
- $\mathbf{u}_r$ radial unit vector $\mathbf{r}/r$.

After this, the vector potential field[1] is approximated as

$$
\mathbf{A}(\mathbf{r}) = \int_V \frac{e^{-jkD}}{4\pi D}\,\mathbf{J}(\mathbf{r}')\,dV' \approx \frac{e^{-jkr}}{4\pi r}\int_V e^{jk\mathbf{u}_r\cdot\mathbf{r}'}\mathbf{J}(\mathbf{r}')\,dV'. \tag{2}
$$

The vector function

$$
\mathbf{F}(\mathbf{u}_r) = \int_V e^{jk\mathbf{u}_r\cdot\mathbf{r}'}\mathbf{J}(\mathbf{r}')\,dV' \tag{3}
$$

sometimes called the radiation vector [4], depends on the direction of propagation $\mathbf{u}_r$ and thus gives the information of the field distribution in the far zone as well as the polarization of the vector potential.

Because of the many approximations involved, one tends to have the idea that some information of the true field is lost in the process. However, as it turns out, the far field carries along all information of the field everywhere outside the source region. This can be understood in terms of the radiation operator, to be introduced in the sequel, which has a simple connection to the radiation vector. Knowing the radiation vector, we can actually express the potential field everywhere outside the source region in terms of the radiation operator operating on the Green function, which is the field from a point source at the origin. Further, we are able to express the source of the radiated field in terms of the radiation operator operating on the point source, i.e., the delta function.

Operating with the radiation operator falls under the title "Heaviside operational calculus" [5], which preceded the modern Laplace and Fourier transformation techniques by a couple of decades. Of course, the operator method could be replaced by one based on integral transformations. However, because the operational technique works in the physical space without introducing a separate transformation space, the concepts appear more simple to grasp. Corresponding to the tables of integral transforms, operational rules for various operators $\mathbf{L}(\nabla)$ in the form $L(\nabla)\delta(\mathbf{r}) = f(\mathbf{r})$ where $f(\mathbf{r})$ is a computable function, are required. Tables of such rules have been collected by this author in the previous papers [6], [7], [20]. In the examples discussed in the present paper, demonstrating the use of the radiation operator, only basic ones of such operational rules are needed.

II. THE RADIATION OPERATOR

A. The Field

Let us consider fields radiated by a time-harmonic source $\mathbf{J}(\mathbf{r})$ occupying a confined region $\mathbf{V}$. The electric field $\mathbf{E}(\mathbf{r})$ can be expressed in terms of the vector potential $\mathbf{A}(\mathbf{r})$ as

$$
\mathbf{E}(\mathbf{r}) = -j\omega\mu\left(\overline{\overline{I}} + \frac{1}{k^2}\nabla\nabla\right)\cdot\mathbf{A}(\mathbf{r}). \tag{4}
$$

Outside V, the vector potential can be expressed as a regular integral

$$
\mathbf{A}(\mathbf{r}) = \int_V G(\mathbf{r}-\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')\,dV' \tag{5}
$$

in terms of the free-space Green function[2]

$$
G(\mathbf{r}-\mathbf{r}') = \frac{e^{-jkD}}{4\pi D} = -\frac{jk}{4\pi}\,h_0(kD) \tag{6}
$$

where D denotes the distance function

$$
D = D(\mathbf{r}-\mathbf{r}') = \sqrt{(\mathbf{r}-\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}. \tag{7}
$$

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[1]For convenience, we denote by $\mathbf{A}$ what is commonly written as $\mathbf{A}/\mu$.

[2]The spherical Hankel functions $h_n^{(2)}(kr)$ are denoted by $h_n(kr)$ for brevity.

Using the shifting operator [5] $e^{-\mathbf{r}'\cdot\nabla}$ we can express the distance and Green functions as

$$D(\mathbf{r}-\mathbf{r}') = e^{-\mathbf{r}'\cdot\nabla} D(\mathbf{r}) = e^{-\mathbf{r}'\cdot\nabla}\, r \tag{8}$$

$$G(\mathbf{r}-\mathbf{r}') = e^{-\mathbf{r}'\cdot\nabla}\, G(\mathbf{r}) = -\frac{jk}{4\pi}\, e^{-\mathbf{r}'\cdot\nabla}\, h_0(kr) \tag{9}$$

and the potential integral as

$$\mathbf{A}(\mathbf{r}) = \int_V e^{-\mathbf{r}'\cdot\nabla} G(\mathbf{r})\mathbf{J}(\mathbf{r}')\, dV' = \mathbf{L}(\nabla)G(\mathbf{r}) \tag{10}$$

where the vector operator $\mathbf{L}(\nabla)$ called the radiation operator is defined by

$$\mathbf{L}(\nabla) = \int_V \mathbf{J}(\mathbf{r}')e^{-\mathbf{r}'\cdot\nabla}\, dV'. \tag{11}$$

This has a form slightly resembling the Laplace or Fourier transformation of the source function. The dimension of the radiation operator is [Am].

Because the previous steps can be reversed, (10) and (5) actually represent the same potential function outside V. The radiation-operator form (10) appears simpler to use for finding the far-field approximation or multipole series expansion when the exponential operator is expressed in Taylor series. On the other hand, (5) remains valid also inside the source region when the integral is interpreted in such a way that the singularity is properly taken care of.

B. The Source

The source of the potential field can be found straightforwardly from the Helmholtz equation, if the potential is known analytically, as

$$(\nabla^2 + k^2)\mathbf{A}(\mathbf{r}) = -\mathbf{J}(\mathbf{r}). \tag{12}$$

The source can also be expressed in operator form by starting from the Green function equation

$$(\nabla^2 + k^2)G(\mathbf{r}) = -\delta(\mathbf{r}). \tag{13}$$

Operating on (10), we obtain the simple result (commutation of constant-coefficient operators is tacitly assumed here)

$$\mathbf{J}(\mathbf{r}) = -(\nabla^2 + k^2)\mathbf{L}(\nabla)G(\mathbf{r}) = \mathbf{L}(\nabla)\delta(\mathbf{r}). \tag{14}$$

This can be also formally verified through (11) as

$$\begin{aligned}\mathbf{L}(\nabla)\delta(\mathbf{r}) &= \int_V \mathbf{J}(\mathbf{r}')e^{-\mathbf{r}'\cdot\nabla}\delta(\mathbf{r})\, dV' \\ &= \int_V \mathbf{J}(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')\, dV' = \mathbf{J}(\mathbf{r}).\end{aligned} \tag{15}$$

It is thus seen that the radiation operator $\mathbf{L}(\nabla)$ contains all information of the radiated potential field. However, the source of the field must be understood in the sense of equivalent sources. Different expressions of $\mathbf{J}(\mathbf{r})$ are not necessarily analytically the same, but, in the field integral they produce the same potential field outside the source region. For example, making the Taylor expansion of the operator $\mathbf{L}(\nabla)$ in (14) in terms of powers of ∇ gives us a multipole source at the origin, equivalent to the original source.

C. Examples

Let us consider a few simple examples of electric current sources $\mathbf{J}(\mathbf{r})$ and the corresponding radiation operators. It is seen that, after a detour through the operator, the original source can be recovered, although other equivalent sources could also be possible results. This shows us that the information of the radiating source is not lost in the operator.

Dipole: For the elementary dipole with moment IL at the origin

$$\mathbf{J}(\mathbf{r}) = \mathbf{u}IL\delta(\mathbf{r}) \tag{16}$$

where $\mathbf{u}$ is a unit vector, the radiation operator (11) reduces to the constant vector

$$\mathbf{L}(\nabla) = IL\mathbf{u}. \tag{17}$$

Both (5) and (10) are seen to give the same potential field

$$\mathbf{A}(\mathbf{r}) = IL\mathbf{u}G(\mathbf{r}). \tag{18}$$

Also, the original source (16) is recovered as $\mathbf{L}(\nabla)\delta(\mathbf{r})$.

More generally, a dipole at the point $\mathbf{r}_o$

$$\mathbf{J}(\mathbf{r}) = \mathbf{u}IL\delta(\mathbf{r}-\mathbf{r}_o) \tag{19}$$

corresponds to the radiation operator

$$\mathbf{L}(\nabla) = \mathbf{u}ILe^{-\mathbf{r}_o\cdot\nabla}. \tag{20}$$

The original source (19) is again recovered as $\mathbf{L}(\nabla)\delta(\mathbf{r})$. By writing the radiation operator in Taylor series results in a multipole series expansion at the origin for the dipole:

$$\begin{aligned}\mathbf{J}(\mathbf{r}) = \mathbf{u}IL\Big[1 - (\mathbf{r}_o\cdot\nabla) + \frac{1}{2!}(\mathbf{r}_o\cdot\nabla)^2 \\ - \frac{1}{3!}(\mathbf{r}_o\cdot\nabla)^3 + \cdots\Big]\delta(\mathbf{r}).\end{aligned} \tag{21}$$

This multipole is equivalent to the original dipole, because inserted in the integral (5) and after partial integrations it yields

$$\mathbf{A}(\mathbf{r}) = \mathbf{u}ILG(\mathbf{r}-\mathbf{r}_o). \tag{22}$$

Linear Array of Dipoles: Taking an array of N parallel dipoles at equal distances a with progressive phase shift ϕ

$$\mathbf{J}(\mathbf{r}) = \mathbf{u}_x IL \sum_{n=0}^{N-1} e^{jn\phi}\delta(z-na)\delta(\rho) \tag{23}$$

the radiation operator (11) becomes

$$\begin{aligned}\mathbf{L}(\nabla) &= \mathbf{u}_x IL \sum_{n=0}^{N-1} e^{jn\phi}e^{-na\partial_z} \\ &= \mathbf{u}_x ILe^{j((N-1)/2)\phi}e^{-((N-1)/2)a\partial_z} \\ &\quad\cdot \frac{\sin\frac{N}{2}(\phi + ja\partial_z)}{\sin\frac{1}{2}(\phi + ja\partial_z)}.\end{aligned} \tag{24}$$

Applied to the Green function in (10) we have

$$\mathbf{A}(\mathbf{r}) = \mathbf{u}_x IL \sum_{n=0}^{N-1} e^{jn\phi} e^{-na\partial_z} G(\mathbf{r}) = \mathbf{u}_x IL \sum_{n=0}^{N-1} e^{jn\phi} G(\mathbf{r} - \mathbf{u}_z na). \quad (25)$$

Half-Wave Dipole: Finally, let us choose the current source as that of a thin dipole of length $2a$,

$$\mathbf{J}(\mathbf{r}) = \mathbf{u}_z I \sin k(a - |z|) U(a^2 - z^2)\delta(\rho) \quad (26)$$

where $U(x)$ denotes the Heaviside unit step function. Assuming a half-wave dipole we have $a = \pi/2k = \lambda/4$. Applying the integral formula

$$\int e^{ax}\sin(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin(bx+c) - b\cos(bx+c)\right] \quad (27)$$

the radiation operator becomes in this case

$$\mathbf{L}(\nabla) = \mathbf{u}_z I \int_{-a}^{a} \sin k(a - |z'|) e^{-z'\partial_z}\,dz' = \mathbf{u}_z 2kI \frac{\cosh(a\partial_z)}{\partial_z^2 + k^2}. \quad (28)$$

As a check, operating on the delta function and using known operator properties [6] we have

$$\begin{aligned}\mathbf{L}(\nabla)\delta(\mathbf{r}) &= \mathbf{u}_z 2jk\eta I \frac{\cosh(a\partial_z)}{\partial_z^2 + k^2}\delta(\mathbf{r}) \\ &= \mathbf{u}_z 2kI \frac{1}{2}\left[e^{a\partial_z} + e^{-a\partial_z}\right]\frac{1}{k}\sin(kz)U(z)\delta(\rho) \\ &= \mathbf{u}_z j\eta I[\sin(k(z+a))U(z+a) \\ &\quad + \sin(k(z-a))U(z-a)]\delta(\rho) \end{aligned} \quad (29)$$

which, again, reproduces the original source (26).

III. Synthesis of Radiation Patterns

It is shown that by requiring certain field pattern and polarization for the vector potential in the far zone uniquely determines the radiation operator. This, in turn, can be used to determine the source of the radiation, provided the operator expression can be interpreted in terms of computable functions.

A. The Far Field

Because the ∇ operator in (10) operates on the function e^{-jkr}/r, in the far zone $r \to \infty$ it can be replaced by $\nabla \to -jk\mathbf{u}_r$ and we have

$$\mathbf{A}(\mathbf{r}) \to \mathbf{L}(-jk\mathbf{u}_r)G(\mathbf{r}) = -\frac{jk}{4\pi}\mathbf{L}(-jk\mathbf{u}_r)h_0(kr). \quad (30)$$

In this limit, the radiation operator becomes an algebraic factor

$$\mathbf{L}(\nabla) \to \mathbf{L}(-jk\mathbf{u}_r) = \int_V \mathbf{J}(\mathbf{r}')e^{jk\mathbf{r}'\cdot\mathbf{u}_r}\,dV' = \mathbf{F}(\mathbf{u}_r) \quad (31)$$

a function of the unit vector $\mathbf{u}_r$, i.e., the direction of radiation. It can be identified as the radiation vector (3) with the dimension [Am]. It also contains the information on the polarization of the radiated potential field. To find the polarization of the corresponding electric field, the function must be multiplied by the projection dyadic $\overline{\overline{I}} - \mathbf{u}_r\mathbf{u}_r$.

It is obvious that information of the fields outside the sources is not lost in this limit process, because from the radiation vector by setting $\mathbf{u}_r \to -\nabla/jk$ we can restore the radiation operator

$$\mathbf{L}(\nabla) = \mathbf{F}(-\nabla/jk). \quad (32)$$

This means that knowledge of the far field gives us knowledge of the field everywhere outside the confined sources. However, as is well known [8], the source of the radiated fields is not unique and can be replaced by equivalent sources. From the far field we are able to determine sources equivalent to the original one and radiating the correct fields outside the source region.

B. The Synthesis

Given a radiation pattern for the vector potential in the form of the radiation vector $\mathbf{F}(\mathbf{u}_r)$, we can find in operational form a source function giving the same radiation pattern. In fact, applying (32), the possible source can be expressed from (14) as

$$\mathbf{J}(\mathbf{r}) = \mathbf{L}(\nabla)\delta(\mathbf{r}) = \mathbf{F}(-\nabla/jk)\delta(\mathbf{r}). \quad (33)$$

Now the problem remains to interpret this in terms of computable functions. This question has been treated in [6], [7], and [20] where tables for some operator families were presented for direct use.

In most practical cases, instead of the radiation vector, only its scalar magnitude and the polarization in the main direction are of interest. This leaves us somewhat more freedom for the synthesis, which is not easily applicable, however. It is not possible to delve into this problem here. Let us simply choose a radiating vector potential function with constant polarization defined by the unit vector $\mathbf{u}$ as

$$\mathbf{F}(\mathbf{u}_r) = \mathbf{u}F(\mathbf{u}_r) \quad (34)$$

where a scalar radiation function $F(\mathbf{u}_r)$ is left for the synthesis. This limits the polarization of the electromagnetic (EM) field to TM with respect to $\mathbf{u}$. Corresponding TE-polarized case can be handled in a similar way by starting from magnetic current sources. If the main beam is perpendicular to the vector $\mathbf{u}$, the polarization of the electric field in the main beam coincides with $\mathbf{u}$. Of course, this choice limits the source to the form $\mathbf{u}J(\mathbf{r})$. Let us consider some simple examples on synthesizing sources of radiated far fields.

1) Gaussian Beam: We wish to synthesize a radiator with the radiation vector of the form

$$\begin{aligned}\mathbf{F}(\mathbf{u}_r) &= \mathbf{u}_x ILe^{-2\kappa(\sin\theta/2)^2} \\ &= \mathbf{u}_x ILe^{-\kappa(1-\cos\theta)} \\ &= \mathbf{u}_x ILe^{-\kappa(1-\mathbf{u}_z\cdot\mathbf{u}_r)} \end{aligned} \quad (35)$$

which, for large values of the parameter κ, can be approximated by the Gaussian function

$$\mathbf{F}(\mathbf{u}_r) \approx \mathbf{u}_x ILe^{-\kappa\theta^2/2}. \quad (36)$$

The procedure above gives the radiation operator

$$\mathbf{L}(\nabla) = \mathbf{F}(-\nabla/jk) = \mathbf{u}_x ILe^{-\kappa(1+\mathbf{u}_z\cdot\nabla/jk)} \quad (37)$$

which leads to the equivalent source

$$\begin{aligned}\mathbf{J}(\mathbf{r}) &= \mathbf{L}(\nabla)\delta(\mathbf{r}) \\ &= \mathbf{u}_x ILe^{-\kappa}e^{-\kappa\mathbf{u}_z\cdot\nabla/jk}\delta(\mathbf{r}) \\ &= \mathbf{u}_x ILe^{-\kappa}\delta(\mathbf{r}-\kappa\mathbf{u}_z/jk).\end{aligned} \tag{38}$$

This is a dipole of moment IL located in the complex space point

$$\mathbf{r} = \mathbf{u}_z\kappa/jk \tag{39}$$

a result first given by Deschamps in 1971 [9]. When κ becomes large, the radiation beam becomes narrow. For $\kappa \to 0$ the dipole at the origin is obtained with constant radiation vector. This corresponds to the isotropic radiation pattern of the vector potential while that of the electric field is modified by the dyadic $\overline{\overline{I}} - \mathbf{u}_r\mathbf{u}_r$.

2) Endfire Pattern: As another example, let us take the radiation vector defined by

$$\begin{aligned}\mathbf{F}(\mathbf{u}_r) &= \mathbf{u}_x IL[\cos(2ka\sin^2(\theta/2))]^N \\ &= \mathbf{u}_x IL[\cos(ka - ka\mathbf{u}_z\cdot\mathbf{u}_r)]^N\end{aligned} \tag{40}$$

where N is a natural number $N = 1, 2, 3, \ldots$. For $ka = \pi/2$ this pattern has a single radiation lobe for $\theta = 0$. The radiation operator now becomes

$$\mathbf{L}(\nabla) = \mathbf{u}_x IL[\cos(ka - ja\mathbf{u}_z\cdot\nabla)]^N \tag{41}$$

and the source in the operator form is

$$\mathbf{J}(\mathbf{r}) = \mathbf{L}(\nabla)\delta(\mathbf{r}) = \mathbf{u}_x IL[\cos(ka - ja\mathbf{u}_z\cdot\nabla)]^N\delta(\mathbf{r}). \tag{42}$$

To interpret this, we apply the binomial expansion

$$\begin{aligned}[\cos\alpha]^N &= \frac{1}{2^N}\left[e^{j\alpha} + e^{-j\alpha}\right]^N \\ &= \frac{1}{2^N}\sum_{n=1}^{N}\binom{N}{n}e^{j(N-2n)\alpha}\end{aligned} \tag{43}$$

and the result can be expressed as

$$\begin{aligned}\mathbf{J}(\mathbf{r}) &= \mathbf{u}_x\frac{IL}{2^N}\sum_{n=1}^{N}\binom{N}{n}e^{j(N-2n)ka}e^{(N-2n)a\mathbf{u}_z\cdot\nabla}\delta(\mathbf{r}) \\ &= \mathbf{u}_x\frac{IL}{2^N}\sum_{n=1}^{N}\binom{N}{n}e^{j(N-2n)ka}\delta(\mathbf{r} + (N-2n)a\mathbf{u}_z).\end{aligned} \tag{44}$$

This corresponds to an array of elementary dipoles with binomial amplitude distribution and progressive phase, a well-known structure in classical antenna theory.

3) Broadside Pattern: As an example of a rotationally symmetric radiation pattern with a beam radiating normal to the axis of symmetry, let us consider the radiation vector

$$\mathbf{F}(\mathbf{u}_r) = \mathbf{u}_z I\frac{\sin(ka\cos\theta)}{k\cos\theta} = \mathbf{u}_z I\frac{\sin(ka\mathbf{u}_z\cdot\mathbf{u}_r)}{k\mathbf{u}_z\cdot\mathbf{u}_r}. \tag{45}$$

For $ka = \pi$ there is a null along the direction of the z-axis. The radiation operator is

$$\mathbf{L}(\nabla) = \mathbf{u}_z I\frac{\sinh(a\mathbf{u}_z\cdot\nabla)}{\mathbf{u}_z\cdot\nabla}. \tag{46}$$

Because of the operational rule [7], [20]

$$\frac{\sinh(a\partial_z)}{\partial_z}\delta(z) = \frac{1}{2}U(a^2 - z^2). \tag{47}$$

The source becomes the constant current line segment $-a \le z \le a$

$$\mathbf{J}(\mathbf{r}) = \mathbf{u}_z\frac{I}{2}U(a^2 - z^2)\delta(\rho). \tag{48}$$

4) Another Broadside Pattern: Let us study another rotationally symmetric pattern:

$$\mathbf{F}(\mathbf{u}_r) = \mathbf{u}_z Ia\frac{1}{\cosh(ka\cos\theta)} = \mathbf{u}_z Ia\frac{1}{\cosh(ka\mathbf{u}_z\cdot\mathbf{u}_r)}. \tag{49}$$

This pattern has only one lobe with maximum for $\theta = \pi/2$ and minimum for $\theta = 0, \pi$. For large ka the minimum is small. The radiation operator is

$$\mathbf{L}(\nabla) = \mathbf{u}_z Ia\frac{1}{\cos(a\mathbf{u}_z\cdot\nabla)}. \tag{50}$$

From the operational rule [7], [20]

$$\frac{1}{\cos(a\partial_z)}\delta(z) = \frac{1}{2a\cosh(\pi z/2a)} \tag{51}$$

we obtain the normalized source

$$\mathbf{J}(\mathbf{r}) = L(\nabla)\delta(\mathbf{r}) = \mathbf{u}_z\frac{I}{2\cosh(\pi z/2a)}\delta(\rho) \tag{52}$$

which is an infinite line current along the z-axis, whose amplitude has maximum at the origin and decays exponentially for $z \to \pm\infty$. For $a \to 0$ it approaches the delta function, i.e., a small current element. This limit corresponds to the constant radiation vector $\mathbf{F}(\mathbf{u}_r) = \mathbf{u}_z Ia$.

IV. MULTIPOLE EXPANSION

The radiation operator gives rise to another straightforward method to find equivalent sources in the form of multipoles by expressing the far field in terms of spherical harmonics. Let us again assume that the source and the vector potential have constant polarization $\mathbf{u}J(\mathbf{r})$, $\mathbf{u}A(\mathbf{u}_r)$, in which case we can work with scalar functions and operators $\mathbf{L}(\nabla) = \mathbf{u}L(\nabla)$, $\mathbf{F}(\mathbf{u}_r) = \mathbf{u}F(\mathbf{u}_r)$. This limits the polarization of the fields to TM with respect to the direction of $\mathbf{u}$. The TE polarization could be considered in a similar way in terms of another vector potential. Another approach through Debye potentials and TE, TM polarized (with respect to the radial direction) fields is also possible, but the definition of the source of a Debye potential is not so straightforward. Also functions $\mathbf{r}\cdot\mathbf{E}(\mathbf{r})$ and $\mathbf{r}\cdot\mathbf{H}(\mathbf{r})$ can be used instead of the potentials with simpler source descriptions, see [10]. However, to keep the analysis simple enough, let us limit to TM field problems and scalar formulation.

A. Spherical Harmonics

Outside the source region the amplitude of the vector potential can be expanded in spherical harmonics as [11]

$$A(\mathbf{r}) = \sum_{n=0}^{\infty}\left(A_n^0 P_n(\cos\theta)h_0(kr) + \sum_{m=1}^{n} \cdot\left[A_n^m e^{jm\varphi} + A_n^{-m}e^{-jm\varphi}\right]P_n^m(\cos\theta)h_n(kr)\right) \tag{53}$$

whose every term satisfies the Helmholtz equation. Because in the far field we have $h_n(kr) \to j^n h_0(kr)$, the vector potential expression is of the form

$$A(\mathbf{r}) \to F(\mathbf{u}_r)G(\mathbf{r}) = -\frac{jk}{4\pi}F(\mathbf{u}_r)h_0(kr) \tag{54}$$

with the radiation function defined as

$$F(\mathbf{u}_r) = -\frac{4\pi}{jk}\sum_{n=0}^{\infty} j^n\left(A_n^0 P_n(\cos\theta) + \sum_{m=1}^{n} \cdot\left[A_n^m e^{jm\varphi} + A_n^{-m}e^{-jm\varphi}\right]P_n^m(\cos\theta)\right). \tag{55}$$

The radiation function is uniquely represented by the coefficients A_n^m, $-n \le m \le n$. This is due to the orthogonality of the spherical harmonics [11]

$$\int_0^{2\pi}\int_0^{\pi}\left(P_n^m(\cos\theta)e^{-jm\varphi}\right)\left(P_{n'}^{m'}(\cos\theta)e^{jm'\varphi}\right)\sin\theta\, d\theta\, d\varphi = \frac{4\pi(n+m)!}{(2n+1)(n-m)!}\delta_{nn'}\delta_{mm'}. \tag{56}$$

The coefficients A_n^m can be obtained from the radiation function as

$$A_n^m = -\frac{jk}{4\pi}\frac{(2n+1)(n-|m|)!}{4\pi j^n(n+|m|)!}\int_0^{2\pi}\int_0^{\pi}F(\mathbf{u}_r) \cdot\left(P_n^m(\cos\theta)e^{-jm\varphi}\right)\sin\theta\, d\theta\, d\varphi. \tag{57}$$

B. Radiation Operator

Now we can find the scalar radiation operator $L(\nabla) = F(-\nabla/jk)$ by replacing

$$\cos\theta = \mathbf{u}_z\cdot\mathbf{u}_r \to -\mathbf{u}_z\cdot\nabla/jk = -\partial_z/jk \tag{58}$$

$$\begin{aligned} e^{\pm j\varphi} &= \cos\varphi \pm j\sin\varphi \\ &= \mathbf{a}_\pm\cdot\mathbf{u}_\rho \\ &= \frac{\mathbf{a}_\pm\cdot\mathbf{u}_r}{\sqrt{1-(\mathbf{u}_z\cdot\mathbf{u}_r)^2}} \to \frac{-\mathbf{a}_\pm\cdot\nabla/jk}{\sqrt{1-(-\partial_z/jk)^2}} \end{aligned} \tag{59}$$

where[3] the two complex vectors are defined as

$$\mathbf{a}_\pm = \mathbf{u}_x \pm j\mathbf{u}_y, \quad \mathbf{a}_+\cdot\mathbf{a}_- = 2, \quad \mathbf{a}_\pm\cdot\mathbf{a}_\pm = 0. \tag{60}$$

[3] A careful reader might note that the product $e^{j\varphi}e^{-j\varphi}$ would be replaced by an operator that is not unity. However, the difference from unity equals an operator containing $\nabla^2 + k^2$, which when operating on $h_0(kr)$ would give a multiple of the delta function. This means that the operator can be replaced by unity without changing the field for $\mathbf{r} \neq 0$.

In fact, substituting these in (55) we obtain the spherical harmonic expansion of the radiation operator as

$$\begin{aligned} L(\nabla) &= F(-\nabla/jk) \\ &= -\frac{4\pi}{jk}\sum_{n=0}^{\infty} j^n \\ &\quad\cdot\left(A_n^0 P_n(-\partial_z/jk) + \sum_{m=1}^{n}\frac{(-1)^m}{[1-(-\partial_z/jk)^2]^{m/2}}\right. \\ &\quad\cdot\left[A_n^m(\mathbf{a}_+\cdot\nabla/jk)^m + A_n^{-m}(\mathbf{a}_-\cdot\nabla/jk)^m\right] \\ &\quad\left.\cdot P_n^m(-\partial_z/jk)\right). \end{aligned} \tag{61}$$

Because of the above, the field defined by

$$A(\mathbf{r}) = -\frac{jk}{4\pi}F(-\nabla/jk)h_0(kr) \tag{62}$$

coincides with the original field outside the source area.

Applying the formula

$$P_n^m(x) = \left(-\sqrt{1-x^2}\right)^m P_n^{(m)}(x) \tag{63}$$

where $P_n^{(m)}(x)$ denotes m-fold differentiation of $P_n(x)$, the source $J(\mathbf{r})$ of the field can be written as

$$\begin{aligned} J(\mathbf{r}) &= L(\nabla)\delta(\mathbf{r}) \\ &= -\frac{4\pi}{jk}\sum_{n=0}^{\infty} j^n \\ &\quad\cdot\left(A_n^0 P_n(-\partial_z/jk) + \sum_{m=1}^{n}\right. \\ &\quad\cdot\left[A_n^m(\mathbf{a}_+\cdot\nabla/jk)^m + A_n^{-m}(\mathbf{a}_-\cdot\nabla/jk)^m\right] \\ &\quad\left.\cdot P_n^{(m)}(-\partial_z/jk)\right)\delta(\mathbf{r}). \end{aligned} \tag{64}$$

This expression represents a multipole source because the operator operating on the delta function is a polynomial operator involving derivatives of different orders. The same operator expression was derived in the classical papers [12], [13] and, for the static case, already by Maxwell [14], [15], through a considerably more complicated analysis than that above, by applying mathematical relations between the spherical harmonic functions.

An operator method similar to the multipole representation above has previously been introduced to the analysis of spherical near-field scanning, see [16]–[18]. In that method, the receiving antenna (probe) was expressed as a differential operator operating on the incident field to produce the measurable voltage.

C. Gaussian Pattern

As an example, let us again apply the multipole expansion method to the synthesis of the rotationally symmetric Gaussian radiation pattern. For simplicity, let us take the main radiation along the z-axis. In this case, the radiation pattern can be defined as

$$\begin{aligned} F(\mathbf{u}_r) &= ILe^{-2\kappa(\sin\theta/2)^2} \\ &= ILe^{-\kappa}e^{\kappa\cos\theta} \\ &= -\frac{4\pi}{jk}\sum_{n=1}^{\infty} j^n A_n^0 P_n(\cos\theta). \end{aligned} \tag{65}$$

Using the orthogonality of Legendre polynomials, the coefficients A_n^m are found to vanish for $m \neq 0$ and otherwise we have

$$\begin{aligned} A_n^0 &= -\frac{jk}{4\pi}\frac{2n+1}{2j^n}\int_0^\pi F(\mathbf{u}_r)P_n(\cos\theta)\sin\theta\,d\theta \\ &= -\frac{jkIL}{4\pi}e^{-\kappa}\frac{2n+1}{2j^n}\int_0^\pi e^{\kappa\cos\theta}P_n(\cos\theta)\sin\theta\,d\theta \\ &= -\frac{jkIL}{4\pi}e^{-\kappa}\frac{2n+1}{2}j^n\sqrt{\frac{2\pi}{\kappa}}I_{n+1/2}(\kappa). \end{aligned} \tag{66}$$

Here, we have applied the integral identity [19, (2.17.5.2)],

$$\int_{-1}^1 e^{\kappa x}P_n(x)\,dx = \sqrt{\frac{2\pi}{\kappa}}I_{n+1/2}(\kappa). \tag{67}$$

The radiation operator can, thus, be represented as the operator series

$$\begin{aligned} L(\nabla) &= F(-\nabla/jk) \\ &= IL\sqrt{\frac{2\pi}{\kappa}}e^{-\kappa}\sum_{n=1}^\infty \frac{2n+1}{2}I_{n+1/2}(\kappa)P_n(-\partial_z/jk). \end{aligned} \tag{68}$$

Invoking a formula from [19, vol. 2, (5.10.3.2)],

$$\sum_{n=0}^\infty (2n+1)I_{n+1/2}(\kappa)P_n(x) = \sqrt{\frac{2\kappa}{\pi}}e^{\kappa x} \tag{69}$$

the series can be summed in closed form and the radiation operator becomes quite simply

$$\begin{aligned} L(\nabla) &= IL\frac{e^{-\kappa}}{2}\sqrt{\frac{2\pi}{\kappa}}\sum_{n=0}^\infty (2n+1)I_{n+1/2}(\kappa)P_n(-\partial_z/jk) \\ &= ILe^{-\kappa}e^{-\kappa\partial_z/jk}. \end{aligned} \tag{70}$$

Thus, the normalized source becomes

$$\begin{aligned} J(\mathbf{r}) &= L(\nabla)\delta(\mathbf{r}) \\ &= ILe^{-\kappa}e^{-\kappa\partial_z/jk}\delta(\mathbf{r}) \\ &= ILe^{-\kappa}\delta(\mathbf{r}-\mathbf{u}_z\kappa/jk) \end{aligned} \tag{71}$$

which is a dipole of moment IL in complex space. The result coincides with that in (38).

V. Conclusion

The concept of radiation operator was introduced to simplify the analysis of EM radiation problems. It was seen that it can be used to express sources of given radiation fields in the far zone in compact operational form. For the interpretation, existing tables of operational rules expressing $L(\nabla)\delta(\mathbf{r})$ in functional form $f(\mathbf{r})$ can be used. The idea was elucidated through several simple examples. Also, an alternative multipole expansion method was discussed in terms of the radiation operator and which avoids the use of operational rules because the operators are polynomial. On the other hand, the coefficients must be found from integral expressions. The multipole series is not too attractive if too many multipoles are involved.

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